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1.8.4

d = row 1
e = row 2
f = row 3

$$A = \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & 2 & a & b \\ b & 2 & a & a \end{array}\right) \rightarrow \begin{array}{c} \\ \\ \frac{a}{b} \end{array} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & 2 & a & b \\ a & \frac{2a}{b} & \frac{a^2}{b} & \frac{a^2}{b} \end{array}\right) \rightarrow \begin{array}{c} \\ e-d \\ f-d \end{array} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & 2 & a-b & b-2 \\ 0 & \frac{2a}{b} & \frac{a^2}{b}-b & \frac{a^2}{b}-2 \end{array}\right) \rightarrow \begin{array}{c} \\ \\ f\left(\frac{b}{a}\right)-e \end{array} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & 2 & a-b & b-2 \\ 0 & 0 & b-\frac{b^2}{a} & a-b+2-\frac{2b}{a} \end{array}\right)$$

★ assuming $a \neq 0$

if $\left(b - \frac{b^2}{a} \neq 0\right) \rightarrow$ UNIQUE SOLUTION

if $\left(b - \frac{b^2}{a} = 0\right)$ & $\left(a - b + 2 - \frac{2b}{a} \neq 0\right) \rightarrow$ NO SOLUTIONS [if ($b=0$ & $a \neq -2$)]

if $\left(b - \frac{b^2}{a} = 0\right)$ & $\left(a - b + 2 - \frac{2b}{a} = 0\right) \rightarrow \infty$ SOLUTIONS [if ($b=a$) or ($b=0$ & $a=-2$)]

There are many cases that can be calculated. Above are three cases. (UNIQUE, ∞ and NO SOLUTIONS)

★ assuming $b=0$

$$A = \left(\begin{array}{ccc|c} a & 0 & 0 & 2 \\ a & 2 & a & 0 \\ 0 & 2 & a & a \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} a & 0 & 0 & 2 \\ 0 & 2 & a & -2 \\ 0 & 2 & a & a \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} a & 0 & 0 & 2 \\ 0 & 2 & a & -2 \\ 0 & 0 & 0 & a+2 \end{array}\right)$$

if $a=0 \rightarrow$ NO SOLUTIONS

if $a=-2 \rightarrow \infty$ SOLUTIONS

★ assuming $a=0$

$$A = \left(\begin{array}{ccc|c} 0 & 0 & b & 2 \\ 0 & 2 & 0 & b \\ b & 2 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} b & 2 & 0 & 0 \\ 0 & 2 & 0 & b \\ 0 & 0 & b & 2 \end{array}\right)$$

if $b \neq 0 \rightarrow$ UNIQUE SOLUTION

if $b=0 \rightarrow$ NO SOLUTION

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1.3.8

c)
$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[\ell_{31}=-1]{\ell_{21}=1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{\text{Swap}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{RANK} = 3$$

* mis on sol

i)
$$\begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{pmatrix} \xrightarrow[\ell_{31}=1]{\ell_{21}=1} \begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 2 & -2 & -2 & -2 \end{pmatrix} \xrightarrow{\ell_{32}=-2} \begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{RANK} = 2$$

HW SOL. HAS +2

1.3.22

f)
$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 2 \\ -3 & 1 & 0 \end{pmatrix} \xrightarrow[L_{31}=-3]{L_{21}=2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{L_{32}=\frac{1}{3}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -13/3 \end{pmatrix} = U$$

$-3 - \frac{4}{3} = \frac{-13}{3}$

* mistake on sol'n

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & \frac{1}{3} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -13/3 \end{pmatrix}$$

check:
$$LU = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 2 \\ -3 & 1 & 0 \end{pmatrix}$$

HW SOLN HAS $-\frac{17}{3}$

1.3.25 LU factorization

a)
$$\begin{pmatrix} 1 & 1 \\ t_1 & t_2 \end{pmatrix} \xrightarrow{L_{21}=t_1} \begin{pmatrix} 1 & 1 \\ 0 & t_2-t_1 \end{pmatrix} = U \qquad L = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}$$

check:
$$LU = \begin{pmatrix} 1 & 1 \\ t_1 & t_2 \end{pmatrix}$$

b)
$$\begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix} \xrightarrow[L_{31}=t_1^2]{L_{21}=t_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 \\ 0 & t_2^2-t_1^2 & t_3^2-t_1^2 \end{pmatrix} \xrightarrow{L_{32}=\frac{t_2^2-t_1^2}{t_2-t_1}=(t_2+t_1)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 \\ 0 & 0 & t_1(t_2-t_3)-t_2(t_3)+t_3^2 \end{pmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_1^2 & (t_1+t_2) & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 \\ 0 & 0 & \underbrace{t_1(t_2-t_3)-t_2(t_3)+t_3^2}_{(t_3-t_1)(t_3-t_2)} \end{bmatrix}$$

