George Francis III — HW# 5

1.9.1c

$$\begin{pmatrix}1&2&3\\2&5&8\\3&8&10\end{pmatrix} \to \begin{matrix}l_{21}=2\\ \\ l_{31}=3\end{matrix}\begin{pmatrix}1&2&3\\0&1&2\\0&2&1\end{pmatrix} \to l_{32}=2\begin{pmatrix}1&2&3\\0&1&2\\0&0&-3\end{pmatrix}$$

$\det = (1)(1)(-3) = -3$

8.2.1 (find eigenvalues & eigen vectors)

a) $\begin{pmatrix}1&-2\\-2&1\end{pmatrix}$ $P_A(\lambda) = (1-\lambda)(1-\lambda)-4 = 1+\lambda^2-2\lambda-4 = \lambda^2-2\lambda-3$

$(\lambda-3)(\lambda+1)$ $\lambda_1=3, \lambda_2=-1$ eigen values

$\lambda=3$ $\begin{pmatrix}1-3&-2\\-2&1-3\end{pmatrix} \to \begin{pmatrix}-2&-2\\-2&-2\end{pmatrix} \to \begin{pmatrix}-2&-2\\0&0\end{pmatrix}$ y free $-2x=2y$, $x=-y$ $\vec{x} = y\begin{pmatrix}-1\\1\end{pmatrix}$

eigen vectors

$\lambda=-1$ $\begin{pmatrix}1-(-1)&-2\\-2&1-(-1)\end{pmatrix} \to \begin{pmatrix}2&-2\\-2&2\end{pmatrix} \to \begin{pmatrix}2&-2\\0&0\end{pmatrix}$ y free $2x=2y$, $x=y$ $\vec{x} = y\begin{pmatrix}1\\1\end{pmatrix}$

c) $\begin{pmatrix}3&1\\-1&1\end{pmatrix}$ $P_A(\lambda) = (3-\lambda)(1-\lambda)-(1)(-1) = \lambda^2-4\lambda+3+1 = \lambda^2-4\lambda+4$

$(\lambda-2)(\lambda-2)$ $\lambda=2$ eigen value

$\lambda=2$ $\begin{pmatrix}3-2&1\\-1&1-2\end{pmatrix} = \begin{pmatrix}1&1\\-1&-1\end{pmatrix} \to \begin{pmatrix}1&1\\0&0\end{pmatrix}$ y free $x=-y$ $\vec{x} = y\begin{pmatrix}-1\\1\end{pmatrix}$

eigen vector

d) $\begin{pmatrix}1&2\\-1&1\end{pmatrix}$ $P_A(\lambda) = (1-\lambda)(1-\lambda)-(-1)(2) = \lambda^2-2\lambda+1+2 = \lambda^2-2\lambda+3$

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-4(1)(3)}}{2(1)} = 1\pm\frac{\sqrt{8}}{2}i = 1\pm\sqrt{2}i$$

$\lambda_1 = 1+\sqrt{2}i$, $\lambda_2 = 1-\sqrt{2}i$ eigen value

$\lambda_1 = 1+\sqrt{2}i$ $\begin{pmatrix}\sqrt{2}i&2\\-1&\sqrt{2}i\end{pmatrix} \to \begin{pmatrix}\sqrt{2}i&2\\0&0\end{pmatrix}$ $x\sqrt{2}i = -2y$

$x = \frac{-2}{\sqrt{2}i} = -\sqrt{2}i$ $\vec{x} = y\begin{pmatrix}-\sqrt{2}i\\1\end{pmatrix}$

eigen vector

$\lambda_2 = 1-\sqrt{2}i$ $\begin{pmatrix}-\sqrt{2}i&2\\-1&-\sqrt{2}i\end{pmatrix} \to \begin{pmatrix}-\sqrt{2}i&2\\0&0\end{pmatrix}$ $-x\sqrt{2}i = -2y$

$x = \frac{2}{\sqrt{2}i}y = \sqrt{2}iy$ $\vec{x} = y\begin{pmatrix}\sqrt{2}i\\1\end{pmatrix}$

