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HW #16.

8.3.15 (c)

$$A = \begin{pmatrix} -4 & -2 \\ 5 & 2 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -4 & -2 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -4-\lambda & -2 \\ 5 & 2-\lambda \end{pmatrix}$$

$$P_A = (-4-\lambda)(2-\lambda) + 10$$
$$= -8 + 4\lambda - 2\lambda + \lambda^2 + 10$$
$$= \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \boxed{\lambda^{(1)} = -1 + i} \quad \boxed{\lambda^{(2)} = -1 - i}$$

For $\lambda = -1 + i$

$$\begin{pmatrix} -4-(-1+i) & -2 \\ 5 & 2-(-1+i) \end{pmatrix} = \begin{pmatrix} -3-i & -2 \\ 5 & 3-i \end{pmatrix} \to \begin{pmatrix} -3-i & -2 \\ 0 & 0 \end{pmatrix}$$

$$(-3-i)x_1 - 2x_2 = 0$$
$$(-3-i)x_1 = 2x_2$$
$$x_1 = \frac{2}{-3-i} x_2$$
$$\boxed{x_1 = \left(-\frac{3}{5} + \frac{1}{5}i\right) x_2}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{3}{5} + \frac{1}{5}i \\ 1 \end{pmatrix}}_{\vec{\xi}^{(1)} \text{ eigenvector}} x_2$$

For $\lambda = -1 - i$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{3}{5} - \frac{1}{5}i \\ 1 \end{pmatrix}}_{\vec{\xi}^{(2)} \text{ eigenvector}} x_2$$

$$\therefore S = \begin{pmatrix} -\frac{3}{5} + \frac{1}{5}i & -\frac{3}{5} - \frac{1}{5}i \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} -\frac{5}{2}i & \frac{1}{2} - \frac{3}{2}i \\ \frac{5}{2}i & \frac{1}{2} + \frac{3}{2}i \end{pmatrix}$$

$$\Lambda = S^{-1} A S = \begin{pmatrix} -\frac{5}{2}i & \frac{1}{2} - \frac{3}{2}i \\ \frac{5}{2}i & \frac{1}{2} + \frac{3}{2}i \end{pmatrix} \begin{pmatrix} -4 & -2 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} \frac{-3}{5} + \frac{1}{5}i & -\frac{3}{5} - \frac{1}{5}i \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix} \quad \#.$$

8.3.15 (g)

$$A = \begin{pmatrix} 2 & 5 & 5 \\ 0 & 2 & 0 \\ 0 & -5 & -3 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 2-\lambda & 5 & 5 \\ 0 & 2-\lambda & 0 \\ 0 & -5 & -3-\lambda \end{pmatrix}$$

$$P_A = -(\lambda-2)^2(\lambda+3) = 0 \Rightarrow \boxed{\lambda^{(1)} = 2} \quad \boxed{\lambda^{(2)} = -3} \text{ eigenvalues}$$

for $\lambda = 2$

$$\begin{pmatrix} 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & -5 & -5 \end{pmatrix} \to \begin{pmatrix} 0 & 5 & 5 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \boxed{5} & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \text{free} & & \text{free} \end{array}$$

$$5x_2 + 5x_3 = 0$$

$$\boxed{x_2 = -x_3}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}_{\vec{\xi}^{(1)}} x_3 + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{\xi}^{(2)}} x_1$$

eigenvector

<u>for $\lambda = -3$</u>

$$\begin{pmatrix} 5 & 5 & 5 \\ 0 & 5 & 0 \\ 0 & -5 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 5 & 5 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$5x_2 = 0 \Rightarrow x_2 = 0$

$5x_1 + 5x_3 = 0$

$x_1 = -x_3$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{\xi}^{(3)} \text{ eigen vector}} x_3$$

$$S = \begin{pmatrix} \vec{\xi}^{(1)} & \vec{\xi}^{(2)} & \vec{\xi}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Lambda = S^{-1} A S = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 & 5 \\ 0 & 2 & 0 \\ 0 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

