

**COMPATIBLE SYSTEMS OF GALOIS REPRESENTATIONS
ASSOCIATED TO THE EXCEPTIONAL GROUP E_6**

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1. INTRODUCTION

In [Ser94], Serre raised the question of whether G_2 or E_8 was the motivic Galois group of a motive M over a number field, and one can evidently ask the same question for the other exceptional simple Lie groups. A slightly weaker version of this question asks for a motive M such that the associated p -adic Galois representations have algebraic monodromy group equal to the exceptional group in question. In this form, Serre’s question was answered in the affirmative by Yun in [Yun14], who also dealt with the case of the exceptional group E_7 . (A stronger version of the question for the group G_2 had previously been answered by Dettweiler and Reiter [DR10].) This left open the cases of E_6 or F_4 . In [Pat16], the last author of this paper succeeded in constructing geometric Galois representations for the remaining exceptional groups using arguments inspired by Ramakrishna’s lifting theorems [Ram02], at least for a set of primes p of density one (improved to all but finitely many in Theorem 1.2 of [Pat17]). While this answered a weak form of (the E_6 analogue of) Serre’s question, the Galois representations constructed in [Pat16, Pat17] did not obviously come from motives M or from compatible systems of Galois representations (although that would certainly be a consequence of the Fontaine–Mazur conjectures [FM95]). The main goal of this paper is to remedy this lacuna for the group E_6 .

Theorem 1.1. *Let F/F^+ be a totally imaginary CM field with maximal totally real subfield F^+ . Let G denote the simply connected form of E_6 , and fix a minuscule representation $G \rightarrow \mathrm{GL}_{27}$. Then there exists a strongly compatible system of Galois representations with coefficients in a number field M such that the representations*

$$r_\lambda : G_F \rightarrow G(\overline{M}_\lambda) \hookrightarrow \mathrm{GL}_{27}(\overline{M}_\lambda)$$

have images with Zariski closure $G(\overline{M}_\lambda)$ for all primes λ . Moreover, this compatible system is potentially automorphic and motivic in the sense that there is a CM extension H/F such that:

- *There a cuspidal automorphic representation π for GL_{27}/H such that $r_\lambda|_{G_H}$ is the compatible system of Galois representations associated to π .*
- *The compatible system r_λ satisfies the conclusion of the Fontaine–Mazur conjecture: there is a smooth projective variety X/F and integers i and j such that r_λ is a G_F -sub-representation of $H^i(X_{\overline{F}}, \overline{\mathbf{Q}}_l(j))$.*

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The main idea of the paper is to follow the strategy of [Pat16], but to replace the lifting theorems inspired by Ramakrishna with those inspired by the work of Khare and Wintenberger [KW09a, KW09b], exploiting the modularity lifting theorems of [BLGGT14]. In order to do this, one must link G -representations with GL_n representations by choosing some (faithful) representation $r: G \rightarrow \mathrm{GL}_n$. The methods of [BLGGT14] require that the corresponding Galois representations have *distinct* Hodge–Tate weights. This imposes a strong restriction on the representation r , namely, that its formal character should be multiplicity-free. In particular, the method of this paper only applies to the exceptional groups G_2 , E_6 , and E_7 , in their quasi-minuscule (G_2) or minuscule (E_6 and E_7) representations (we concentrate on E_6 because other methods are available in the other cases). As in [Pat16, Pat17], we require a seed representation $\bar{\rho}: G_F \rightarrow G(\overline{\mathbf{F}}_p)$ from which to construct geometric lifts. In [Pat16], suitable representations $\bar{\rho}$ came from composing representations associated to modular forms with the principal SL_2 . These representations are not suitable for our purposes, because their composition with the minuscule representation is reducible. Instead, we construct a representation related to the action of the Weyl group of E_6 on the weight space of our representation. The fact that the representation we consider is irreducible in GL_n relies on the assumption that r is minuscule. The reason we succeed in controlling the monodromy groups at *all* primes is a consequence of elementary combinatorial properties of the formal character of E_6 (using ideas of Larsen and Pink [LP92]) together with our ability to exploit independence of p results in compatible systems of Galois representations associated to automorphic forms ([TY07, Shi11, Car12]).

