MATH 410 (BAYLY) HW1 SOLUTIONS:

1.8.1 a

$$x - 2y = 1$$
$$3x + 2y = -3$$

Augmented matrix: $\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & -3 \end{array}\right)$

Row reduces to ECHELON FORM (subtract 3× row 1 from row 2)

$\left(\begin{array}{cc|c} \overset{\text{pivot}}{1} & -2 & 1 \\ 0 & \underset{\text{pivot}}{8} & -6 \end{array}\right)$

Rank = 2 is FULL
⇒ solution guaranteed.

NO free variables or null vectors.

SOLUTION $8y = -6 \Rightarrow \boxed{y = -6/8 = -3/4}$

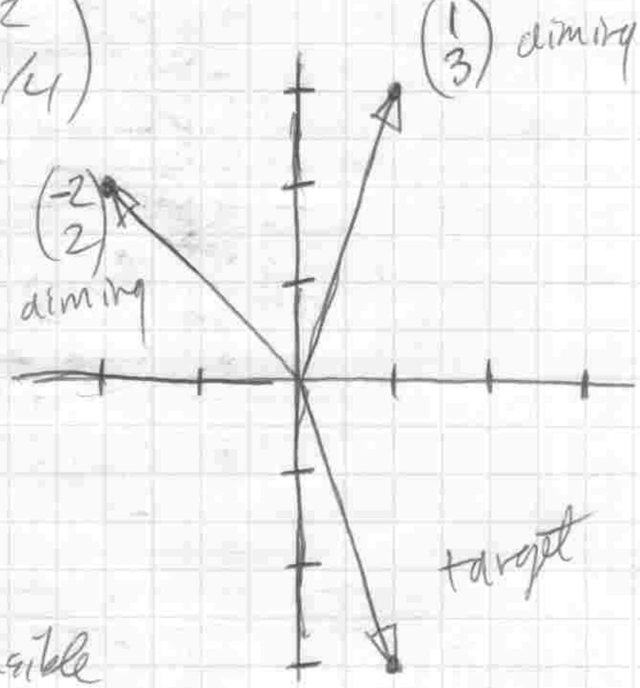
$x - 2\left(\frac{-3}{4}\right) = 1 \quad \boxed{x = -\frac{1}{2}}$

in vector form $\vec{x} = \begin{pmatrix} -1/2 \\ -3/4 \end{pmatrix}$

GEOMETRICALLY

$$x\begin{pmatrix}1\\3\end{pmatrix} + y\begin{pmatrix}-2\\2\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$$

aiming vectors   target vector



From the picture it is pretty sensible that you have to go in NEGATIVES of both aiming directions to reach the target.

1.8.2 a $2x + y + 3z = 1$

$x + 4y - 2z = -3$

Augmented matrix $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & 4 & -2 & -3 \end{array}\right)$

FOR CONVENIENCE ~~[illegible]~~ EXCHANGE rows 1,2

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{from row 2}]{\text{subtract } 2\times\text{row 1}} \underbrace{\left(\begin{array}{ccc|c} \overset{\text{pivot}}{1} & 4 & -2 & -3 \\ 0 & \overset{\text{pivot}}{-7} & 7 & 7 \end{array}\right)}_{\text{ECHELON FORM}}$$

~~ECHELON FORM~~ Rank = 2 is FULL

Solution GUARANTEED, __no__ compatibility conditions,
Variable z is FREE $\Rightarrow \infty$ solution

