MATH 410 (BAYLY) HW 2 SOLUTIONS

1.8.4 Analyze $\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & 2 & a & b \\ b & 2 & a & a \end{array}\right)$

There are a lot of cases!

① If $a=0$ get $\left(\begin{array}{ccc|c} 0 & 0 & b & 2 \\ 0 & 2 & 0 & b \\ b & 2 & 0 & 0 \end{array}\right) \xrightarrow{\text{SWAP}} \left(\begin{array}{ccc|c} b & 2 & 0 & 0 \\ 0 & 2 & 0 & b \\ 0 & 0 & b & 2 \end{array}\right)$

ⓐ If $b \neq 0$ solution will be UNIQUE

ⓑ If $b=0$ Last row is INCOMPATIBLE $\Rightarrow$ NO solutions

② If $a \neq 0$ we can do normal elimination:

$$\rightarrow \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & 2 & a-b & b-2 \\ 0 & 2 & a-\frac{b^2}{a} & a-2\frac{b}{a} \end{array}\right)$$

using $l_{21}=1$, $l_{31}=b/a$ multipliers

$$\xrightarrow{l_{32}=1} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & 2 & b-a & b-2 \\ 0 & 0 & b-\frac{b^2}{a} & a-b+2-2\frac{b}{a} \end{array}\right)$$

ⓐ If $b - b^2/a \neq 0$ then UNIQUE solution

ⓑ If $b - b^2/a = 0$ it depends on RIGHT SIDE entry

The relevant values of b are

$b(1-b/a)=0 \Rightarrow b=0$ or $b=a$.

1.8.4 cont. If $b=a$ then right side is

$a-a+2-2=0 \Rightarrow \infty$ SOLUTIONS!

If $b=0$ right side is $a+2$

$\Rightarrow \infty$ solution if $a=-2$, 0 solutions if $a\neq -2$

PHEW!

1.8.8 Find rank of matrices in (c), (i):

(c) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[L_{31}=-1]{L_{21}=1} \begin{pmatrix} \boxed{1} & -1 & 1 \\ 0 & \boxed{2} & 1 \\ 0 & 0 & \boxed{1} \end{pmatrix}$ ECHELON FORM! RANK $=3$

pivots

(i) $\begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & 2 \end{pmatrix} \xrightarrow[L_{31}=1]{L_{21}=1} \begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 2 & -2 & -2 & 2 \end{pmatrix}$

$\xrightarrow{L_{32}=-2} \begin{pmatrix} \boxed{1} & 2 & -1 & 3 & 0 \\ 0 & \boxed{-1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \boxed{4} \end{pmatrix}$ RANK $=3$ again!

pivots

☆ To review LU factorization:

in (c) $L=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, U=\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(i) $L=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$ $U=$ as above.

1.3.22 (f)

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 2 \\ -3 & 1 & 0 \end{pmatrix} \xrightarrow[l_{31}=-3]{l_{21}=2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 1 & -3 \end{pmatrix}$$

$$l_{32}=\frac{1}{3} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -17/3 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 1/3 & 1 \end{pmatrix} \quad \text{check } LU = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 2 \\ -3 & 1 & 0 \end{pmatrix} \checkmark$$

1.3.25 (a)

$$\begin{pmatrix} 1 & 1 \\ t_1 & t_2 \end{pmatrix} \xrightarrow{l_{21}=t_1} \begin{pmatrix} 1 & 1 \\ 0 & t_2-t_1 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix} \xrightarrow[l_{31}=t_1^2]{l_{21}=t_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 \\ 0 & t_2^2-t_1^2 & t_3^2-t_1^2 \end{pmatrix}$$

$$\xrightarrow[t_2+t_1]{l_{32}=} \begin{pmatrix} 1 & 1 & 1 \\ 0 & t_2-t_1 & t_3-t_1 \\ 0 & 0 & \underbrace{t_3^2-t_1^2-(t_2+t_1)(t_3-t_1)} \end{pmatrix}$$

$$\rightarrow t_3^2 - \cancel{t_1^2} - t_2t_3 - t_1t_3 + t_1t_2 + \cancel{t_1^2}$$

$$= (t_3-t_1)(t_3-t_2)$$

So $L = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_1^2 & t_1+t_2 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & (t_2-t_1) & (t_3-t_1) \\ 0 & 0 & (t_3-t_1)(t_3-t_2) \end{pmatrix}$,

Looks like a pattern, but can't exactly see it.

