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Math 410 (Prof. Bayly) FINAL EXAM (Eigentheory part): Wednesday 11 August 2004

There are problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

NOTE Problems 3 and 4 are rather lengthy; I have included a blank sheet between them to give you extra writing space.

(1)(10 points) The variables $\vec{x}(t) = (x(t), y(t))^T$ describing a vibrating system satisfy

$$\frac{d^2x}{dt^2} = 2y - 6x \quad , \quad \frac{d^2y}{dt^2} = 2x - 9y.$$

Find the general solution $\vec{x}(t)$ and identify the frequencies at which the vibrations occur.

Write $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $A = \begin{pmatrix} -6 & 2 \\ 2 & -9 \end{pmatrix} \Rightarrow \frac{d^2\vec{x}}{dt^2} = A\vec{x}$

We expect solutions in form $\vec{x}(t) = e^{rt}\vec{\xi}$

where $r^2\vec{\xi} = A\vec{\xi}$, i.e. $(A - r^2 I)\vec{\xi} = \vec{0}$

Therefore $r^2 = \lambda$ = eigenvalue of A; $\vec{\xi}$ = eigenvector of A.

Here $p_A(\lambda) = \det\begin{pmatrix} -6-\lambda & 2 \\ 2 & -9-\lambda \end{pmatrix} = \lambda^2 + 15\lambda + 54 - 4$

$= \lambda^2 + 15\lambda + 50$

$= (\lambda+5)(\lambda+10) \Rightarrow$ roots are $\lambda = -5, -10$

POSSIBLE r's are $\pm i\sqrt{5}, \pm i\sqrt{10}$

Eigen vector: $\lambda^{(1)} = -5$ $\quad A - \lambda^{(1)} I = \begin{pmatrix} -1 & 2 \\ \cancel{2} & \cancel{-4} \end{pmatrix}$ $\quad$ y free

$0 \Leftarrow$ $\quad -x + 2y = 0$

$x = 2y$

$$\vec{x} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \vec{\xi}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda^{(2)} = -10$ $\quad A - \lambda^{(2)} I = \begin{pmatrix} 4 & 2 \\ \cancel{2} & \cancel{1} \end{pmatrix}$ $\quad$ y free

$0 \Leftarrow$ $\quad 4x + 2y = 0 \quad x = -y/2$

$$\vec{x} = y \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \Rightarrow \vec{\xi}^{(2)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ if } y = 2$$

General solution is combination of all possibilities

$$\vec{x}(t) = \vec{\xi}^{(1)} \left[C_1 e^{i\sqrt{5}t} + C_2 e^{-i\sqrt{5}t} \right] + \vec{\xi}^{(2)} \left[C_3 e^{i\sqrt{10}t} + C_4 e^{-i\sqrt{10}t} \right]$$

OR

$$\vec{x}(t) = \vec{\xi}^{(1)} \left[D_1 \cos t\sqrt{5} + D_2 \sin t\sqrt{5} \right] + \vec{\xi}^{(2)} \left[D_3 \cos t\sqrt{10} + D_4 \sin t\sqrt{10} \right]$$

(Radian)
FREQUENCIES are $\sqrt{5}, \sqrt{10}$.

(2)(10 points) The *Goose and Gherkin* and *No Octopi* are neighboring restaurants that start the year with 75 customers each. The Goose regularly presents live music, with the result that 80 per cent of the patrons one night return on the next night, with the other 20 per cent going to No Octopi for some quiet pizza. Meanwhile 60 per cent of the customers at No Octopi return the next night, with 40 per cent going over to the Goose.

As weeks and weeks go by (i.e. as time goes to infinity), what are the expected numbers of customers at the two restaurants?

