PROF. BAYLY SOLUTIONS

Math 410 (Prof. Bayly) EXAM 4: Monday 8 August 2005

There are 4 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can.

You will not need a calculator on this exam. If your calculations become numerically awkward and time-consuming, you may describe the steps you would take if you had a calculator.

It is EXTREMELY important to show your work! Correct answers without documented support will have points deducted.

By complex conjugation, since $\lambda_2 = 1-i = \lambda_1^*$

$\Rightarrow \vec{\xi}_2 = \vec{\xi}_1^* = \begin{pmatrix} +i \\ 1 \end{pmatrix}$ belongs to λ_2.

(1)(15 points) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

$$p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1 - (-1)$$

$$= \lambda^2 - 2\lambda + 2$$

ROOTS $\lambda = \frac{1}{2}\left[2 \pm \sqrt{4-4(2) = -4}\right]$

$$\lambda = \frac{1}{2}[2 \pm 2i] = 1 \pm i \quad \text{Eigenvalues}$$

$\lambda_1 = 1+i$ $\quad (A-\lambda_1 I)\vec{x} = \vec{0}$ is $\left(\begin{array}{cc|c} 1-(1+i) & 1 & 0 \\ -1 & 1-(1+i) & 0 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right) \xrightarrow{l_{21} = \frac{-1}{-i} = -i} \left(\begin{array}{cc|c} -i & 1 & 0 \\ 0 & \cancel{i}\,-i\;0 & 0 \end{array}\right)$$

y free

$-ix + y = 0$

$+ix = +y \quad x = y/i = -iy \Rightarrow \vec{x} = y\begin{pmatrix} -i \\ 1 \end{pmatrix}$

SO $\boxed{\vec{\xi}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \text{ belongs to } \lambda_1 = 1+i}$.

(2)(20 points) A travelling saleswoman's territory consists of the three towns Arivaca, Benson, and Cochise. Each day she decides which town to be at the next day according to the probabilities:

Today Arivaca: tomorrow Benson(.3), Cochise (.7).

Today Benson: tomorrow Arivaca (.5), Cochise (.5).

Today Cochise: tomorrow Arivaca (.4), Benson (.6).

(a)(5 points) If $\vec{p}(t) = (p_A(t), p_B(t), p_C(t))^T$ represents the probabilities that she is in towns Arivaca, Benson, or Cochise respectively on day t, write down a matrix T for which $\vec{p}(t+1) = T\vec{p}(t)$.

(b)(10 points) In the long term, what are the relative amounts of her time that she spends in each town?

(c)(5 points) Assuming she never takes a day off, how many days does she spend in each town in a year? (LEAVE AS A FRACTION!)

(a) ~~TT~~ Note that the saleswoman NEVER stays 2 nights in the same town!

$$\Rightarrow T = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & .5 & .4 \\ .3 & 0 & .6 \\ .7 & .5 & 0 \end{pmatrix} \end{array}$$

(b) We KNOW ~~Expect~~ ~~the~~ eigenvalue $\lambda = 1$ will have an INVARIANT distribution given by its eigenvector.

$$(T - 1I)\vec{x} = 0 \quad \begin{pmatrix} -1 & .5 & .4 \\ .3 & -1 & .6 \\ .7 & .5 & -1 \end{pmatrix} \xrightarrow[L_3 = -.7]{L_2 = -.3} \begin{pmatrix} -1 & .5 & .4 \\ 0 & -.85 & .72 \\ \cancel{0} & \cancel{.85} & \cancel{-.72} \end{pmatrix} \Rightarrow 0\text{s}$$

$$\Rightarrow z \text{ free}, \quad -.85y + .72z = 0 \Rightarrow y = \frac{72}{85}z$$

$$-x + .5\left(\frac{72}{85}z\right) + .4z = 0 \quad \Rightarrow$$

$x = .5\frac{72}{85}z + .4z = \frac{72}{170}z + \frac{4\times 17 = 68}{10\times 17}z = \frac{14\cancel{0}}{17\cancel{0}}z = \frac{70}{85}z$

so $\vec{x} = z\begin{pmatrix} 70/85 \\ 72/85 \\ 1 = 85/85 \end{pmatrix}$ z free

To get a probability vector, need to get entries in $\vec{x}$ to add to 1 $\Rightarrow z\left(\frac{70}{85} + \frac{72}{85} + \frac{85}{85}\right) = 1$

$$\begin{array}{r} 70 \\ 72 \\ 85 \\ \hline 227 \end{array}$$

$\Rightarrow z = \frac{85}{227}$ $\vec{p} = \begin{pmatrix} 70/227 \\ 72/227 \\ 85/227 \end{pmatrix}$ = fraction in ARIVACA, BENSON, COCHISE

ⓒ Total days per year $= \frac{365 \text{ days}}{\text{year}} \times \vec{p} = \begin{pmatrix} 365\times 70/227 \\ 365\times 72/227 \\ 365\times 85/227 \end{pmatrix}$

* I did not realize the fractions would be so ugly! You don't need to work out all multiplication to get full credit.

(3)(30 points) The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is not square but can still be represented by its Singular Value Decomposition $P\Sigma Q^T$, where

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad , \quad \Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad Q = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -\sqrt{3} \\ 2 & 0 \\ 1 & \sqrt{3} \end{pmatrix}.$$

(a)(10 points) If A represents a digital image, we can compress the image by deleting the 1 in the lower right corner of Σ. What is the resulting approximate A? How similar is the approximate image to the exact?