To find actual solutions, last row $\Rightarrow -7y + 7z = 7$

$\boxed{y = z - 1}$

1st row row swap

$x + 4y - 2z = -3$

$x + 4(z-1) - 2z = -3 \rightarrow \boxed{x = 1 - 2z}$

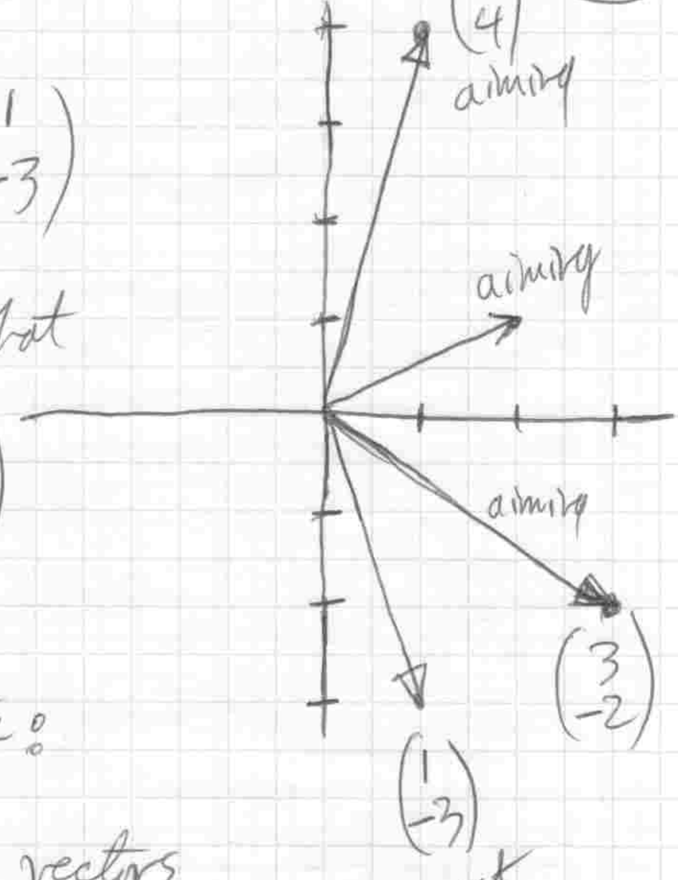
z is free, remember

$\vec{n}$ = null vector

General solution $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2z \\ z-1 \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\vec{x}^{(p)}} + \underbrace{z \overbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}^{\vec{n}}}_{\vec{x}^{(c)}}$

partic solution $\vec{x}^{(p)}$ $\qquad$ $\vec{x}^{(c)}$ complementary solution

GEOMETRICALLY:



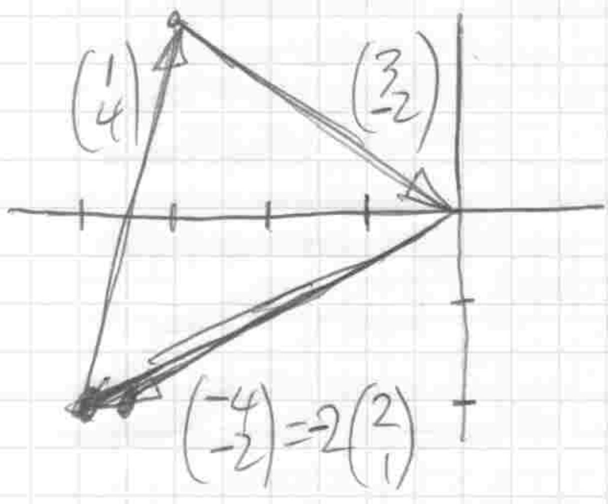
$$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}1\\4\end{pmatrix}+z\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}1\\-3\end{pmatrix}$$

The null vector $\begin{pmatrix}x=-2\\y=1\\z=1\end{pmatrix}$ implies that

$$-2\begin{pmatrix}2\\1\end{pmatrix}+1\begin{pmatrix}1\\4\end{pmatrix}+1\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

which is clearly true in arithmetic

We can illustrate this in a picture:



The three vectors $\begin{pmatrix}-4\\-2\end{pmatrix}$, $\begin{pmatrix}1\\4\end{pmatrix}$, and $\begin{pmatrix}3\\-2\end{pmatrix}$ form a cycle that starts & ends at the origin.

1.8.3(a) $\begin{matrix}6x_1+3x_2=12\\4x_1+2x_2=9\end{matrix}$ Lets use $\begin{pmatrix}b_1\\b_2\end{pmatrix}$ on right

$\Rightarrow$ AM is $\left(\begin{array}{cc|c}6&3&b_1\\4&2&b_2\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}6&3&b_1\\0&0&b_2-\frac{2}{3}b_1\end{array}\right)$ Zero row! comp. condition $b_2-\frac{2}{3}b_1=0$

~~See PAGE 6 for solution.~~

☆ Here is solution to 1.8.3(a) in general case: (4)

$$6x_1 + 3x_2 = b_1 \qquad x_2 \text{ free}$$

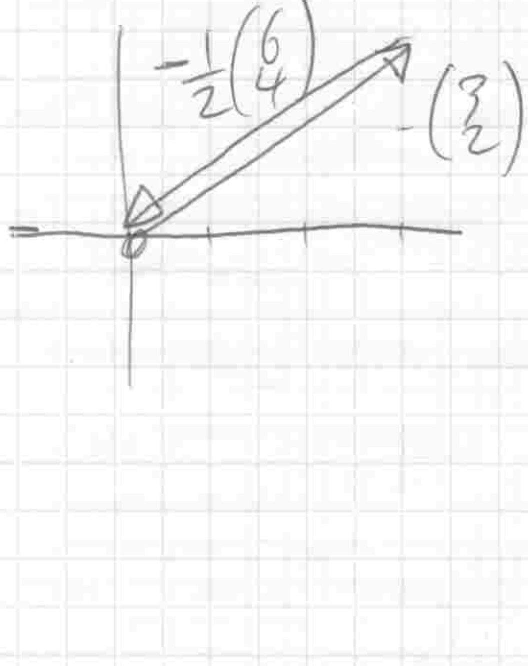
(ASSUMING $b_2 - \frac{2}{3}b_1 = 0$ is satisfied.)