Markov transition matrix $M = \begin{array}{cc} \begin{pmatrix} .8 & .4 \\ .2 & .6 \end{pmatrix} & \begin{array}{l} \text{Goose} \\ \text{Octopi} \end{array} \end{array}$

Goose Octopi

We expect $\vec{x}^{(p)} \longrightarrow \vec{q}$ = eigenvector belonging to $\lambda = 1$ as $p \to \infty$

where $\begin{pmatrix} x^{(p)} \\ y^{(p)} \end{pmatrix} = \begin{pmatrix} \text{\# people in Goose} \\ \text{\# people in Octopi} \end{pmatrix}$ on night p.

For $\lambda = 1$ $M - \lambda I = \begin{pmatrix} -.2 & .4 \\ \cancel{.2} & \cancel{-.4} \end{pmatrix} \Rightarrow -.2x + .4y = 0$

0's ←

y free

$x = 2y$

$\vec{x} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. CHOOSE y so that $x + y = 150$

= total number of clients

Therefore $2y + y = 150$ $3y = 150$ $y = 50$

$\vec{q} = 50 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \\ 50 \end{pmatrix}$ = expected # at Goose

= expected # at Octopi

(3)(32 points) A model of guerrilla warfare assumes that the numbers $x(t), y(t)$ of combatants from sides X and Y satisfy

$$\frac{dx}{dt} = -y \quad , \quad \frac{dy}{dt} = -4x.$$

The difference in coefficients indicates that the X forces are 4 times as well-trained and equipped as the Y forces.

(a)(2 points) Express these equations as $\frac{d\vec{x}}{dt} = A\vec{x}$, where $\vec{x}(t) = (x(t), y(t))^T$ and A is the coefficient matrix that reproduces the above equations.

(b)(10 points) Find the eigenvalues and eigenvectors of A.

(c)(10 points) Find an invertible matrix S for which you expect (but don't have to check) $S^{-1}AS$ is diagonal. Then calculate the matrix exponential e^{tA}.

(c)(5 points) The general of X assumes that their factor of 4 advantage means that with a company of initial strength $x(0) = 100$ they can prevail over an adversary Y of initial strength $y(0) = 300$. Use e^{tA} to find $\vec{x}(t)$.

(d)(5 points) The battle ends when one of the variables becomes equal to zero. Which side will actually be the winner?

(a) $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $\quad \frac{d\vec{x}}{dt} = \underbrace{\begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}}_{=A} \vec{x}$

(b) E-values: $p_A(\lambda) = \det\begin{pmatrix} -\lambda & -1 \\ -4 & -\lambda \end{pmatrix} = \lambda^2 - 4 \Rightarrow \boxed{\lambda = \pm 2}$

E-vector: $\lambda^{(1)} = 2$: $\begin{pmatrix} -2 & -1 \\ \cancel{-4} & \cancel{-2} \end{pmatrix}$ 0's $\quad$ y free $\quad x = -\frac{1}{2}y$

$-2x - y = 0$

$\vec{x} = y\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ choose $y = 2 \Rightarrow \boxed{\vec{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$

For $\lambda^{(2)} = -2$ $\quad \begin{pmatrix} 2 & -1 \\ \cancel{-4} & \cancel{2} \end{pmatrix}$ 0's $\quad$ y free $\quad x = 1/2 y \quad \vec{x} = y\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$2x - y = 0$

choose $y = 2 \Rightarrow \boxed{\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

Extra workspace for problems 3, 4!

old c $S = \left(\vec{\xi}^{(1)}\ \vec{\xi}^{(2)}\right) = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$

$$S^{-1} = \frac{1}{-2-2}\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{pmatrix}$$

This ought to give $S^{-1}AS = \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$\Rightarrow A = S\Lambda S^{-1}$ but we don't have to check!