We end the introduction with some remarks on what the methods of this paper cannot do.

Remark 1.2. Galois representations for \mathbf{Q} versus imaginary quadratic fields. Our construction gives compatible systems of E_6 -representations over any imaginary quadratic field F ; these extend to representations of $G_{\mathbf{Q}}$ whose image is Zariski-dense in the L-group $G \rtimes \mathrm{Out}(G)$ of an outer form of E_6 . We leave open the question as to whether actual E_6 -systems exist over \mathbf{Q} , noting that the methods of this paper will not succeed in constructing them. Indeed, our methods require that the corresponding Galois representations have regular weight, and there do not exist any such compatible systems of Galois representations over \mathbf{Q} (see Remark 5.1).

Remark 1.3. Motives versus motives with coefficients. We ultimately construct compatible systems of 27-dimensional Galois representations for a coefficient field L over which we have little control. One can ask the more refined question of whether there exists a motive M with coefficients over \mathbf{Q} of type E_6 (Yun’s result [Yun14] answer this question in the affirmative for E_8). One reason that this refinement is interesting is that it would have applications to the to the inverse Galois problem (see Corollary 6.1).

It seems to us that the Galois theoretic methods of either this paper or of [Pat16, Pat17] are unsuited to answering such a question. It is illustrative to consider the simpler case of GL_2 . By constructing a geometric Galois representation $\rho: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Q}_p)$, one can hope to prove it is automorphic and hence associated to a modular form f , and thus to construct a corresponding motive M_f ([Sch90]). But it seems very hard to impose conditions on ρ to ensure that the form f has coefficients in \mathbf{Q} . For example, in weight 2 (on the modular form side) one would want to put conditions on ρ to ensure that it actually come from an elliptic curve rather than

an abelian variety of GL_2 -type. In practice, we actually work in highly regular weight, and there is a certain amount of numerical evidence [Rob17] pointing to the fact that there may not exist any motives M at all with coefficients in \mathbf{Q} and monodromy group GL_2 with Hodge–Tate weights $[0, k - 1]$ when $k > 50$.

2. THE MOD p REPRESENTATION

In this section, we construct the mod p representations that we will lift in the next section using potential automorphy theorems.

Definition 2.1. Let G be the split simply-connected reductive group scheme over \mathbf{Z} of type E_6 . Fix a pinned based root datum of G , and let ${}^L G = G \rtimes \mathrm{Out}(G)$ with the non-trivial element τ of $\mathrm{Out}(G) = \mathbf{Z}/2\mathbf{Z}$ acting through the corresponding pinned automorphism. The induced action of τ on W_G is conjugation by the longest element w_0 , since τ acts on T , the maximal torus of the pinning, by the opposition involution $-w_0$. Let $W = W_G \rtimes \mathrm{Out}(G)$, and continue to write τ for the non-trivial element of $\mathrm{Out}(G) \subset W$. We will write B for the Borel subgroup of G associated to the based root datum.

Remark 2.2. *There is an isomorphism $W \simeq W_G \times \mathbf{Z}/2\mathbf{Z}$ given by the identity on W_G and sending τ to $(w_0, 1)$.*

We fix, once and for all, a choice of minuscule representation $r_{\min}: G \rightarrow \mathrm{GL}_{27}$, writing Λ_{\min} for the weights of T in r_{\min} . Now we explain how to extend r_{\min} to ${}^L G$. Let $\mathcal{G}_{27} = (\mathrm{GL}_{27} \times \mathrm{GL}_1) \rtimes \mathbf{Z}/2\mathbf{Z}$, where the non-trivial element $j \in \mathbf{Z}/2\mathbf{Z}$ acts via $j(g, \mu)j^{-1} = (\mu \cdot {}^t g^{-1}, \mu)$. The representation $g \mapsto r_{\min}(\tau g \tau^{-1})$ is isomorphic to the dual minuscule representation of G , so there exists $A \in \mathrm{GL}_{27}$ such that $r_{\min}(\tau g \tau^{-1}) = A \cdot {}^t r_{\min}(g)^{-1} A^{-1}$. Iterating, we see that $A \cdot {}^t A^{-1}$ commutes with r_{\min} , so must be a scalar: $A = {}^t A \cdot \varepsilon$. Clearly $\varepsilon \in \{\pm 1\}$, and since 27 is odd, we must in fact have $\varepsilon = 1$ by considering determinants.