$$\Rightarrow \; x_1 = \frac{b_1}{6} - \frac{x_2}{2}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1/6 - x_2/2 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} b_1/6 \\ 0 \end{pmatrix}}_{\vec{x}^{(p)}} + \underbrace{x_2 \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}}_{\vec{x}^{(c)}}$$

Null vector $\vec{n} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

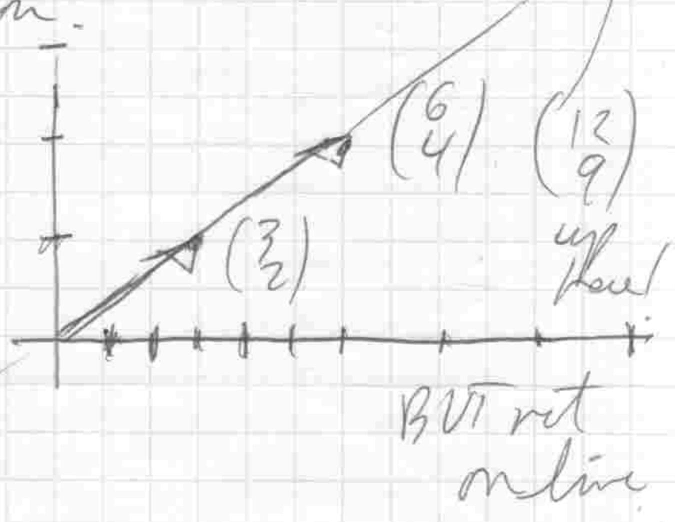
GEOMETRICALLY this null vector says $-\frac{1}{2}\begin{pmatrix} 6 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is obvious



Starts at & ends $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$!

Note that for $b_1 = 12$, $b_2 = 9$, $b_2 - \frac{2}{3} b_1 = 1 \neq 0$

$\Rightarrow$ given system has no solution.



GEOMETRICALLY

$$x_1 \underbrace{\begin{pmatrix} 6 \\ 4 \end{pmatrix}}_{} + x_2 \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$$

Two target vectors define the same LINE in the plane (with slope 2/3)

$\Rightarrow$ BUT the TARGET POINT $\begin{pmatrix} 12 \\ 9 \end{pmatrix}$ is NOT on this line!

1.8.3 (c)

$x_1 + 2x_2 = 1$ or b_1

$2x_1 + 5x_2 = 2$ or b_2 in general

$3x_1 + 6x_2 = 3$ or b_3

$$\text{A.M.} \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 5 & b_2 \\ 3 & 6 & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 3b_1 \end{array}\right)$$

ZERO ROW! $\Rightarrow$ compat. condition

requires $b_3 - 3b_1 = 0$ for solution to exist

Note that ~~there~~ every COLUMN has a PIVOT
$\Rightarrow$ no free variables or null vectors.

General solution would then be $\boxed{x_2 = b_2 - 2b_1}$

$x_1 + 2x_2 = b_1 \Rightarrow x_1 + 2(b_2 - 2b_1) = b_1$

$\boxed{x_1 = 5b_1 - 2b_2}$

In specific case $b_1 = 1$, $b_2 = 2$, $b_3 = 3$ $\quad b_3 - 3b_1 = 3 - 3(1) = 0$
$\Rightarrow$ solution exists!

$x_2 = 2 - 2(1) = 0 \qquad x_1 = 5 - 4 = 1$

$\Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is general solution.

GEOMETRICALLY $\quad x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Kind of hard to sketch in 3d. BUT pretty clear that since target vector is same as 1st driving vector, then $x_1 = 1$, $x_2 = 0$ ought to work!