(b)(10 points) Calculate the pseudoinverse $A^+ = Q\Sigma^{-1}P^T$.

(c)(10 points) Calculate A^+A and AA^+. Did you expect one or both of these to be the identity?

Check SVD! $P\Sigma Q^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ -\sqrt{3} & 0 & \sqrt{3} \end{pmatrix}\frac{1}{\sqrt{6}}$

$= \frac{1}{\sqrt{2}\sqrt{6}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 2\sqrt{3} & \sqrt{3} \\ -\sqrt{3} & 0 & \sqrt{3} \end{pmatrix} = \frac{1}{\underbrace{\sqrt{2}\sqrt{6}}_{2\sqrt{3}}}\begin{pmatrix} 2\sqrt{3} & 2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} & 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

(THIS WAS NOT ASKED in the question!)

(a) Delete $\sigma_2 = 1 \Rightarrow A_{approx} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ -\sqrt{3} & 0 & \sqrt{3} \end{pmatrix}\frac{1}{\sqrt{6}}$

$= \frac{1}{\sqrt{2}\sqrt{6}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 2\sqrt{3} & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix} \sqrt{3} & 2\sqrt{3} & \sqrt{3} \\ \sqrt{3} & 2\sqrt{3} & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 1 & 1/2 \end{pmatrix}$

SAME middle column! And AVERAGE dark ness of side columns. BUT dark & light pixels have blurred together.

(3b) $$A^+ = Q\Sigma^{-1}P^T = \frac{1}{\sqrt{6}}\begin{pmatrix}1 & -\sqrt{3}\\ 2 & 0\\ 1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}\sqrt{6}}\begin{pmatrix}1 & -\sqrt{3}\\ 2 & 0\\ 1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -1 & 1\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\frac{1}{\sqrt{3}}+\sqrt{3} & \frac{1}{\sqrt{3}}-\sqrt{3}\\ 2/\sqrt{3} & 2/\sqrt{3}\\ 1/\sqrt{3}-\sqrt{3} & 1/\sqrt{3}+\sqrt{3}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}\frac{1}{3}+1 & \frac{1}{3}-1\\ 2/3 & 2/3\\ 1/3-1 & 1/3+1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}4/3 & -2/3\\ 2/3 & 2/3\\ -2/3 & 4/3\end{pmatrix} = \begin{pmatrix}2/3 & -1/3\\ 1/3 & 1/3\\ -1/3 & 2/3\end{pmatrix} = A^+$$

(3c) $$AA^+ = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}2/3 & -1/3\\ 1/3 & 1/3\\ -1/3 & 2/3\end{pmatrix} = \begin{pmatrix}2/3+1/3 & -1/3+1/3\\ 1/3-1/3 & 1/3+2/3\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I \checkmark$$

$$A^+A = \begin{pmatrix}2/3 & -1/3\\ 1/3 & 1/3\\ -1/3 & 2/3\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix} = \begin{pmatrix}2/3 & 1/3 & -1/3\\ 1/3 & 2/3 & 1/3\\ -1/3 & 1/3 & 2/3\end{pmatrix}$$

Sort of like a BLURRED 3×3 I!

(4)(35 points) The matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

(a)(5 points) Find the characteristic (eigen)polynomial of A, and identify the eigenvalues.

(b)(10 points) Find the corresponding eigenvectors of A.

(c)(10 points) Find a matrix S and its inverse S^{-1} for which $S^{-1}AS = \Lambda$ is expected to be diagonal. You don't have to calculate this product, but you should calculate the inverse explicitly, using

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{a\cancel{b}_{d} - b\cancel{d}_{c}} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

(d)(10 points) Calculate e^{tA}. You may use the diagonal representation $A = S\Lambda S^{-1}$, with S and Λ from part (c).

(a) $p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - 3$

$= \lambda^2 - 6\lambda + 8 - 3 = \lambda^2 - 6\lambda + 5 \overset{\text{FACTOR!}}{=} (\lambda-1)(\lambda-5)$

$\Rightarrow \boxed{\lambda_1 = 1, \ \lambda_2 = 5 \quad \text{EIGENVALUES}}$

(b) $(A - \lambda_1 I)\vec{x} = \vec{0} \Rightarrow \begin{pmatrix} 1 & 1 \\ \cancel{3} & \cancel{3} \end{pmatrix} \to 0\text{s}$ y free $x + y = 0$ $x = -y$ $\vec{x} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\boxed{\vec{\xi}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

$(A - \lambda_2 I)\vec{x} = \vec{0} \Rightarrow \begin{pmatrix} -3 & 1 \\ \cancel{3} & \cancel{-1} \end{pmatrix} \to 0\text{s}$ y free $-3x + y = 0$ $x = \tfrac{1}{3}y$ $\vec{x} = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix} y$

choose $y = 3 \Rightarrow \boxed{\vec{\xi}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$

(c) $S = \text{eigenvector matrix} = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$

$$S^{-1} = \frac{1}{-3-1}\begin{pmatrix} 3 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

CHECK
$\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} =$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(d) $e^{tA} = S e^{t\Lambda} S^{-1}$ since $A = S \Lambda S^{-1}$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{4}$$

$$= \frac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -3e^{t} & e^{t} \\ e^{5t} & e^{5t} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3e^{t}+e^{5t} & -e^{t}+e^{5t} \\ -3e^{t}+3e^{5t} & e^{t}+3e^{5t} \end{pmatrix}$$