8.4.12 (b)

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ -1 & 4-\lambda \end{pmatrix}$$

$$\begin{aligned} P_A &= (2-\lambda)(4-\lambda) - 1 \\ &= 8 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 7 \end{aligned}$$

$\boxed{\lambda^{(1)} = 3+\sqrt{2}}$ $\boxed{\lambda^{(2)} = 3-\sqrt{2}}$ eigenvalues

For $\lambda = 3-\sqrt{2}$

$$\begin{pmatrix} \sqrt{2}-1 & -1 \\ -1 & \sqrt{2}+1 \end{pmatrix} \to \begin{pmatrix} \sqrt{2}-1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$(\sqrt{2}-1)x_1 - x_2 = 0$$

$$x_1 = \frac{1}{\sqrt{2}-1} x_2$$

$$\boxed{x_1 = \sqrt{2}+1\ x_2}$$

$$\vec{x} = \underbrace{\begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix}}_{\xi^{(1)}} x_2$$

$\xi^{(1)}$ eigenvector

length $= \sqrt{2\sqrt{2}+4}$

$$\vec{q}^{(1)} = \begin{pmatrix} \dfrac{\sqrt{2}+1}{\sqrt{2\sqrt{2}+4}} \\ \dfrac{1}{\sqrt{2\sqrt{2}+4}} \end{pmatrix}$$

For $\lambda = 3+\sqrt{2}$

$$\begin{pmatrix} -\sqrt{2}-1 & -1 \\ -1 & -\sqrt{2}+1 \end{pmatrix} \to \begin{pmatrix} -\sqrt{2}-1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$(-\sqrt{2}-1)x_1 - x_2 = 0$$

$$x_1 = \frac{1}{-\sqrt{2}-1} x_2$$

$$\boxed{x_1 = -\sqrt{2}+1\ x_2}$$

$$\vec{x} = \underbrace{\begin{pmatrix} -\sqrt{2}+1 \\ 1 \end{pmatrix}}_{\xi^{(2)}} x_2$$

$\xi^{(2)}$ eigenvector

length $= \sqrt{-2\sqrt{2}+4}$

$$\vec{q}^{(2)} = \begin{pmatrix} \dfrac{-\sqrt{2}+1}{\sqrt{-2\sqrt{2}+4}} \\ \dfrac{1}{\sqrt{-2\sqrt{2}+4}} \end{pmatrix}$$

continue 8.4.12 (b)

$$Q = \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{2\sqrt{2}+4}} & \frac{-\sqrt{2}+1}{\sqrt{-2\sqrt{2}+4}} \\ \frac{1}{\sqrt{2\sqrt{2}+4}} & \frac{1}{\sqrt{-2\sqrt{2}+4}} \end{pmatrix}$$

8.4.14 (c)

$\lambda_1 = 3 \quad V_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{length}_{(1)} = \frac{1}{\sqrt{5}}$

$\lambda_2 = -1 \quad V_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{length}_{(2)} = \frac{1}{\sqrt{5}}$

* Note :- This problem is wrongly stated, The eigenvectors are not orthognal $Q^T \neq Q^{-1}$

$$\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{q}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} \qquad \vec{q}_2 = \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \qquad Q^T = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \neq Q^{-1}$$

$$A = Q \Lambda Q^T = \begin{pmatrix} \frac{11}{5} & -\frac{4}{5} \\ \frac{-4}{5} & -1/5 \end{pmatrix}$$

9.4.1 (a)

