PROF. BAYLY

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Math 410 (Prof. Bayly) EXAM 2: Monday 26 July 2004

There are 4 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

(1)(20 points) Consider the quadratic function of 3 variables $\vec{x} = (x, y, z)^T$:

$$f(\vec{x}) = x^2 - 2xy + \frac{1}{2}y^2 + 4xz - 3yz + \frac{9}{2}z^2 - 4x + 4y - 10z + 751.$$

(a)(10 points) To find maximum or minimum value of f, we set its partial derivatives equal to zero. What is the resulting system of equations for $\vec{x}$?

(b)(10 points) Will the solution $\vec{x}$ be unique? Will it be a maximum or minimum of f? If not, how would you describe it? (You don't need to solve for $\vec{x}$ but you should give calculations supporting your answers.)

ⓐ $0 = \frac{\partial f}{\partial x} = 2x - 2y + 4z - 4 = 0$

$\frac{\partial f}{\partial y} = -2x + y - 3z + 4 = 0$

$\frac{\partial f}{\partial z} = 4x - 3y + 9z - 10 = 0$

$$\underbrace{\begin{pmatrix} 2 & -2 & 4 \\ -2 & 1 & -3 \\ 4 & -3 & 9 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 10 \end{pmatrix}$$

ⓑ To answer these questions, ROW-REDUCE A:

$$\begin{pmatrix} 2 & -2 & 4 \\ -2 & 1 & -3 \\ 4 & -3 & 9 \end{pmatrix} \xrightarrow[\ell_{31}=2]{\ell_{21}=-1} \begin{pmatrix} 2 & -2 & 4 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\ell_{32}=-1} \begin{pmatrix} 2 & -2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

CONCLUDE ~~the~~ QUADRATIC PART of $f(\vec{x})$ is INDEFINITE because there are POSITIVE & NEGATIVE pivots $\Rightarrow$ Neither max nor min but SADDLE POINT!

(2)(25 points) Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(a)(5 points) Find the general solution of this system and express its $(length)^2$ as a function of the free variable.

(b)(5 points) Find the value of the free variable that minimizes the $(length)^2$, and write the corresponding solution vector.

(c)(5 points) Calculate the so-called *right pseudoinverse* $R = A^T(AA^T)^{-1}$.

(d)(5 points) Calculate AR and RA. Which yields the identity matrix?

(e)(5 points) Calculate $R\vec{b}$; is it the same minimum-length solution you found in (b)?

(a) Augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$ already in ECHELON FORM

$\Rightarrow \boxed{z \text{ free}}$, $y - z = -1 \Rightarrow \boxed{y = z - 1}$, $\vec{x} = \begin{pmatrix} 1-2z \\ z-1 \\ z \end{pmatrix}$

$x + 2z = 1 \Rightarrow \boxed{x = 1 - 2z}$

$\text{LENGTH}^2 = (1-2z)^2 + (z-1)^2 + z^2$

(b) First simplify $\text{LENGTH}^2 = (1 - 4z + 4z^2) + (z^2 - 2z + 1) + z^2$

$= 6z^2 - 6z + 2$

Derivative $= 12z - 6 = 0$ if $z = 1/2$

$\Rightarrow$ min. length solution is $\begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$.

2c $AA^T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$

$\Rightarrow (AA^T)^{-1} = \frac{1}{10-4} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

$A^T(AA^T)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & 5 \\ 2 & -1 \end{pmatrix}$

2d $AR = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & 5 \\ 2 & -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= I$!

$RA = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & 5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{pmatrix} \neq I$

2e $R\vec{b} = \frac{1}{6} \begin{pmatrix} 2 & 2 \\ 2 & 5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$

YES!

(3)(30 points) Four teams A, B, C, and D play a tournament, and the results may be described as follows: In game a B beats A by 1 at B's home field, in game b C beats B by 2 at C's home field, and in game c D beats B by 2 at D's home field. We would like to find a vector $\vec{p}$ of team "potentials" such that the differences between potentials equal the score differences in the actual games.

(a)(5 points) Draw a network representing the tournament, in which nodes are teams and edges represent games. The head of the arrow should indicate the home team, the tail the visiting team, and the numerical value = (home score) - (visitor score).

(b)(5 points) If the potential p_D of team D is a free variable, show that you can quickly solve for the potentials of the other teams. (You DO NOT need to use the edge-node incidence matrix!)