8.2.1 $\det(A-\lambda I)$

e) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$P_A(\lambda) = (3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right] - (-1)\left[(-1)(3-\lambda)-0\right]$$

$$(A-\lambda I) = \begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right] - (3-\lambda)$$

$$= (3-\lambda)\left[(\lambda^2-5\lambda+5)-1\right]$$

$$= (3-\lambda)(\lambda^2-5\lambda+4)$$

$$= (3-\lambda)(\lambda-4)(\lambda-1) \rightarrow \boxed{\begin{matrix}\lambda_1 = 3 \\ \lambda_2 = 4 \\ \lambda_3 = 1\end{matrix}}$$ eigen values

$\lambda_1 = 3$ $\begin{pmatrix} 3-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 3-3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $z =$ free, $y = 0$, $x = -z$, $\vec{x} = z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

eigen vectors

$\lambda_2 = 4$ $\begin{pmatrix} 3-4 & -1 & 0 \\ -1 & 2-4 & -1 \\ 0 & -1 & 3-4 \end{pmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{l_{21}=1} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$z =$ free
$x = -y = z$
$y = -z$
$z = z$

$\vec{x} = z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$ $\begin{pmatrix} 3-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 3-1 \end{pmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{l_{21}=-\frac{1}{2}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$z =$ free
$x = \frac{y}{2} = z$
$\frac{y}{2} = z \rightarrow y = 2z$
$z = z$

$\vec{x} = z\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

<u>NOTE</u>: In section 8.4 and in lecture, Prof Bayly said eigen values of a symmetric matrix are always REAL #'s and eigen vectors are always orthogonal.

For example, $(-1\ 0\ 1)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$, $(-1\ 0\ 1)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$, $(1\ -1\ 1)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$

9.1.10 By using the (long) elimination method:

$\frac{du}{dt} = 3u + 4v$ initial conditions $u(0)=3$ $v(0) = -2$

$\frac{dv}{dt} = 4u - 3v$

$u = \left(\frac{1}{4}\right)\frac{dv}{dt} + \left(\frac{3}{4}\right)v$

LHS $\frac{du}{dt} = 3\left(\frac{1}{4}\frac{dv}{dt} + \frac{3}{4}v\right) + 4v = \frac{3}{4}\frac{dv}{dt} + \frac{9}{4}v + 4v = \cancel{\frac{3}{4}\frac{dv}{dt}} + \frac{25}{4}v$

RHS $\frac{d}{dt}\left(\frac{1}{4}\frac{dv}{dt} + \frac{3}{4}v\right) = \frac{1}{4}\frac{d^2v}{dt^2} + \cancel{\frac{3}{4}\frac{dv}{dt}}$

$\frac{1}{4}\frac{d^2v}{dt^2} = \frac{25}{4}v \rightarrow \frac{d^2v}{dt^2} = 25v \rightarrow \frac{d^2v}{dt^2} - 25v = 0$

guess $v(t) = e^{rt}$

$p(r) = r^2 - 25 \Rightarrow (r-5)(r+5)$ $r_1 = 5$ $r_2 = -5$ eigen values

$v(t) = C_1 e^{5t} + C_2 e^{-5t}$ C_1, C_2 arbitrary

back-subs.

$u(t) = \left(\frac{1}{4}\right)\left(5C_1 e^{5t} - 5C_2 e^{-5t}\right) + \left(\frac{3}{4}\right)\left(C_1 e^{5t} + C_2 e^{-5t}\right)$

$u(t) = 2C_1 e^{5t} - \frac{1}{2}C_2 e^{-5t}$

9.1.10 cont...

$$\vec{x}(t) = \begin{pmatrix} 2C_1e^{5t} - \frac{1}{2}C_2e^{-5t} \\ C_1e^{5t} + C_2e^{-5t} \end{pmatrix} = C_1e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2e^{-5t}\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

eigen vectors

using initial conditions ($u(0)=3$, $v(0)=-2$) to solve C_1, C_2

$$u(0) = 2C_1 - \tfrac{1}{2}C_2 = 3$$
$$v(0) = C_1 + C_2 = -2$$

$$\left(\begin{array}{cc|c} 2 & -\frac{1}{2} & 3 \\ 1 & 1 & -2 \end{array}\right) \xrightarrow{L_{2,1}=\frac{1}{2}} \left(\begin{array}{cc|c} 2 & -\frac{1}{2} & 3 \\ 0 & 5/4 & -7/2 \end{array}\right)$$

$$\tfrac{5}{4}C_2 = -\tfrac{7}{2} \rightarrow C_2 = -\tfrac{14}{5}$$

check: $\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} = 0$

"orthogonal"

$$2C_1 - \tfrac{1}{2}C_2 = 3$$
$$2C_1 - \tfrac{1}{2}\left(-\tfrac{14}{5}\right) = 3$$
$$2C_1 + \tfrac{7}{5} = 3 \rightarrow C_1 = \tfrac{4}{5}$$

