SOLUTIONS by PROF. BAYLY

Math 410 (Prof. Bayly) FINAL EXAM (comprehensive part): Wednesday 11 August 2004

There are 4 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

NOTE Problems 3 and 4 are rather lengthy; I have included a blank sheet between them to give you extra writing space.

(1)(20 points) Consider the functions of $\vec{x} = (x, y)^T$:

$$q(\vec{x}) = x^2 + 4xy + 5y^2 \quad , \quad d(\vec{x}) = x^2 + y^2 \quad , \quad r(\vec{x}) = q(\vec{x})/d(\vec{x}).$$

(a)(10 points) Express $q(\vec{x})$ as $\vec{x}^T A\vec{x}$ for a symmetric matrix A. Factor A into the product of a lower triangular matrix L and upper triangular U, and determine the possible range of values of $q(\vec{x})$.

(b)(5 points) Evaluate $r(\vec{x})$ at the points (1,0), (0,1), (1,1), (1,-1), and one other point of your choosing.

(c)(5 points) Determine the maximum and minimum values of the function r, and check that they are larger and smaller, respectively, than the largest and smallest values you found in (b).

(a) $q(\vec{x}) = (x\ y)\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

Row reduce: $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \xrightarrow{l_{21}=2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = U \quad ; \; L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$\Rightarrow q(\vec{x}) = (x\ y)\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x+2y)^2 + y^2$

ALWAYS ≥ 0 ; $=0$ only if $x=y=0$.

(b), (c) on next page:

1b $r(x,y) = \dfrac{x^2+4xy+5y^2}{x^2+y^2}$ 2

$\vec{x}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}$
$r(\vec{x})$	1	5	$\frac{10}{2}=5$	$\frac{2}{2}=1$	$\frac{29}{5}\approx 5.8$

MAX = 5.8
MIN = 1
SO FAR

1c Max / Min of $r(\vec{x})$ are largest / smallest eigenvalues of A

$$p_A(\lambda) = \det\begin{pmatrix}1-\lambda & 2\\ 2 & 5-\lambda\end{pmatrix} = \lambda^2 - 6\lambda + 5 - 4$$
$$= \lambda^2 - 6\lambda + 1$$

Roots $\lambda = \frac{1}{2}\left[6 \pm \sqrt{36-4=32}\right] = \frac{1}{2}\left[6 \pm 4\sqrt{2}\right]$

$\lambda = 3 \pm 2\sqrt{2}$

$$\boxed{\begin{aligned}\text{Max} &= 3+2\sqrt{2}\\ \text{Min} &= 3-2\sqrt{2}\end{aligned}}$$

Since $\sqrt{2} \approx 1.4$ plus a little bit

Max $\lambda = 3+2\sqrt{2} \approx 3 + 2.8\ldots = 5.8$ plus a little bit

Min $\lambda = 3-2\sqrt{2} \approx 3-2.8$ − a little bit

$\approx .2$ − a little bit

SURE ENOUGH, agrees with results in b.

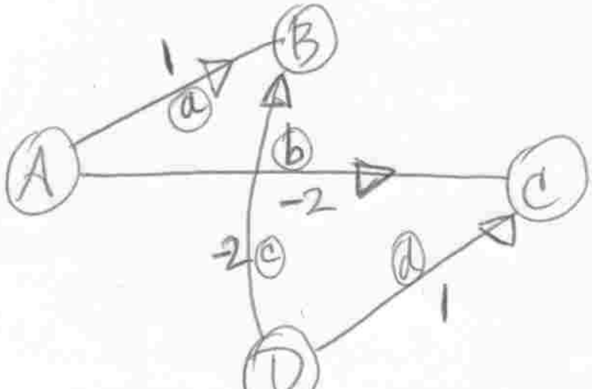
(2)(20 points) Four teams A, B, C, and D play a tournament, and the results may be described as follows: In game a B beats A by 1 at B's home field, in game b A beats C by 2 at C's home field, in game c D beats B by 2 at B's home field, and in game d C beats D by 1 at C's home field. We would like to find a vector $\vec{p}$ of team "potentials" such that the differences between potentials equal the score differences in the actual games.

(a)(5 points) Draw a network representing the tournament, in which nodes are teams and edges represent games. The head of the arrow should indicate the home team, the tail the visiting team, and the numerical value = (home score) - (visitor score).

(b)(10 points) Write down the edge-node matrix A for the network, the vector $\vec{b}$ of score differences. Then, by looking at the network, guess the null vector $\vec{m}$ for A^T, and check by actually multiplying. Can you tell from $\vec{m}$ and $\vec{b}$ if a solution exists to the system $A\vec{x} = \vec{b}$?

(c)(5 points) If there was no exact solution to $A\vec{x} = \vec{b}$, find a related linear system for the least squares best approximate solution. (You do NOT have to solve it! Just find the related matrix and right-hand side.)