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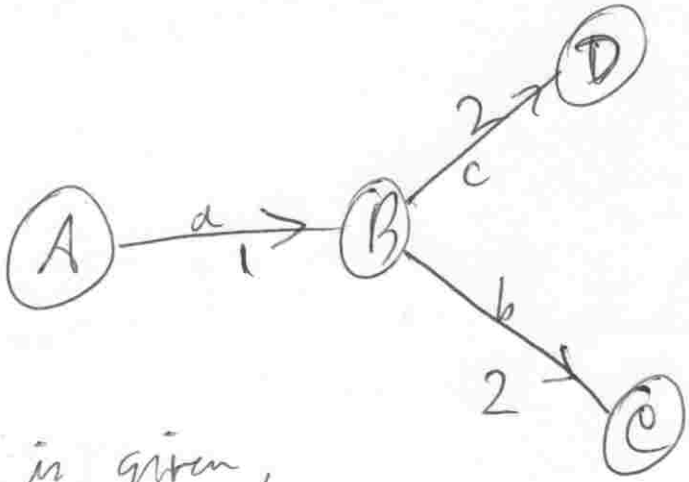
(a)



(b) If P_D is given,

then game c $\Rightarrow$ $P_D - P_B = 2$ $\Rightarrow$ $\boxed{P_B = P_D - 2}$

game b $\Rightarrow$ $P_C - P_B = 2$ $\Rightarrow$ $\boxed{P_C = P_B + 2 = P_D}$

game a $\Rightarrow$ $P_B - P_A = 1$ $\Rightarrow$ $\boxed{P_A = P_B - 1 = P_D - 3}$

C, D tied for first; B next, A last,

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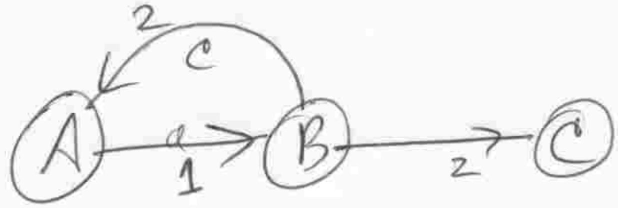
(3)(c)(5 points) It is suddenly discovered that team D is really team A in disguise. Draw the network representing the new situation, and write down the corresponding matrix.

(d)(5 points) By looking at the network, guess the null vector for the transpose of the matrix, and check by actually multiplying them.

(e)(10 points) Is there still an exact solution for the potential vector $\vec{p}$? If not, find the least squares best approximate solution, and give the resulting ranking of the teams.

Extra Credit (5 points) Do you expect to be able to find an exact solution to the potential equations for any network with no loops? Give a convincing argument.

(c) If D is A then game c goes back to node A



$$\text{Matrix } A = \begin{array}{c|ccc} & A & B & C \\ \hline a & -1 & 1 & 0 \\ b & 0 & -1 & 1 \\ c & 1 & -1 & 0 \end{array}$$

(d) There is clearly a loop along a & c edges, both in + direction

$$\Rightarrow \vec{m} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow \text{edge a} \\ \\ \leftarrow \text{edge c} \end{matrix}$$

Check

$$A^T\vec{m} = \begin{pmatrix} -1 & 0 & +1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark !$$

(e) See next page for calculation

$$\vec{p} = \begin{pmatrix} -3/2 \\ -2 \\ 0 \end{pmatrix} + p_c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad p_c \text{ free}$$

$\Rightarrow$ C Best, then A, then B.

3e We want $A\vec{p} = \vec{d}$ = vector of score differences:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} p_A \\ p_B \\ p_C \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

row 1, row 3 are INCOMPATIBLE !

Least Squares: $A^T A \vec{p} = A^T \vec{d}$

$$A^T A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$A^T \vec{d} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & -2 & 0 & 1 \\ -2 & 3 & -1 & -3 \\ 0 & -1 & 1 & 2 \end{array}\right) \xrightarrow[\ell_{31}=0]{\ell_{21}=-1} \left(\begin{array}{ccc|c} 2 & -2 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & 1 & 2 \end{array}\right) \rightarrow \begin{pmatrix} \text{same} \\ \text{same} \\ \leftarrow 0 \rightarrow \end{pmatrix}$$