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{pmatrix}$$

$$P_A = \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda^{(1)} = -2 \quad \lambda^{(2)} = 1$$

for $\lambda = -2$

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -1 \\ 0 & 0 \end{pmatrix}$$

$$4x_1 - x_2 = 0$$
$$4x_1 = x_2$$
$$x_1 = \frac{1}{4} x_2$$

$$\vec{x} = \underbrace{\begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}}_{\xi^{(1)} \text{ eigen-vector}} x_2$$

for $\lambda = 1$

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$x_1 - x_2 = 0$$
$$x_1 = x_2$$

$$\vec{x} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\xi^{(2)} \text{ eigenvector}} x_2$$

$$S = \begin{pmatrix} \frac{1}{4} & 1 \\ 1 & 1 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} -4/3 & 4/3 \\ 4/3 & -1/3 \end{pmatrix}$$

$$e^{t\Lambda} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix}$$

$$e^{tA} = S \; e^{t\Lambda} \; S^{-1} = \begin{pmatrix} \frac{1}{4} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} -4/3 & 4/3 \\ 4/3 & -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}\left(4e^{t} - e^{-2t}\right) & \frac{1}{3}\left(e^{-2t} - e^{t}\right) \\ \frac{4}{3}\left(e^{t} - e^{-2t}\right) & \frac{1}{3}\left(4e^{-2t} - e^{t}\right) \end{pmatrix}$$

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9.4.1 (e)

$$A = \begin{pmatrix} -1 & 2 \\ -5 & 5 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -1-\lambda & 2 \\ -5 & 5-\lambda \end{pmatrix}$$

$$p_A = (-1-\lambda)(5-\lambda) + 10$$

$$= \lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \boxed{\lambda = 2 \pm i} \text{ eigenvalues}$$

for $\lambda = 2+i$

$$\begin{pmatrix} -3-i & 2 \\ -5 & 3-i \end{pmatrix} \to \begin{pmatrix} -3-i & 2 \\ 0 & 0 \end{pmatrix}$$

, so

$$(-3-i)x_1 + 2x_2 = 0$$

$$x_1 = \frac{-2}{-3-i} x_2$$

$$= \left(\frac{3}{5} - \frac{1}{5}i\right) x_2$$

$$\vec{x} = \underbrace{\begin{pmatrix} \frac{3}{5} - \frac{1}{5}i \\ 1 \end{pmatrix}}_{\vec{\xi}^{(1)} \text{ eigenvector}} x_2$$

for $\lambda = 2-i$

$$\vec{\xi}^{(2)} = \begin{pmatrix} \frac{3}{5} + \frac{1}{5}i \\ 1 \end{pmatrix} \quad \Rightarrow \quad S = \begin{pmatrix} \frac{3}{5} - \frac{1}{5}i & \frac{3}{5} + \frac{1}{5}i \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} \frac{5}{2}i & \frac{1}{2} - \frac{3}{2}i \\ -\frac{5}{2}i & \frac{1}{2} + \frac{3}{2}i \end{pmatrix}, \quad e^{t\Lambda} = \begin{pmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{pmatrix}$$

$$e^{tA} = S e^{t\Lambda} S^{-1} = \begin{pmatrix} \frac{3}{5} - \frac{1}{5}i & \frac{3}{5} + \frac{1}{5}i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{pmatrix} \begin{pmatrix} \frac{5}{2}i & \frac{1}{2} - \frac{3}{2}i \\ -\frac{5}{2}i & \frac{1}{2} + \frac{3}{2}i \end{pmatrix}$$

$$= \begin{pmatrix} (\cos t - 3\sin t)e^{2t} & 2\sin t \, e^{2t} \\ -5\sin t \, e^{2t} & (\cos t + 3\sin t)e^{2t} \end{pmatrix}$$

9.4.2 a)

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -\lambda & 0 & 0 \\ 2 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{pmatrix}$$

$$P_A = -\lambda(\lambda^2+1) \qquad \lambda = 0 \quad \text{or} \quad \lambda = \pm i$$

for $\lambda = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} -x_2 = 0 \\ 2x_1 + x_3 = 0 \\ \boxed{x_1 = -\frac{1}{2}x_3} \end{array} \qquad \vec{x} = \underbrace{\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}}_{\vec{\xi}^{(1)} \text{ eigenvector}} x_3$$

for $\lambda = i$

$$\begin{pmatrix} -i & 0 & 0 \\ 2 & -i & 1 \\ 0 & -1 & -i \end{pmatrix} \xrightarrow{l_{21} = 2i} \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{pmatrix} \xrightarrow{l_{32} = -i} \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-ix_2 + x_3 = 0$$

