NAME: Prof. BAYLY

Math 410 (Prof. Bayly) EXAM 1: Monday 19 July 2004

There are ~~6~~ 5 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

(1)(10 points) Let $A = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$. Evaluate AA^T and A^TA, and show that

$$trace(A^TA) = trace(AA^T)$$

for any matrix of this form. Do you expect this to be true for matrices of other dimensions?

$$AA^T = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a^2+c^2+e^2 & ab+cd+ef \\ ba+dc+fe & b^2+d^2+f^2 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd & ae+bf \\ ca+db & c^2+d^2 & ce+df \\ ea+fb & ec+fd & e^2+f^2 \end{pmatrix}$$

$$\text{Trace}(AA^T) = (a^2+c^2+e^2)+(b^2+d^2+f^2)$$

$$\text{Trace}(A^TA) = (a^2+b^2)+(c^2+d^2)+(e^2+f^2) = \text{TRACE}(AA^T)$$

IN GENERAL $\text{Trace}(AA^T) = \text{Trace}(A^TA)$ = sum of squares of entries of A, for ANY matrix A.

(2)(20 points) Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

(a)(10 points) Find a lower-triangular matrix L and an echelon-form matrix U for which $A = LU$ (you don't need to check this).

(b)(10 points) Solve $L\vec{c} = \vec{b}$ for $\vec{c}$, and then solve $U\vec{x} = \vec{c}$ for $\vec{x}$.

Extra Credit (5 points): If the matrix A has n rows and columns, with 2's down the diagonal and -1's in a band just above and just below, and $\vec{b}$ an n-vector of 2's, what will L and U and $\vec{c}$ be? (I.e. guess the patterns!)

(a) Row reduce, saving multipliers:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow[\text{all others } = 0 \text{ in column } 1]{l_{21} = -\frac{1}{2}} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow[\text{all others} = 0 \text{ in col } 2]{l_{32} = -\frac{2}{3}}$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{l_{43} = \frac{-3}{4}} \underbrace{\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}}_{U}$$

ECHELON FORM! This is U

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix}$$

Pattern: Diagonal of U will continue $6/5, 7/6, 8/7, \ldots$

Below diagonal of L will continue $-4/5, -5/6, -6/7, \ldots$

(2b) $L\vec{c} = \vec{b}$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ -1/2 & 1 & 0 & 0 & 2 \\ 0 & -2/3 & 1 & 0 & 2 \\ 0 & 0 & -3/4 & 1 & 2 \end{array}\right) \Rightarrow \begin{array}{l} c_1 = 2 \\ -\frac{1}{2}c_1 + c_2 = 2 \\ \Rightarrow c_2 = 3 \end{array}$$

row 3 : $-\frac{2}{3}c_2 + c_3 = 2 \Rightarrow c_3 = 2+2 = 4$

row 4: $-\frac{3}{4}c_3 + c_4 = 2 \quad c_4 = 2+3 = 5$

~~EXCR~~ Pattern: $c_5 = 6, \; c_6 = 7, \; \ldots$

$U\vec{x} = \vec{c}$

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 2 \\ 0 & 3/2 & -1 & 0 & 3 \\ 0 & 0 & 4/3 & -1 & 4 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right)$$

$$\boxed{\vec{x} = \begin{pmatrix} 4 \\ 6 \\ 6 \\ 4 \end{pmatrix}}$$

Excr does NOT ask for $\vec{x}$ pattern!

row 4 $\Rightarrow$ $5/4\, x_4 = 5 \quad x_4 = 4$

row 3 $\quad \frac{4}{3}x_3 - x_4 = 4 \quad \frac{4}{3}x_3 = 8 \quad x_3 = 6$

row 2 $\quad \frac{3}{2}x_2 - x_3 = 3 \quad \frac{3}{2}x_2 = 9 \quad x_2 = 6$

row 1 $\quad 2x_1 - x_2 = 2 \quad 2x_1 = 8 \quad x_1 = 4$

(3)(30 points) Consider the linear system

$$x + y + 2z = b_1 \quad , \quad 2x + y + 3z = b_2 \quad , \quad 4x + 3y + 7z = b_3.$$

(a)(10 points) Write down the augmented matrix for this system, and use Gaussian elimination to reduce it to echelon form. Identify the pivot(s) and free variable(s).