We can now extend r_{\min} to

$$r_{\min}: {}^L G \rightarrow \mathcal{G}_{27}$$

by $r_{\min}(\tau) = (A, \varepsilon, j) = (A, 1, j)$. The fact that $\varepsilon = 1$ has the following consequence, which we will need later:

Lemma 2.3. Let $\nu: \mathcal{G}_{27} \rightarrow \mathbf{G}_m$ be the character given by $\nu(g, a, 0) = a$, $\nu(j) = -1$. Let $x \in {}^L G$ be any element with non-trivial projection to $\mathrm{Out}(G)$ (the case of interest will be x such that conjugation by x induces a split Cartan involution of G). Then $\nu \circ r_{\min}(x) = -1$.

Lemma 2.4. Let E^+/F^+ be a Galois extension of totally real fields whose Galois group is identified with a subgroup P of W_G , and let F/F^+ be a quadratic totally imaginary extension. Let $E = F \cdot E^+$. Then the composite

$$\mathrm{Gal}(E/F^+) \xrightarrow{\sim} \mathrm{Gal}(E^+/F^+) \times \mathrm{Gal}(F/F^+) \xrightarrow{\sim} P \times \mathbf{Z}/2\mathbf{Z} \subset W_G \times \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} W$$

sends complex conjugation to $(w_0, \tau) \in W = W_G \rtimes \mathrm{Out}(G)$. In this setting we will write ${}^L P$ for the image of $P \times \mathbf{Z}/2\mathbf{Z}$ in W .

Proof. This follows from the definition of the isomorphism $W_G \times \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} W$ (Remark 2.2). □

We now construct the mod p Galois representation. Let P be a Sylow 3-subgroup of W_G ; we note that $|W_G| = 2^7 \cdot 3^4 \cdot 5$. Let F/F^+ denote a fixed totally imaginary quadratic extension of our fixed totally real field F^+ . Our starting point will be to construct P as a Galois group over F^+ , with some local restrictions whose significance will become apparent when we apply potential automorphy theorems.

Lemma 2.5. There exists a totally real Galois extension E^+/F^+ together with an equality $\text{Gal}(E^+/F^+) = P$ (we write “=” to denote a fixed isomorphism) and all the primes of F^+ which ramify in E^+ are split in F/F^+ .

Proof. The Scholz–Reichardt theorem ([Ser08, Thm 2.1.1]) guarantees the existence of a number field L with $\text{Gal}(L/\mathbf{Q}) = P$, and, because $|P|$ is odd, such an extension will automatically be totally real. It suffices to show that we can construct such an extension ramified only at primes which are totally split in F (equivalently, in the Galois closure of F). This forces the intersection of L with the Galois closure of F to be unramified everywhere over \mathbf{Q} and hence trivial, and thus $E^+ = L.F^+$ will have Galois group P over F and produce the desired extension. The result follows immediately by induction and from the following lemma, which is extremely close to [Ser08, Thm 2.1.3]:

Sublemma 2.6. Let $\tilde{A} \rightarrow A$ be a central extension of a finite group A by $\mathbf{Z}/p\mathbf{Z}$, and assume that the exponent of \tilde{A} divides p^n . Let L/\mathbf{Q} be Galois with Galois group A , and assume that every prime l which ramifies in L has the following properties:

- (1) $l \equiv 1 \pmod{p^n}$.
- (2) The inertia group(s) of l in A coincides with the decomposition group(s) at l .
- (3) l splits completely in F .