1.3.25

c)
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{pmatrix} \to \begin{array}{l} L_{21} = t_1 \\ L_{31} = t_1^2 \\ L_{41} = t_1^3 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 & t_4-t_1 \\ 0 & t_2^2-t_1^2 & t_3^2-t_1^2 & t_4^2-t_1^2 \\ 0 & t_2^3-t_1^3 & t_3^3-t_1^3 & t_4^3-t_1^3 \end{bmatrix}$$

$$L_{32} = \frac{t_2^2-t_1^2}{t_2-t_1} = t_2+t_1$$

$$L_{42} = \frac{t_2^3-t_1^3}{t_2-t_1} = t_1^2+t_1t_2+t_2^2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 & t_4-t_1 \\ 0 & 0 & (t_1(t_2-t_3)-t_2t_3+t_3^2) & t_4^2-t_4(t_1+t_2)+t_1t_2 \\ 0 & 0 & (t_1^2(t_2-t_3)+t_1(t_2^2-t_2t_3)-t_2^2t_3+t_3^3) & t_4(t_1^2+t_1t_2+t_2^2)+t_1(t_1+t_2)t_2 \end{bmatrix}$$

$$L_{43} = \frac{X_2}{X_1} = t_1+t_2+t_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 & t_4-t_1 \\ 0 & 0 & (t_3-t_1)(t_3-t_2) & (t_4-t_1)(t_4-t_2) \\ 0 & 0 & 0 & (t_4-t_3)(t_4-t_2)(t_4-t_1) \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 \\ t_1^2 & t_2+t_1 & 1 & 0 \\ t_1^3 & t_1^2+t_1t_2+t_2^2 & t_1+t_2+t_3 & 1 \end{bmatrix} = L$$

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1.5.4

$$L = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_{31}=b]{L_{21}=a} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & -b & 0 & 1 \end{array}\right) \to L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$$

$$M = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ b & c & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_{31}=b]{L_{21}=a} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & c & 1 & -b & 0 & 1 \end{array}\right) \xrightarrow{L_{32}=c} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & ac-b & -c & 1 \end{array}\right)$$

$$-b-(ac) = ac-b$$

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{pmatrix}$$

1.5.24 Gauss Jordan Method

e) $$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 & 1 & 0 \\ -2 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_{31}=-2]{L_{21}=3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & +6 & -3 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{L_{11}=-1} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -1 & 1 & 1 \end{array}\right)$$

$$\begin{array}{c} \\ -1 \\ \frac{1}{5} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 9/5 & 1/5 & 6/5 \\ 0 & 0 & 1 & -1/5 & 1/5 & 1/5 \end{array}\right) \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 3/5 & 2/5 & 2/5 \\ 9/5 & 1/5 & 6/5 \\ -1/5 & 1/5 & 1/5 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

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2.5.21 Find Range, corange, kernel, cokernel

c) $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix} \xrightarrow{l_{21}=1,\ l_{31}=2} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & 2 \\ 0 & 1 & 3 & -2 \end{pmatrix} \xrightarrow{l_{32}=-1} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ RANK = 2, DIM of RANGE = 2

2 Free variables (x_3, x_4) → DIM of KERNEL = 2

Basis for Range = $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

HW SOL'N HAS $5X_3 - X_4$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{matrix} x_1 = -x_2 - 2x_3 - x_4 = 3x_3 - 2x_4 - 2x_3 - x_4 = \boxed{x_3 - 3x_4} \\ x_2 = 2x_4 - 3x_3 \\ x_3 = x_3 \\ x_4 = x_4 \end{matrix} = x_3 \underbrace{\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}}_{\vec{n}^{(1)}} + x_4 \underbrace{\begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{n}^{(2)}}$ * mistake on sol'n

Basis for Kernel (A)

HW SOL'N HAS $l_{32} = 1$

$A^T = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & -1 & 7 \\ 1 & 3 & 0 \end{pmatrix} \xrightarrow{l_{21}=1,\ l_{31}=2,\ l_{41}=1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{\boxed{l_{32}=3},\ l_{42}=-2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ → RANK = 2 = DIM of CORANGE * mistake

Basis of CORANGE = $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \end{pmatrix}$

1 Free variable (x_3) → Dim of cokernel = 1

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{matrix} x_1 = -x_3 - 2x_3 = -3x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{matrix} = x_3 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}}_{\vec{m}}$ * mistake on sol'n (corange)

Basis of $\boxed{\text{COKERNEL}}$ (A)

SOL'N STATES THIS IS CORANGE

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show that $A^{-1} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = (aI + bJ)^{-1} = \frac{(aI - bJ)}{(a^2+b^2)}$

using formula for 2x2 matrices:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2-(-b^2)}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2}\,\underbrace{aI - bJ} = \boxed{\frac{aI-bJ}{a^2+b^2}}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = aI + bJ = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$aI - bJ = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + (-b)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$