1.8.3 d $\quad 2x_1 - 6x_2 + 4x_3 = 2$ or b_1 $\qquad$ 7

$-x_1 + 3x_2 - 2x_3 = -1$ or b_2

AM $\left(\begin{array}{ccc|c} 2 & -6 & 4 & b_1 \\ -1 & 3 & -2 & b_2 \end{array}\right) \xrightarrow[\text{rows}]{\text{exchange}} \left(\begin{array}{ccc|c} -1 & 3 & -2 & b_2 \\ 2 & -6 & 4 & b_1 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} \boxed{-1} & 3 & -2 & b_2 \\ 0 & 0 & 0 & b_1+2b_2 \end{array}\right)$ $\quad$ ~~add~~ subtract -2 times top from bottom

There is only $\underline{1}$ pivot, & 2 free variables x_2, x_3

RANK = 1

COMPATIBILITY condition $\quad b_1 + 2b_2 = 0$

If C.C. satisfied, then top row says

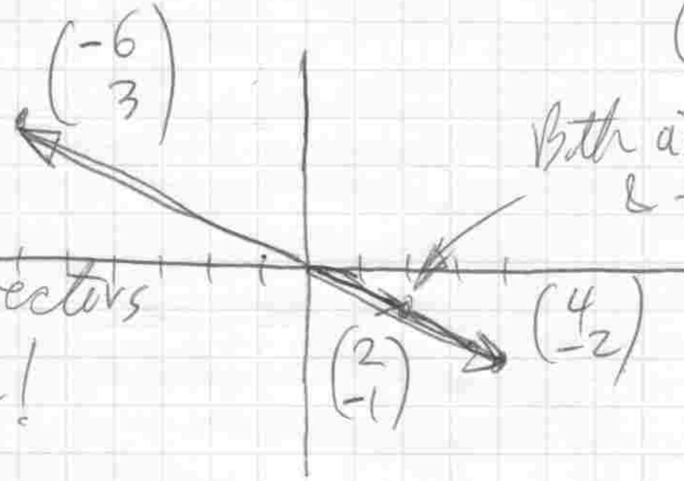
$-x_1 + 3x_2 - 2x_3 = b_2 \Rightarrow x_1 = -b_2 + 3x_2 - 2x_3$

$$\vec{x} = \begin{pmatrix} -b_2+3x_2-2x_3 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -b_2 \\ 0 \\ 0 \end{pmatrix}}_{\vec{x}^{(p)}} + \underbrace{x_2 \overset{\vec{n}^{(1)}}{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}} + x_3 \overset{\vec{n}^{(2)}}{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}}_{\vec{x}^{(c)}}$$

For specific case $b_1 = 2$, $b_2 = -1$ $\quad b_1 + 2b_2 = 2-2 = 0$

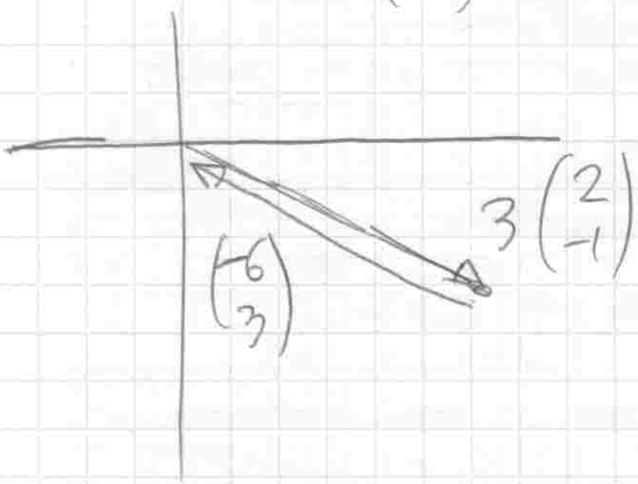
$\Rightarrow$ Solution exists! (Given above)

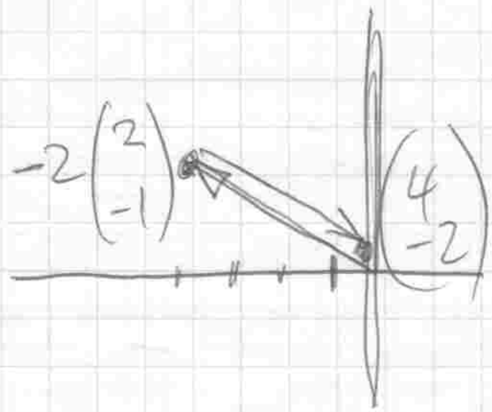
GEOMETRICALLY:



ALL THREE aiming vectors lie on the same line!

Null vector $\begin{pmatrix}3\\1\\0\end{pmatrix}$ says $3\begin{pmatrix}2\\-1\end{pmatrix}+1\begin{pmatrix}-6\\3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$





Null vector $\begin{pmatrix}-2\\0\\1\end{pmatrix}$ says $-2\begin{pmatrix}2\\-1\end{pmatrix}+1\begin{pmatrix}4\\-2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$