Then there exists an extension \tilde{L}/L which is Galois over \mathbf{Q} with Galois group \tilde{A} and such that the primes l which ramify in \tilde{L} satisfy the same conditions as above.

The only difference between this statement and Theorem 2.1.3 of [Ser08] is the extra requirement that the primes l splits completely in F . There are two inductive steps in the proof of Theorem 2.1.3 of [Ser08], and we indicate the required argument to show that the new auxiliary prime q may be chosen to split completely in F .

Suppose first that $\tilde{A} = A \times \mathbf{Z}/p\mathbf{Z}$ is a split extension. Pick any prime $q \equiv 1 \pmod{p^n}$ which is totally split in the Galois closure of F and also totally split in the field $L(\zeta_{p^n}, \{l^{1/p}\}_{l \in \text{Ram}(L/\mathbf{Q})})$ given by adjoining the p th roots of primes l which ramify in L . Then take \tilde{L} to be the composite of L and the sub-extension of $\text{Gal}(\mathbf{Q}(\zeta_q)/\mathbf{Q})$ with Galois group $\mathbf{Z}/p\mathbf{Z}$.

Now suppose that \tilde{A} is a non-split extension. The argument in [Ser08] proceeds by first finding an extension \tilde{L} and then modifying \tilde{L} so that it is ramified at the same places as L . Hence the ramified primes l automatically satisfy the required splitting condition in F . The final step is to modify the field further so that it has property (2). This is achieved by choosing an auxiliary prime $q \equiv 1 \pmod{p}$ along with a character $\chi : (\mathbf{Z}/q\mathbf{Z})^\times \rightarrow \mathbf{Z}/p\mathbf{Z}$ satisfying the following properties:

- (1) The prime $q \equiv 1 \pmod{p^n}$.
- (2) For every prime l which ramifies in L , there is an equality $\chi(l) = c_l$ where c_l is determined from the extension L .

- (3) The prime q splits completely in L .
- (4) The prime q splits completely in F .

Only the last condition is new. The first three conditions are Čebotarev conditions in the field $L(\zeta_{p^n}, \{l^{1/p}\}_{l \in \text{Ram}(L/\mathbf{Q})})$, whereas the fourth condition is a Čebotarev condition in the Galois closure of F . By construction, the primes l are totally split in the Galois closure of F . Hence the intersection of the Galois closure of F with $L(\zeta_{p^n}, \{l^{1/p}\}_l)$ must be contained inside $\mathbf{Q}(\zeta_{p^n})$. Since the first condition implies that q splits completely in $\mathbf{Q}(\zeta_{p^n})$, there is no obstruction to finding such primes q satisfying all four conditions using the Čebotarev density theorem provided there is no obstruction without the last hypothesis. But this is exactly what follows from the proof of Theorem 2.1.3 of [Ser08]. \square

As in Lemma 2.4, let ${}^L P$ be the image of $P \times \mathbf{Z}/2\mathbf{Z}$ in W ; explicitly, it is the product $P \times \langle (w_0, \tau) \rangle$ inside W . Our reason for working with the group P is the combination of the following two properties:

Lemma 2.7.

- (1) The restriction of the extension

$$1 \rightarrow T(\mathbf{Z}) \rightarrow N_G(T)(\mathbf{Z}) \rtimes \text{Out}(G) \xrightarrow{\pi} W_G \rtimes \text{Out}(G) \rightarrow 1$$

to ${}^L P$ splits.

- (2) P acts transitively on the set Λ_{\min} of weights of the minuscule representation r_{\min} .

Proof. Since P is a 3-group while $T(\mathbf{Z})$ is an \mathbf{F}_2 -module, the Hochschild–Serre spectral sequence

$$H^i(P, H^j(\langle (w_0, \tau) \rangle, T(\mathbf{Z}))) \implies H^{i+j}({}^L P, T(\mathbf{Z}))$$

degenerates at the E_2 -page, and restriction induces an isomorphism

$$H^2({}^L P, T(\mathbf{Z})) \xrightarrow{\sim} H^2(\langle (w_0, \tau) \rangle, T(\mathbf{Z}))^P.$$

The image of our extension under this restriction isomorphism is trivial, because the element (w_0, τ) lifts to an order two element of $N(T)(\mathbf{Z}) \rtimes \text{Out}(G)$ (see for instance [AH16, Lemma 3.1], noting that Z_G has order 3).