(a)



(b)

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

(b) cont. We know Null vector of A^T correspond to LOOPS in the network. There is a loop if you go along

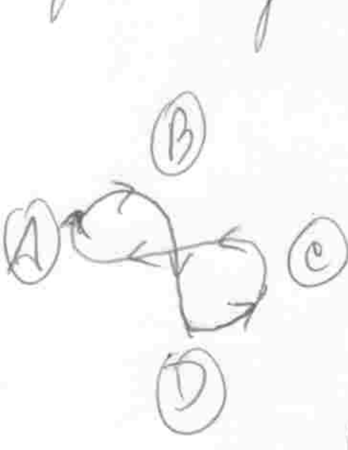
(a) in + direction

(b) in − direction

(c) in − direction

(d) in + direction

$$\Rightarrow \vec{m} = \begin{pmatrix} +1 \\ -1 \\ -1 \\ +1 \end{pmatrix}$$



(LAZY FIGURE-8!)

Check $A^T \vec{m} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

(b) Vector of score differences is $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}$

$A\vec{x} = \vec{b}$ has a solution only if $\vec{m}^T \vec{b} = 0$.

BUT here $\vec{m}^T \vec{b} = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} = 6$

$\Rightarrow$ NO SOLUTION!

(c) We could seek LEAST SQUARES best approx:

$(A^T A)\vec{x} = A^T \vec{b}$, is guaranteed to have a solution.

Here $A^T A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$

$A^T \vec{b} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.

5

(3)(30 points) Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & -1 & 0 \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}.$$

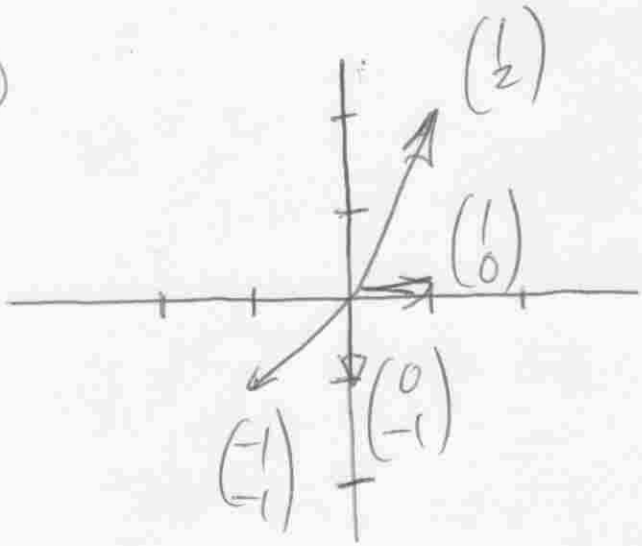
(a)(5 points) Draw the aiming vectors in the 2d plane.

(b)(10 points) Find the general solution to the homogeneous system $A\vec{x} = \vec{0}$ and express as a combination of free variables multiplying null vectors.

(c)(5 points) The entries in each null vector indicate a combination of aiming vectors that, if placed head-to-tail, will return to the starting point. Choose one of the null vectors and sketch a picture indicating this behavior.

(d)(10 points) Suppose $\vec{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$. The minimum-length solution to $A\vec{x} = \vec{b}$ can be sought indirectly as $\vec{x} = A^T\vec{u}$, where $\vec{u}$ satisfies $(AA^T)\vec{u} = \vec{b}$. Find this min length solution.

(a)



(b) Row reduce augmented matrix

pivots, free, free

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 & 0 \end{array}\right) \xrightarrow{\ell_{21}=2} \left(\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & -2 & 0 \end{array}\right)$$

y free, z free

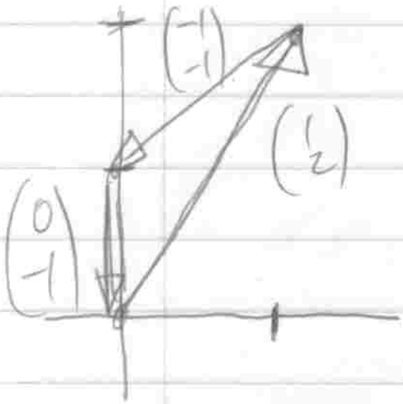
$x - y - 2z = 0 \Rightarrow x = y + 2z$

$w - x + z = 0 \Rightarrow w = (y+2z) - z = y + z$

so $\vec{x} = \begin{pmatrix} y+z \\ y+2z \\ y \\ z \end{pmatrix} = y \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{n}^{(1)}} + z \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{n}^{(2)}}$

Null vectors:

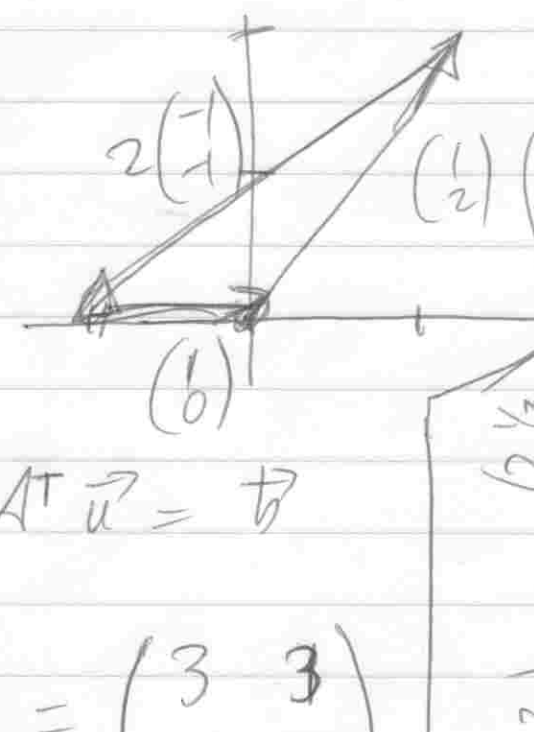
(c) The null vector $\vec{n}^{(1)}$ indicates $(\text{col } 1) + (\text{col } 2) + (\text{col } 3) = \vec{0}$



$\vec{n}^{(2)}$ indicates

$(\text{col } 1) + 2(\text{col } 2) + (\text{col } 4) = \vec{0}$

$\Rightarrow$



(d) Min Length solution

$\vec{x}_{ml} = A^T \vec{u}$, $AA^T \vec{u} = \vec{b}$

$$AA^T = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 3 & 4 \\ 3 & 6 & 7 \end{array}\right) \xrightarrow{L_{21}=1} \left(\begin{array}{cc|c} 3 & 3 & 4 \\ 0 & 3 & 3 \end{array}\right) \Rightarrow \begin{array}{l} u_2 = 1 \\ 3u_1 + 3 = 4 \quad u_1 = 1/3 \end{array}$$

$$\vec{x}_{ml} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\,1/3 \\ -1\,1/3 \\ -1 \\ 1/3 \end{pmatrix}$$

Let $\vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ for working with A^T! (7)

(4)(30 points) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ -2 & 3 \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad , \quad \vec{b}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \vec{b}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(a)(10 points) Find the null vector(s) $\vec{m}$ of A^T, and use to determine for which $\vec{b}^{(1) or (2)}$ the system $A\vec{x} = \vec{b}$ has a solution.

(b)(5 points) Find the general solution to the system in the case when a solution exists.

(c)(10 points)Find the left pseudoinverse $(A^T A)^{-1} A^T$,

(d)(5 points) In the case when no solution exists, use the l.p.i. to find the least squares best approximate solution.

w free

$l_{21} = -2$

ⓐ $A^T \vec{u} = \vec{0}$ $\quad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -2 & -1 & +3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$

2nd row $\Rightarrow$ $v - w = 0$ $\quad (v = w)$

1st row $\quad u + v - 2w = 0 \quad (u = 2w - (w) = w$

$\Rightarrow \vec{u} = \begin{pmatrix} w \\ w \\ w \end{pmatrix} = w \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{m} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

We know $A\vec{x} = \vec{b}$ has a solution ONLY if $\vec{m}^T \vec{b} = 0$

Here $\vec{m}^T \vec{b}^{(1)} = (1 \ 1 \ 1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2 \Rightarrow$ NO SOLUTION!

$\vec{m}^T \vec{b}^{(2)} = (1 \ 1 \ 1) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow$ SOLUTION EXISTS!

4b

$A\vec{x} = \vec{b}^{(2)}$ has augmented matrix

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 1 & -1 & 0 \\ -2 & 3 & 1 \end{array}\right) \xrightarrow[l_{31}=-2]{l_{21}=1} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right)$$

$$\xrightarrow{l_{32}=-1} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x-2y=-1 \Rightarrow x=1 \\ \Rightarrow y=1 \\ \Leftarrow \text{ok!} \end{array} \qquad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(UNIQUE!)

c $LPI = (A^TA)^{-1}A^T$. Here

$$A^TA = \begin{pmatrix} 1 & 1 & -2 \\ -2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -9 \\ -9 & 14 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{6\times 14 - 9\times 9}\begin{pmatrix} 14 & 9 \\ 9 & 6 \end{pmatrix} = \frac{1}{\underbrace{84-81}_{=3}}\begin{pmatrix} 14 & 9 \\ 9 & 6 \end{pmatrix} = \begin{pmatrix} \frac{14}{3} & 3 \\ 3 & 2 \end{pmatrix}$$

$$\text{So } (A^TA)^{-1}A^T = \begin{pmatrix} 14/3 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -2 \\ -2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{14}{3}-6 & \frac{14}{3}-3 & -\frac{28}{3}+9 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4/3 & 2/3 & -1/3 \\ -1 & 1 & 0 \end{pmatrix}$$

$\vec{x}_{LS}$!

d
$$\vec{x}_{LS} = (A^TA)^{-1}A^T\vec{b} = \begin{pmatrix} -4/3 & 2/3 & -1/3 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix}$$