$\boxed{p_C \text{ free}}$

Row 2 $\Rightarrow p_B - p_C = -2 \Rightarrow p_B = p_C - 2$

Row 1 $2p_A - 2p_B = 1$

$\Rightarrow p_A = p_B + 1/2$

$p_A = p_C - 3/2$

(4)(25 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

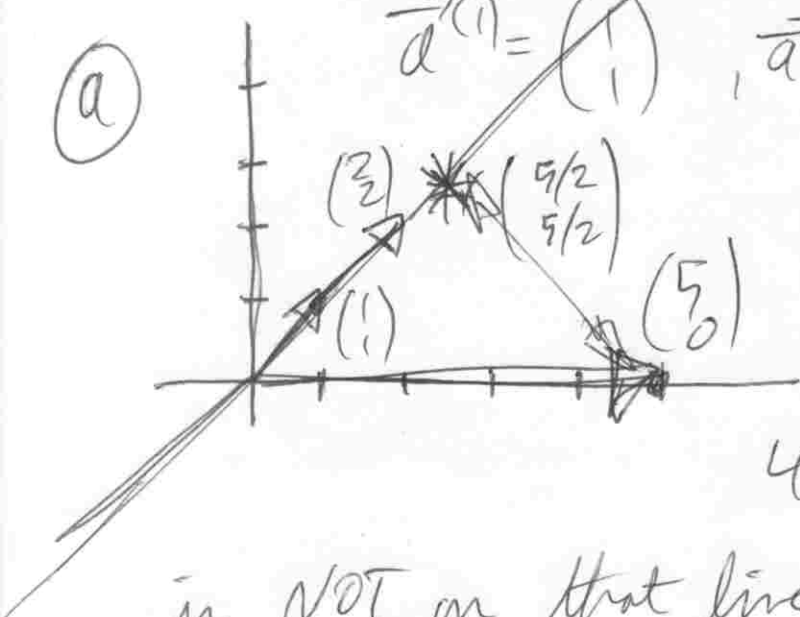
(a)(5 points) Sketch the aiming and target vectors and indicate graphically why there is no solution.

(b)(10 points) Find the least squares best approximate solution $\vec{x}$. Is it unique?

(c)(5 points) Draw the "closest point in the range" $A\vec{x}$, and the error vector $A\vec{x} - \vec{b}$, on your diagram. Is the error orthogonal to the range of A?

(d)(5 points) Find the minimum-length least-squares solution. Is it unique?

(a) $\vec{a}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{a}^{(2)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$



RANGE of A consists only of points on the 45° line. WHILE $\vec{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ is NOT on that line.

(b) $A^TA = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$, $A^T\vec{b} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2 & 4 & 5 \\ 4 & 8 & 10 \end{array}\right) \xrightarrow{L_{21}=2} \left(\begin{array}{cc|c} 2 & 4 & 5 \\ 0 & 0 & 0 \end{array}\right)$$

$\boxed{y \text{ free}}$ $\quad 2x + 4y = 5 \Rightarrow \boxed{x = \frac{5}{2} - 2y}$

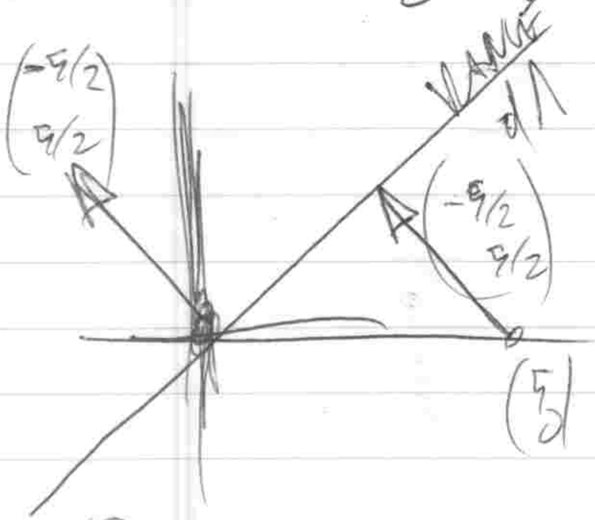
$$\vec{x} = \begin{pmatrix} 5/2 \\ 0 \end{pmatrix} + y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$ is NOT UNIQUE!

(4) (c) Closest point is $A\vec{x} = \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}\frac{5}{2}-2y\\ y\end{pmatrix}$ (8)

$$= \begin{pmatrix}5/2 \cancel{4} -2y+2y\\ 5/2-2y+2y\end{pmatrix} = \begin{pmatrix}5/2\\ 5/2\end{pmatrix}$$

Error vector $A\vec{x}-\vec{b} = \begin{pmatrix}5/2\\ 5/2\end{pmatrix} - \begin{pmatrix}5\\ 0\end{pmatrix} = \begin{pmatrix}-5/2\\ 5/2\end{pmatrix}$

Sure LOOKS $\perp$ to range of A!



check $\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}-5/2\\ 5/2\end{pmatrix}=0$

$\begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}-5/2\\ 5/2\end{pmatrix}=0$

(d) LENGTH^2 of $\vec{x}$ is $\left(\frac{5}{2}-y\right)^2+y^2$

$$= \frac{25}{4}-5y+y^2+y^2 = 2y^2-5y+\frac{25}{4}$$

DERIVATIVE is $4y-5 = 0$ if $y = 5/4$

$\Rightarrow x = 5/2 - 2(5/4) = 0$

$\boxed{\text{MIN LENGTH SOL} = \begin{pmatrix}0\\ 5/4\end{pmatrix}}$ & UNIQUE !!