For the second part of the lemma, note that W_G acts transitively on Λ_{\min} , so the 3-part of the stabilizer of any element $\lambda \in \Lambda_{\min}$ has order 3 (recall $|W_G| = 2^7 \cdot 3^4 \cdot 5$ and $|\Lambda_{\min}| = 3^3$). It follows that the orbit of P on λ has order at least $|P|/3 = 3^3$, and thus (equality holds and) P acts transitively. (More generally, a finite group G acts transitively on a set X of p -power order if and only if a p -Sylow subgroup P acts transitively on X .) \square

The work of Shafarevich on the inverse Galois problem for solvable groups implies that every split embedding problem with nilpotent kernel has a proper solution; we need some precise local control so will not be able to invoke this theorem, and unfortunately the following construction, guided by the demands of the local deformation theory as in [Pat16] and of automorphy lifting as in [BLGGT14], is somewhat technical.

Applying Lemma 2.7, let us fix a splitting of the extension

$$1 \rightarrow T(\mathbf{Z}) \rightarrow \pi^{-1}({}^L P) \xrightarrow{\pi} {}^L P \rightarrow 1$$

and write $s: \text{Gal}(E/F^+) \rightarrow \pi^{-1}({}^L P)$ for the resulting lift. This representation is not yet suitable for potential automorphy theorems, so we modify it in the following Proposition.

We first establish some notation. For a fixed prime p , we write ϵ for the p -adic cyclotomic character and $\bar{\epsilon}$ for its mod p reduction. For a prime l exactly dividing $p - 1$, let $\text{pr}(l)$ be the projection from \mathbf{F}_p^\times onto the l -torsion subgroup $\mathbf{F}_p^\times[l]$ restricting to the identity on $\mathbf{F}_p^\times[l]$. Finally, let $\bar{\epsilon}[l] = \text{pr}(l) \circ \bar{\epsilon}: G_{F^+} \rightarrow \mathbf{F}_p^\times[l]$. For any homomorphism $\bar{\rho}: G_{F^+} \rightarrow G(\mathbf{F}_p)$ and any place v of F^+ , let $\bar{\rho}_v := \bar{\rho}|_{G_{F_v^+}}$. Finally, for any homomorphism of groups $\rho: \Gamma \rightarrow \Gamma'$, and any $\rho(\Gamma)$ -module M , let $\rho(M)$ denote M regarded as a Γ -module.

Proposition 2.8. Consider pairs of primes (l, p) such that:

- All primes above l split in F/F^+ .
- p splits in E/\mathbf{Q} and $p - 1$ is divisible by l but not by l^2 .

Then there exist infinitely many primes l such that there exist infinitely many pairs (l, p) such that there exists a homomorphism

$$\bar{\rho}: G_{F^+} \rightarrow T(\mathbf{F}_p)[l] \cdot \pi^{-1}({}^L P) \subset N_G(T)(\mathbf{F}_p) \rtimes \text{Out}(G)$$

lifting our fixed identification $\text{Gal}(E/F^+) = {}^L P$ and satisfying the following:

- (1) The restriction $\bar{\rho}|_{G_F}$ is ramified only at places split in F/F^+ .
- (2) For any choice of complex conjugation $c \in G_{F^+}$, $\bar{\rho}(c)$ is a split Cartan involution of G , i.e. $\dim(\mathfrak{g}^{\text{Ad}(\bar{\rho}(c))=1}) = \dim(G/B) = 36$.
- (3) For all places $v|p$ of F^+ , fix any choice of integers $n_{v,\alpha}$ indexed by simple roots $\alpha \in \Delta = \Delta(G, B, T)$. Then $\bar{\rho}|_{G_{F_v^+}}$ is equal to

$$\prod_{\alpha \in \Delta} \alpha^\vee \circ \bar{\epsilon}[l]^{n_{v,\alpha}}.$$