I'll just give the answer for 4×4
(calculation very long!)

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 \\ t_1^2 & t_1+t_2 & 1 & 0 \\ t_1^3 & t_1^2+t_1t_2+t_2^2 & t_1+t_2+t_3 & 1 \end{pmatrix}$$

☆ I STILL DON'T SEE THE WHOLE PATTERN!

$$U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & (t_2-t_1) & (t_3-t_1) & (t_4-t_1) \\ 0 & 0 & (t_3-t_1)(t_3-t_2) & (t_4-t_1)(t_4-t_2) \\ 0 & 0 & 0 & (t_4-t_3)(t_4-t_2)(t_4-t_1) \end{pmatrix}$$

1.5.4 You can check $L(L^{-1}) = I$!

For $M_{..}$, aug. matrix is
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ b & c & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_{31}=b]{L_{21}=a}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & c & 1 & -b & 0 & 1 \end{array}\right) \xrightarrow{L_{32}=c}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & ca-b, & -c & 1 \end{array}\right)$$

There's an extra $+ca$ term!

1.5.24 (e)
$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 & 1 & 0 \\ -2 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_{31}=-2]{L_{21}=3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{L_{32}=-1} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -1 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1/5 & 1/5 & 1/5 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/5 & 2/5 & 2/5 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -1/5 & 1/5 & 1/5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/5 & 2/5 & 2/5 \\ 0 & 1 & 0 & 9/5 & 1/5 & 6/5 \\ 0 & 0 & 1 & -1/5 & 1/5 & 1/5 \end{array}\right)$$

This is A^{-1} !

2.5.21 (b) $\cancel{A = \begin{pmatrix} 0 & 0 & -8 \\ 1 & 2 & -1 \\ 2 & 4 & 6 \end{pmatrix}}$ OOPS! Not (b), (c)!

(c) $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix}$ REDUCE using GE: $L_{21}=1$, $L_{31}=2$

$$\to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & 2 \\ 0 & 1 & 3 & -2 \end{pmatrix} \xrightarrow{L_{32}=-1} \begin{pmatrix} \boxed{1} & 1 & 2 & 1 \\ 0 & \boxed{-1} & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots

RANK = 2 $\Rightarrow$ dim RANGE = 2,

BASIS for RANGE is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$.

Free variables are $x_3, x_4 \Rightarrow$ Dim (Kernel) = 2

$-x_2 - 3x_3 + 2x_4 = 0 \Rightarrow x_2 = -3x_3 + 2x_4$

$x_1 + x_2 + 2x_3 + x_4 = 0 \Rightarrow$ use $x_2 = -3x_3 + 2x_4$

$x_1 = -(-3x_3 + 2x_4) + 2x_3 + x_4 = 5x_3 - x_4$ ok

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5x_3 - x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} 5 \\ -3 \\ 1 \\ 0 \end{pmatrix}}_{\vec{n}^{(1)}} + x_4 \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{n}^{(2)}}$$

Vectors $\vec{n}^{(1)}$ and $\vec{n}^{(2)}$ form a BASIS for KERNEL(A).

2.5.21 ⓔ NOW consider A^T for CORANGE & COKERNEL

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$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & -1 & 7 \\ 1 & 3 & 0 \end{pmatrix} \xrightarrow[\substack{l_{31}=2 \\ l_{41}=1}]{l_{21}=1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & +1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow[l_{42}=-2]{l_{32}=1}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ECHELON FORM!
RANK = 2
# free variables = 1

BASIS for CORANGE is $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \end{pmatrix}$ Dimension of CORANGE = 2

Free variable x_3, $-x_2 + x_3 = 0 \Rightarrow x_2 = x_3$

$x_1 + x_2 + 2x_3 = 0 \Rightarrow x_1 + 3x_3 = 0 \quad x_1 = -3x_3$

$$\vec{x} = \begin{pmatrix} -3x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}}_{\vec{m}}$$

The vector $\vec{m}$ is a basis for CORANGE of A, which is 1-dimensional

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COMPLEX VARIABLES: If $A = aI + bJ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Then $A^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2}(aI - bJ)$

using MEMORIZED Formula!