(b)(10 points) Under what compatibility conditions on $\vec{b}$ does a solution exist? Find the general solution, identify the particular and complementary parts, and the null vector(s).

(c)(10 points) What are the dimensions of the range and kernel of the matrix in this problem? Write down a basis for each space.

(a)
$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 2 & 1 & 3 & b_2 \\ 4 & 3 & 7 & b_3 \end{array}\right) \xrightarrow[\ell_{31}=4]{\ell_{21}=2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2-2b_1 \\ 0 & -1 & -1 & b_3-4b_1 \end{array}\right)$$

$$\xrightarrow{\ell_{32}=1} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 2 & b_1 \\ 0 & \boxed{-1} & -1 & b_2 \\ 0 & 0 & 0 & (b_3-4b_1)-(b_2-2b_1) = -2b_1-b_2+b_3 \end{array}\right)$$

2 PIVOTS in columns 1, 2, FREE VARIABLE is z

(b) Compatibility condition: Need $-2b_1 - b_2 + b_3 = 0$ for solution to exist.

IF SO: 2nd row $\Rightarrow$ $-y - z = b_2 \Rightarrow \boxed{y = -z - b_2}$

1st row $\Rightarrow$ $x + y + 2z = b_1 \quad x + (-z - b_2) + 2z = b_1$

$$\boxed{x = b_1 + b_2 - z}$$

3b continued $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 + b_2 - z \\ -z - b_2 \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 + b_2 \\ -b_2 \\ 0 \end{pmatrix}}_{\vec{x}^{(p)}} + \underbrace{z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{\vec{x}^{(c)}}$

Basic Null vector $\vec{n} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

3c Dim (Range) = RANK = # pivots = 2

BASIS for range is $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Nullspace = Kernel has dimension 1, Basis is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

(4)(20 points) Write the system of linear equations

$$9x + ay = -3 \quad , \quad ax + y = 1$$

in matrix-vector form (a is assumed to be a given real number).

$$\left(\begin{array}{cc|c} 9 & a & -3 \\ a & 1 & 1 \end{array}\right)$$

(a)(10 points) For what values of a are there no solutions, one unique solution, or many (i.e. an infinite number) solutions?

(b)(5 points) For the value of a giving no solutions, sketch the aiming vectors and the target vector, and indicate graphically why there is no solution.

(c)(5 points) For the value of a giving many solutions, sketch the aiming vectors and the target vector, and indicate graphically why there are many solutions.

(a) Row reduce: $L_{21} = \frac{a}{9} \longrightarrow \left(\begin{array}{cc|c} 9 & a & -3 \\ 0 & 1-\frac{a^2}{9} & 1+3a/9 \end{array}\right)$