- (4) Let ρ_G^\vee denote the half-sum of the positive coroots of (G, B, T) ; it lies in $X_\bullet(T)$ since $\#Z_G = 3$. There is a set S_{reg} of two primes q split in E/\mathbf{Q} and of order l modulo p such that for some place $v|q$ of F^+ , $\bar{\rho}|_{G_{F_v^+}}$ is unramified with Frobenius mapping to $\rho_G^\vee(q)$ (which lands inside the group $T(\mathbf{F}_p)[l]$ because $q^l \equiv 1 \pmod{p}$).

Moreover, in addition to satisfying the above conditions, we can choose $p > 56 = 2 \cdot (27 + 1)$, $l > h_G = 12$ (the Coxeter number of G), $\{n_{v,\alpha}\}$, and $\bar{\rho}$ such that

- (1) For all $v|p$, the composite $r_{\min} \circ \bar{\rho}|_{G_{F_v^+}}$ is a direct sum of distinct powers of $\bar{\epsilon}[l]$.
- (2) For all $v|p$ and for any Borel subgroup B (with Lie algebra \mathfrak{b}) containing T , the cohomology groups $H^0(G_{F_v^+}, \bar{\rho}_v(\mathfrak{g}/\mathfrak{b}))$ and $H^0(G_{F_v^+}, \bar{\rho}_v(\mathfrak{g}/\mathfrak{b})(1))$ both vanish. Moreover, for any Borel subgroup B_{27} containing the maximal torus of SL_{27} that stabilizes the weight spaces of r_{\min} , $H^0(G_{F_v^+}, (r_{\min} \circ \bar{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})) = 0$ and $H^0(G_{F_v^+}, (r_{\min} \circ \bar{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})(1)) = 0$.
- (3) The composite $r_{\min} \circ \bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.

Proof. For now let l be any odd prime such that the places of F^+ above l are all split in F/F^+ , and E and $\mathbf{Q}(\mu_{l^2})$ are linearly disjoint over \mathbf{Q} (eg, take l split in E/\mathbf{Q}); later in the argument we will require l to be larger than some absolute bound depending only on the group G . Let p be any prime split in E/\mathbf{Q} such that l , but not l^2 , divides $p - 1$ (such p exist by Čebotarev). We will modify the

original lift $s: \text{Gal}(E/F^+) \rightarrow \pi^{-1}({}^L P)$ by an element of $H^1(G_{F^+}, T(\mathbf{F}_p)[l])$ so as to satisfy the local conditions (here G_{F^+} continues to act via the quotient $\text{Gal}(E/F^+)$; note that $T(\mathbf{F}_p)[l]$ is τ -stable). Let $T(\mathbf{F}_p)[l] = \bigoplus W_i$ be the decomposition into irreducible $\mathbf{F}_l[P]$ -modules. Recall that $(w_0, c) \in {}^L P$ acts on T by -1 , so this is also a decomposition as ${}^L P$ -module, and the splitting field $F^+(W_i)$ of the $\text{Gal}(E/F^+)$ -module W_i contains F for each i (since $l \neq 2$). Let Σ be the set of places of F^+ that are either split or ramified in F/F^+ (implicitly including the infinite places in the former condition); in particular, the action of G_{F^+} on $T(\mathbf{F}_p)[l]$ factors through $G_{F^+, \Sigma}$. For any finite subset T of Σ , [NSW00, Theorem 9.2.3(v)] implies that the restriction map

$$H^1(G_{F^+, \Sigma}, W_i) \rightarrow \bigoplus_T H^1(G_{F_v^+}, W_i)$$

is surjective for all i . Assembling the different i (with a common set T), the restriction map

$$H^1(G_{F^+, \Sigma}, T(\mathbf{F}_p)[l]) \rightarrow \bigoplus_T H^1(G_{F_v^+}, T(\mathbf{F}_p)[l])$$

is also surjective. We apply this observation to the following set T and the following local cohomology classes: let T be the union of the following sets of places of F^+ :