$$e^{tA} = S e^{t\Lambda} S^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} -e^{2t} & e^{-2t} \\ 2e^{2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} & -\frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} \\ -e^{2t} + e^{-2t} & \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} \end{pmatrix}$$

new c $\vec{x}(t) = e^{tA}\vec{x}(0) = \begin{pmatrix} \frac{e^{2t}+e^{-2t}}{2} & \frac{e^{-2t}-e^{2t}}{4} \\ e^{-2t}-e^{2t} & \frac{e^{2t}+e^{-2t}}{2} \end{pmatrix}\begin{pmatrix} 100 \\ 300 \end{pmatrix}$

$$= \begin{pmatrix} 50(e^{2t}+e^{-2t}) + 75(e^{-2t}-e^{2t}) \\ 100(e^{-2t}-e^{2t}) + 150(e^{2t}+e^{-2t}) \end{pmatrix}$$

(c) Rewrite $\vec{x}(t) = \begin{pmatrix} 125e^{-2t} - 25e^{2t} \\ 50e^{2t} + 250e^{-2t} \end{pmatrix}$ (6)

(d) Because 50 & 250 both >0, $y(t)$ will never $=0$ $\Rightarrow$ Y will survive battle

HOWEVER $125e^{-2t} - 25e^{2t}$ will eventually become <0 because $25e^{2t}$ is growing while $125e^{-2t}$ is decaying.

$\Rightarrow$ X will lose battle

[I did not ask for duration of battle, but it ends when $125e^{-2t} - 25e^{2t} = 0$

Ie. $25e^{2t} = 125e^{-2t} \Rightarrow e^{4t} = 125/25$

$e^{4t} = 5$ $\qquad$ $4t = \ln 5$

$t = \frac{1}{4}\ln 5$

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(4)(30 points) Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(a)(10 points) Calculate AA^T and find its eigenvalues $\lambda^{(1)}, \lambda^{(2)}$. What are the singular values?

(b)(10 points) Find the corresponding eigenvectors $\vec{\xi}^{(1)}_{AA^T}, \vec{\xi}^{(2)}_{AA^T}$ and check that they are orthogonal.

(c)(5 points) The eigenvectors $\vec{\xi}^{(1)}_{A^TA}, \vec{\xi}^{(2)}_{A^TA}$ of A^TA are found by multiplying the corresponding eigenvectors of AA^T by A or A^T, I can never remember which. Make the correct choice and calculate the new eigenvectors. (Do NOT normalize the lengths to unit magnitude!)

(d)(5 points) Check that the length of each eigenvector you just found for A^TA equals the product of the length of the corresponding eigenvector of AA^T and the corresponding singular value.

(a) $AA^T = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$

Eigenvalues: $P_{AA^T}(\lambda) = \det\begin{pmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{pmatrix}$

$= (5-\lambda)^2 - 1$ Roots $\lambda = 6, 4$ $\boxed{\lambda^{(1)} = 6, \lambda^{(2)} = 4}$

SING. VALUES $\boxed{\sigma^{(1)} = \sqrt{6}}$, $\boxed{\sigma^{(2)} = \sqrt{4} = 2}$

(b) For $\lambda^{(1)} = 6$ $AA^T - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ y free $-x+y=0$ $x=y$

$\Rightarrow \vec{x} = y\begin{pmatrix} +1 \\ 1 \end{pmatrix}$ $\boxed{\vec{\xi}^{(1)}_{AA^T} = \begin{pmatrix} +1 \\ 1 \end{pmatrix}}$

$\lambda^{(2)} = 4$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\vec{x} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\boxed{\vec{\xi}^{(2)}_{AA^T} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

$\cancel{\begin{matrix} 1 & 1 \end{matrix}}$

OS

NOTE: $\vec{\xi}^{(2)T}_{AA^T}\,\vec{\xi}^{(1)}_{AA^T} = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = -1+1 = 0$ ✓ 8

$\Rightarrow$ orthogonal

Extra workspace for problems 3, 4!