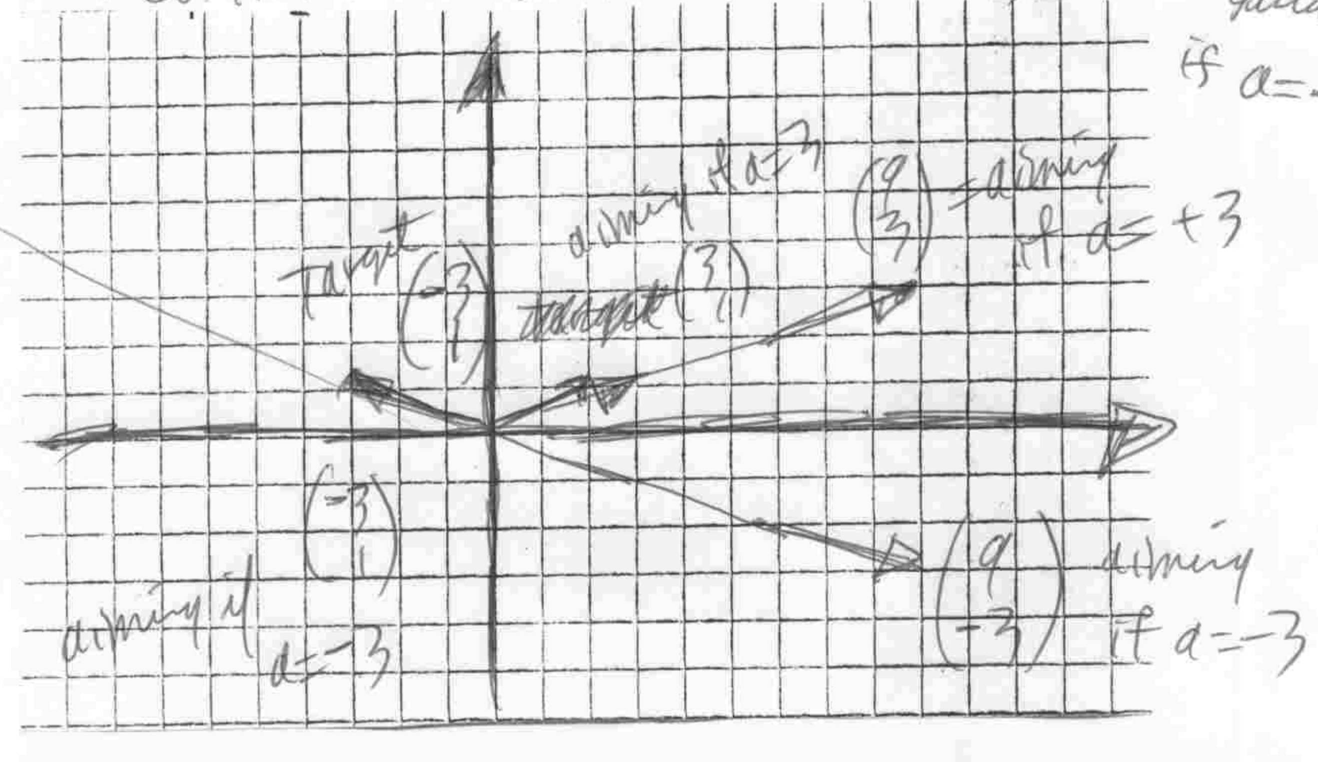
If $a^2 \neq 9$ then RANK $= 2$, no zero rows, no free variables

$\Rightarrow$ UNIQUE SOLUTION.

If $a^2 = 9 \Rightarrow a = \pm 3$ need to be considered separately (then Y is FREE VARIABLE)

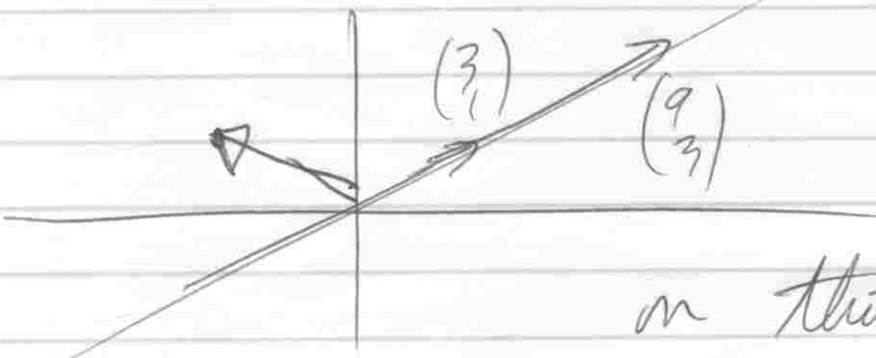
If $1 + 3a/9 = 0$ then 2nd row is $0\ 0 | 0$ is COMPATIBLE $\Rightarrow$ solution exists $\Rightarrow \infty$ solutions if $a=-3$