$$-ix_2 = -x_3$$

$$x_2 = \frac{1}{i}x_3$$

$$\boxed{x_2 = -ix_3}$$

$$\vec{x} = \underbrace{\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}}_{\vec{\xi}^{(2)} \text{ eigenvector}} x_3 \qquad \therefore \vec{\xi}^{(3)} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

then $S = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -i & i \\ 1 & 1 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2}i & \frac{1}{2} \\ 1 & -\frac{1}{2}i & \frac{1}{2} \end{pmatrix},$

$$e^{t\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-it} \end{pmatrix}$$

$$A = S e^{t\Lambda} S^{-1} = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -i & i \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2}i & \frac{1}{2} \\ 1 & -\frac{1}{2}i & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 2\sin t & \cos t & \sin t \\ 2\cos t + 2 & -\sin t & \cos t \end{pmatrix}$$

9.4.2 (d)

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$P_A = -\lambda(\lambda^2) - 1 = -\lambda^3 + 1 \qquad \boxed{\lambda = 1 \quad \lambda = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i \quad \lambda = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i} \text{ eigenvalues}$$

for $\lambda = 1$

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$-x_2 + x_3 = 0$

$x_2 = x_3$

$-x_1 + x_3 = 0$

$-x_1 = -x_3$

$x_1 = x_3$

$$\vec{x} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{v}^{(1)}} x_3 \quad \text{eigenvector}$$

for $\lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$$\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 & 1 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix} \xrightarrow{L_{21} = \frac{1}{2} + \frac{\sqrt{3}}{2}i} \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 & 1 \\ 0 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ 0 & 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix}$$

$$\xrightarrow{L_{32} = \frac{1}{2} + \frac{\sqrt{3}}{2}i} \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 & 1 \\ 0 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right) x_2 + \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right) x_3 = 0$$

$$\boxed{x_2 = \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right) x_3}$$

$$\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right) x_1 + x_3 = 0$$

$$x_1 = -\frac{1}{\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right)} x_3$$

$$x_1 = \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right) x_2$$

$$\vec{x} = \underbrace{\begin{pmatrix} -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \\ -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \\ 1 \end{pmatrix}}_{\vec{\xi}^{(2)} \text{ eigenvector}}$$

for $\lambda = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

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$$\vec{\xi}^{(3)} = \begin{pmatrix} -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \\ -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \\ 1 \end{pmatrix}$$

$$\therefore S = \begin{pmatrix} 1 & -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i & -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \\ 1 & -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i & -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \\ 1 & 1 & 1 \end{pmatrix}$$

$$, S^{-1} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/6 + \tfrac{\sqrt{3}}{6} i & -\tfrac{1}{6} - \tfrac{\sqrt{3}}{6} i & 1/3 \\ -1/6 - \tfrac{\sqrt{3}}{6} i & -1/6 + \tfrac{\sqrt{3}}{6} i & 1/3 \end{pmatrix}, \quad e^{t\Lambda} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t} & 0 \\ 0 & 0 & e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t} \end{pmatrix}$$

$$e^{tA} = S e^{t\Lambda} S^{-1} = \begin{pmatrix} 1 & -1/2 - \tfrac{\sqrt{3}}{2} i & -1/2 + \tfrac{\sqrt{3}}{2} i \\ 1 & -1/2 + \tfrac{\sqrt{3}}{2} i & -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t} & 0 \\ 0 & 0 & e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t} \end{pmatrix} *$$

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -\tfrac{1}{6} + \tfrac{\sqrt{3}}{6} i & -1/6 - \sqrt{3}/6\, i & 1/3 \\ -1/6 - \sqrt{3}/6\, i & -1/6 + \sqrt{3}/6\, i & 1/3 \end{pmatrix} =$$