(c) Since A is ~~3×2~~ 2×3 and $\vec{\xi}_{AA^T}$'s are 2×1,

it must be A^T that we multiply by!

$$\vec{\xi}^{(1)}_{A^TA} = \begin{pmatrix}1 & 1\\ 2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix}$$

$$\vec{\xi}^{(2)}_{A^TA} = \begin{pmatrix}1 & 1\\ 2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\-2\\2\end{pmatrix}$$

NOTE these also have $\vec{\xi}^{(2)T}_{A^TA}\,\vec{\xi}^{(1)}_{A^TA} = 0$

Though I did NOT ask you to do this.

(d) LENGTH of $\vec{\xi}^{(1)}_{A^TA} = \sqrt{4+4+4} = \cancel{12} = \sqrt{12}$

while length of $\vec{\xi}^{(1)}_{AA^T} = \sqrt{1+1} = \sqrt{2}$, $\sigma^{(1)} = \sqrt{6}$

$\sqrt{12} = \sqrt{2}\sqrt{6}$ checks!

SIMILARLY Length of $\vec{\xi}^{(2)}_{A^TA} = \sqrt{4+4} = \sqrt{8}$

Length of $\vec{\xi}^{(2)}_{AA^T} = \sqrt{1+1} = \sqrt{2}$ $\quad\sigma^{(1)} = \sqrt{4} = 2$

$\sqrt{8} = \sqrt{2}\sqrt{4}$ checks!

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(5)(20 points) Consider the system of linear equations for the vector $\vec{x} = (x, y)^T$:

$$2x + y = 0 \quad , \quad x + y = 2.$$

(a)(10 points) Rearrange these equations into a form in which you can make a reasonable guess for $\vec{x}^{(1)}$ and then systematically obtain better approximations. Calculate the next two approximations $\vec{x}^{(2)}, \vec{x}^{(3)}$ explicitly.

(b)(10 points) By what factor do you expect the errors in the successive approximation method to diminish as the number of repetitions increases? (Give reasons, of course!)

(a) $2x+y=0 \Rightarrow x = -y/2$
$x+y=2 \Rightarrow y = 2-x$ is one possible rearrangement.

(Matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is ALMOST diagonally dominant!)

So 1st GUESS is $\vec{x}^{(1)} = \begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Subsequent guesses $x^{(p+1)} = -y^{(p)}/2$

$y^{(p+1)} = 2 - x^{(p)}$

$$\vec{x}^{(p+1)} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} \vec{x}^{(p)}$$

Next couple of approximations

$\vec{x}^{(1)}$	$\vec{x}^{(2)}$	$\vec{x}^{(3)}$	$\vec{x}^{(4)}$	$\vec{x}^{(5)}$	$\vec{x}^{(6)}$	$\vec{x}^{(7)}$
$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 3 \end{pmatrix}$	$\begin{pmatrix} -3/2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} -3/2 \\ 7/2 \end{pmatrix}$	$\begin{pmatrix} -7/4 \\ 7/2 \end{pmatrix}$	$\begin{pmatrix} -7/4 \\ 15/4 \end{pmatrix}$

(b) Since the exact solution satisfies

$$\vec{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} \vec{x}$$

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the error $\left(\vec{x}^{(p+1)} - \vec{x}\right) = \underbrace{-\begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix}}_{=K} \left(\vec{x}^{(p)} - \vec{x}\right)$

Rate of convergence depends on size of eigenvalue of K

$$\det\left(K - \lambda I\right) = \det\begin{pmatrix} -\lambda & -1/2 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 - 1/2$$

$$\Rightarrow \lambda = \pm 1/\sqrt{2} = \pm \sqrt{2}/2 \approx \pm .7$$

SO error goes down by factor of $\frac{1}{\sqrt{2}}$ each step

OR by factor of $\frac{1}{2}$ every 2 steps.

DID NOT ASK FOR EXACT SOLUTION, but here it is: $\begin{pmatrix} y = 4 \\ x = -2 \end{pmatrix} \Rightarrow \vec{x}_{exact} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Iteration getting close!