0 solution if $a=+3$ since $1+\frac{3a}{9} = 2 \neq 0$



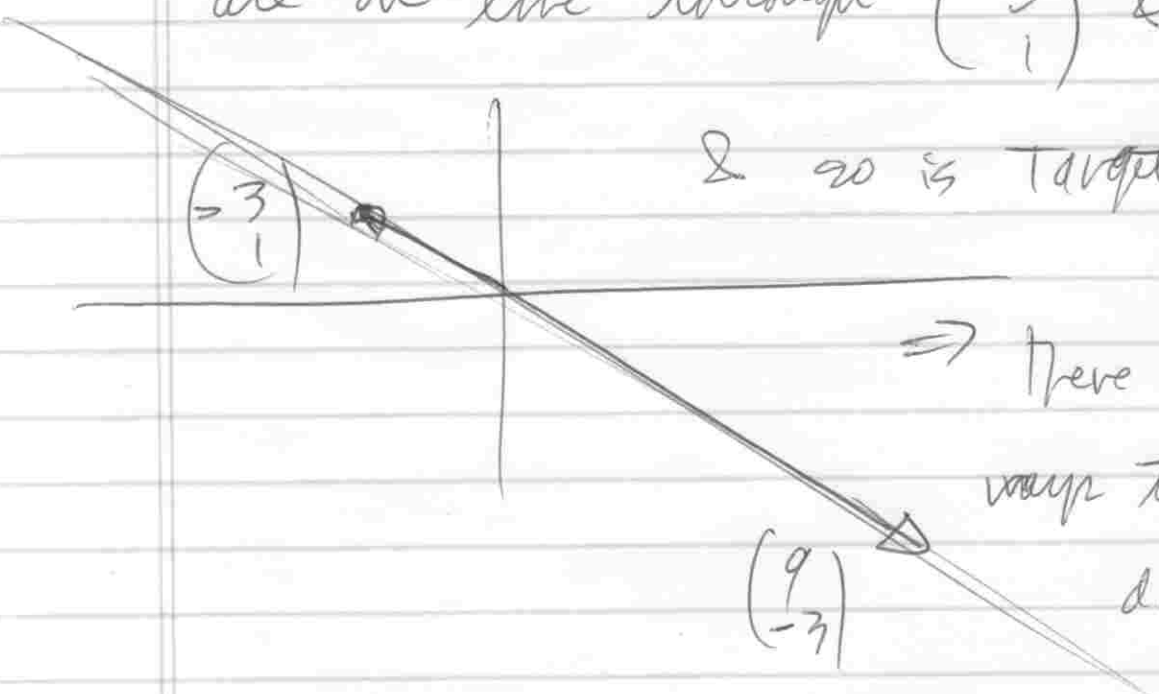
④ continued

ⓑ when $a=3$ both aiming vectors are on line through $\begin{pmatrix}3\\1\end{pmatrix}$ & $\begin{pmatrix}9\\3\end{pmatrix}$



BUT Target is NOT on this line!

ⓒ When $a=-3$ BOTH aiming vectors are on line through $\begin{pmatrix}-3\\1\end{pmatrix}$ & $\begin{pmatrix}9\\-3\end{pmatrix}$



& so is Target $= \begin{pmatrix}-3\\1\end{pmatrix}$!

$\Rightarrow$ There are many ways to combine driving vectors.

(5)(20 points) The infinite series $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ applies to square matrices also:

$$(I+X)^{-1} = I - X + X^2 - X^3 + \cdots,$$

though it's not clear what's the use of a formula with an infinite number of terms. Let's investigate this formula when

$$X = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}.$$

(a)(10 points) Calculate X^2 and X^3. What can you say about higher powers of X?

(b)(10 points) Evaluate the resulting formula for $(I+X)^{-1}$ (which should look familiar from the homework), and check by multiplying by $I+X$.

(a) $X^2 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ac & 0 & 0 \end{pmatrix}$

$X^3 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ac & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ All Higher powers ALSO $= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$!

(b) Conclude $(I+X)^{-1} = I - X + X^2 + \underbrace{0 + 0 + 0 \ldots}_{}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ac & 0 & 0 \end{pmatrix}$ all zero

$= \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{pmatrix}$ just as on HW for $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}^{-1}$!