☆ using matrices, we know that the solution will be combinations of: $e^{\lambda t}\vec{\xi}$ with λ = eigen value, $\vec{\xi}$ = eigen vector

write D.E. as $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Find λ

$$\det\begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} = (3-\lambda)(-3-\lambda) - 16 = 0 \rightarrow \lambda^2 - 25$$

$\lambda_1 = 5$, $\lambda_2 = -5$ eigen values

$\lambda_1 = 5$: $\begin{pmatrix} 3-5 & 4 \\ 4 & -3-5 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix}$ $\quad -2u = -4v$, $u = 2v$ $\quad \vec{X}_1(t) = v\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\vec{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

eigen vectors

$\lambda_2 = -5$: $\begin{pmatrix} 3+5 & 4 \\ 4 & -3+5 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} 8 & 4 \\ 0 & 0 \end{pmatrix}$ $\quad 8u = -4v$, $u = -\frac{1}{2}v$ $\quad \vec{X}_1(t) = v\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

$\vec{\xi}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

therefore, gen sol'n is

$$\vec{u}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = C_1e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2e^{-5t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

C_1, C_2 arbitrary, just as before.

9.1.11 Using elimination method:

a) $$\dot{u}_1 = \frac{du_1}{dt} = u_1 + 9u_2$$

$$\dot{u}_2 = \frac{du_2}{dt} = u_1 + 3u_2$$

$$u_1 = \frac{du_2}{dt} - 3u_2$$

$$\frac{d}{dt}\left(\frac{du_2}{dt} - 3u_2\right) = \left(\frac{du_2}{dt} - 3u_2\right) + 9u_2$$

$$\frac{d^2u_2}{dt^2} - 3\frac{du_2}{dt} = \frac{du_2}{dt} + 6u_2$$

$$\frac{d^2u_2}{dt^2} - 4\frac{du_2}{dt} - 6u_2 = 0 \rightarrow r^2 - 4r - 6 = 0 \rightarrow \frac{+4 \pm \sqrt{16 - 4(1)(-6)}}{2(1)}$$

$$= 2 \pm \frac{2\sqrt{10}}{2} = 2 \pm \sqrt{10}$$

$$r_1 = 2 + \sqrt{10}$$
$$r_2 = 2 - \sqrt{10}$$

eigen values

$$u_2(t) = C_1 e^{(2+\sqrt{10})t} + C_2 e^{(2-\sqrt{10})t}$$

$$u_1(t) = \frac{du_2}{dt} - 3u_2 = (2+\sqrt{10})C_1 e^{(2+\sqrt{10})t} + (2-\sqrt{10})C_2 e^{(2-\sqrt{10})t} - 3C_1 e^{(2+\sqrt{10})t} - 3C_2 e^{(2-\sqrt{10})t}$$

$$u_1(t) = (\sqrt{10}-1)C_1 e^{(2+\sqrt{10})t} - (\sqrt{10}+1)C_2 e^{(2-\sqrt{10})t}$$

$$C_1 e^{(2+\sqrt{10})t}\begin{pmatrix} \sqrt{10}-1 \\ 1 \end{pmatrix} + C_2 e^{(2-\sqrt{10})t}\begin{pmatrix} -\sqrt{10}-1 \\ 1 \end{pmatrix}$$

eigen vectors

$$\vec{u} = \begin{pmatrix} (\sqrt{10}-1)C_1 e^{(2+\sqrt{10})t} - (\sqrt{10}+1)C_2 e^{(2-\sqrt{10})t} \\ C_1 e^{(2+\sqrt{10})t} + C_2 e^{(2-\sqrt{10})t} \end{pmatrix}$$

9.1.11 Using matrices:

a) $\dot{u}_1 = u_1 + 9u_2$ $\qquad e^{\lambda t}\vec{\xi}$ $\qquad \lambda$ = eigen value