9.4.4 (e)

$$A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -\lambda & 1 & -2 \\ -1 & -\lambda & 2 \\ 2 & -2 & -\lambda \end{pmatrix}$$

$$P_A = -\lambda(\lambda^2 + 9) \qquad \boxed{\lambda = 0 \quad \lambda = 3i \quad \lambda = -3i} \text{ eigenvalues}$$

for $\lambda = 0$

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & -2 \\ 2 & -2 & 0 \end{pmatrix} \xrightarrow{L_{31} = -2} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

$$\xrightarrow{L_{32} = -2} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & -2 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \qquad x_2 - 2x_3 = 0 \qquad -x_1 + 2x_3 = 0$$

$$\boxed{x_2 = 2x_3} \qquad \boxed{x_1 = 2x_3}$$

$$\vec{x} = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}_{\xi^{(1)} \text{ eigenvector}} x_3$$

for $\lambda = 3i$

$$\begin{pmatrix} -3i & 1 & -2 \\ -1 & -3i & 2 \\ 2 & -2 & -3i \end{pmatrix} \xrightarrow[L_{31} = \frac{2}{3}i]{L_{21} = -\frac{1}{3}i} \begin{pmatrix} -3i & 1 & 2 \\ 0 & -8/3\, i & 2 - \frac{2}{3}i \\ 0 & -2 - \frac{2}{3}i & -\frac{5}{3}i \end{pmatrix} \xrightarrow{L_{32} = \frac{1}{4} - \frac{3}{4}i} \begin{pmatrix} -3i & 1 & -2 \\ 0 & -8/3\, i & 2 - \frac{2}{3}i \\ 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{8}{3}i\, x_2 + \left(2 - \frac{2}{3}i\right) x_3 = 0 \quad , \quad -3i\, x_1 - 2x_3 = 0$$

$$x_2 = \left(-\frac{1}{4} - \frac{3}{4}i\right) x_3 \qquad x_1 = \frac{2}{3}i\, x_3$$

$$\vec{x} = \underbrace{\begin{pmatrix} \frac{2}{3}i \\ -\frac{1}{4} - \frac{3}{4}i \\ 1 \end{pmatrix}}_{\xi^{(2)} \text{ eigenvector}} x_3$$

Hence, $\xi_3^{(3)} = \begin{pmatrix} -\frac{2}{3i} \\ -\frac{1}{4} + \frac{3}{4} i \\ 1 \end{pmatrix}$, $S = \begin{pmatrix} 2 & \frac{2}{3} i & -\frac{2}{3} i \\ 2 & -\frac{1}{4} - \frac{3}{4} i & -\frac{1}{4} + \frac{3}{4} i \\ 1 & 1 & 1 \end{pmatrix}$

$$S^{-1} = \begin{pmatrix} 1/4 & 2/9 & 1/18 \\ -\frac{1}{8} - \frac{3}{8} i & -1/9 + \frac{1}{3} i & \frac{17}{36} + \frac{1}{12} i \\ -\frac{1}{8} + \frac{3}{8} i & -1/9 - 1/3 i & \frac{17}{36} - \frac{1}{12} i \end{pmatrix}$$

$$e^{t\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{3it} & 0 \\ 0 & 0 & e^{-3it} \end{pmatrix}$$ and since $t = 1$

$$e^{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{3i} & 0 \\ 0 & 0 & e^{-3i} \end{pmatrix} \Rightarrow e^{A} = S e^{\Lambda} S^{-1}$$

$$e^{A} = \begin{pmatrix} 2 & \frac{2}{3} i & -\frac{2}{3} i \\ 2 & -\frac{1}{4} - \frac{3}{4} i & -\frac{1}{4} - \frac{3}{4} i \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{3i} & 0 \\ 0 & 0 & e^{-3i} \end{pmatrix} \begin{pmatrix} 1/4 & 2/9 & 1/18 \\ -1/8 - 3/8 i & -1/9 + 1/3 i & \frac{17}{36} + 1/12 i \\ -1/8 + 3/8 i & -1/9 - 1/3 i & 17/36 - 1/12 i \end{pmatrix}$$

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