$\dot{u}_2 = u_1 + 3u_2$ $\qquad \vec{\xi}$ = eigen vector

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \begin{pmatrix} 1-\lambda & 9 \\ 1 & 3-\lambda \end{pmatrix} \qquad P_A(\lambda) = (1-\lambda)(3-\lambda) - 9$$

$$= \lambda^2 - 4\lambda - 6$$

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-6)}}{2(1)} = 2 \pm \sqrt{10}$$

$$\lambda_1 = 2+\sqrt{10}$$
$$\lambda_2 = 2-\sqrt{10}$$

eigen values

For,
$\lambda_1 = 2+\sqrt{10}$

$$\begin{pmatrix} 1-(2+\sqrt{10}) & 9 \\ 1 & 3-(2+\sqrt{10}) \end{pmatrix} = \begin{pmatrix} -1-\sqrt{10} & 9 \\ 1 & 1+\sqrt{10} \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} -1-\sqrt{10} & 9 \\ 0 & 0 \end{pmatrix}$$

y free

$x(-1-\sqrt{10}) = -9y$

$x = \left(\frac{9}{1+\sqrt{10}}\right)y$

$$\vec{x} = y\begin{pmatrix} \frac{9}{1+\sqrt{10}} \\ 1 \end{pmatrix} \rightarrow {}^{*}\ \frac{9}{1+\sqrt{10}} = \sqrt{10}-1, \text{ so } \vec{x} = y\begin{pmatrix} \sqrt{10}-1 \\ 1 \end{pmatrix}$$

For
$\lambda_2 = 2-\sqrt{10}$

$$\begin{pmatrix} 1-(2-\sqrt{10}) & 9 \\ 1 & 3-(2-\sqrt{10}) \end{pmatrix} = \begin{pmatrix} -1+\sqrt{10} & 9 \\ 1 & 1-\sqrt{10} \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} -1+\sqrt{10} & 9 \\ 0 & 0 \end{pmatrix}$$

$x(-1+\sqrt{10}) = -9y$

$x = \left(\frac{9}{1-\sqrt{10}}\right)y$

$$\vec{x} = y\begin{pmatrix} \frac{9}{1-\sqrt{10}} \\ 1 \end{pmatrix} = y\begin{pmatrix} -\sqrt{10}-1 \\ 1 \end{pmatrix}$$

$$\vec{\xi}_1 = \begin{pmatrix} \sqrt{10}-1 \\ 1 \end{pmatrix} \qquad \vec{\xi}_2 = \begin{pmatrix} -\sqrt{10}-1 \\ 1 \end{pmatrix}$$

eigen vectors

* just as before

9.11.1

d) $\dot{y}_1 = y_2$

$\dot{y}_2 = 3y_1 + 2y_3$

$\dot{y}_3 = -y_2$

$$\rightarrow \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}}_{A}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

know $\lambda = 0$ will be an eigenvalue

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 & 0 \\ 3 & -\lambda & 2 \\ 0 & -1 & -\lambda \end{bmatrix} = (-\lambda)\left[-\lambda(-\lambda) - 2(-1)\right] - (1)\left[3(-\lambda) - 0\right]$$

$$= -\lambda(\lambda^2 + 2) + 3\lambda = -\lambda^3 - 2\lambda + 3\lambda = -(\lambda^3 - \lambda)$$

$$= -\lambda(\lambda^2 - 1) = -\lambda(\lambda + 1)(\lambda - 1)$$

$\lambda_1 = 0$

$\lambda_2 = 1$

$\lambda_3 = -1$

eigen values

$\lambda_1 = 0$
$$\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$3y_1 = -2y_3 \rightarrow y_1 = -\frac{2}{3}y_3$

$y_2 = 0$

$y_3 = y_3$ free

$$\vec{\xi}_1 = \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_2 = 1$
$$\begin{pmatrix} -1 & 1 & 0 \\ 3 & -1 & 2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{l_{21} = -3} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$y_1 = y_2 = -y_3$

$y_2 = -y_3$

$y_3 = \text{free} = y_3$

$$\vec{\xi}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda_3 = -1$
$$\begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{l_{21} = 3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$y_1 = -y_2 = -y_3$

$y_2 = +y_3$

$y_3 = \text{free} = y_3$

$$\vec{\xi}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

eigen vectors

therefore, gen sol'n is

$$\vec{y} = C_1\begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$