- places which are ramified in F/F^+ ;
- places dividing p ;
- places above an auxiliary rational prime $q \in S_{\text{reg}}$ that is split in E/\mathbf{Q} and has order l modulo p ; (A positive density of such q exist since E is linearly disjoint from $\mathbf{Q}(\mu_p)$ over \mathbf{Q} , by comparing ramification at p .)
- an auxiliary place w lying above a rational prime r that splits completely in $E(\zeta_p)/\mathbf{Q}$.

Consider the following local classes in $H^1(G_{F_v^+}, T(\mathbf{F}_p)[l])$ for $v \in T$:

- trivial at places which ramify in F/F^+ ;
- the prescribed homomorphism $\prod_{\alpha \in \Delta} \alpha^\vee \circ \bar{\epsilon}[l]^{n_{v, \alpha}}$ for $v|p$,
- the unramified homomorphism $\text{Fr}_v \mapsto \rho_G^\vee(q)$ for $v|q$ and $q \in S_{\text{reg}}$, where ρ_G^\vee denotes the half-sum of the positive coroots of G , which is in fact a cocharacter of G (we will only need this construction for one of the places above q).
- the unramified homomorphism $\text{Fr}_w \mapsto t$, where t is any element of $T(\mathbf{F}_p)[l]$ such that the values $\lambda(t)$ for $\lambda \in \Lambda_{\min}$ are all distinct (for l sufficiently large, such t exist).

Let $\phi \in H^1(G_{F^+, \Sigma}, T(\mathbf{F}_p)[l])$ be a class with these local restrictions, and set $\bar{\rho} = \phi \cdot s$. We claim the conclusions of the proposition hold for this $\bar{\rho}$. The conditions at finite places are all evident from the construction, and the condition on complex conjugation is satisfied because any order two element $(\tilde{w}_0, \tau) \in N_G(T)(\mathbf{F}_p)$ lifting $(w_0, \tau) \in W_G \rtimes \text{Out}(G)$ gives a split Cartan involution of \mathfrak{g} : it acts by -1 on $\text{Lie}(T)$, and it sends a root space \mathfrak{g}_α to $\mathfrak{g}_{-\alpha}$, so we get precisely $\dim(G/B) = \dim(\mathfrak{g}^{\text{Ad}((\tilde{w}_0, \tau))=1})$.

For the second list of assertions, note that for $v|p$, $r_{\min} \circ \bar{\rho}|_{G_{F_v^+}}$ is equal to

$$\bigoplus_{\lambda \in \Lambda_{\min}} \bar{\epsilon}[l]^{\sum_{\alpha \in \Delta} n_{v, \alpha} \langle \lambda, \alpha^\vee \rangle},$$

where Λ_{\min} is the set of weights of r_{\min} . For varying λ , we want these exponents to be distinct modulo l . We simply choose the $(n_{v,\alpha})_\alpha$ so that the exponents are distinct in \mathbf{Z} , and then any l sufficiently large will do. To evaluate the cohomology groups appearing in the conclusion of the proposition, note that as a $G_{F_v^+}$ -module, $\bar{\rho}_v(\mathfrak{g}/\mathfrak{b})$ is a direct sum of characters of the form

$$\bar{\epsilon}[l]^{\sum_{\alpha \in \Delta} n_{v,\alpha} \langle \beta, \alpha^\vee \rangle},$$

where β is a negative root of G . Clearly the choice of l can be modified if necessary to ensure that these exponents (which don't depend on p) are all integers between 1 and $l - 1$, and so regardless of how p is chosen the group $H^0(G_{F_v^+}, \bar{\rho}_v(\mathfrak{g}/\mathfrak{b}))$ will vanish. The vanishing of $H^0(G_{F_v^+}, \bar{\rho}_v(\mathfrak{g}/\mathfrak{b})(1))$ is even more straightforward: the order of $\bar{\epsilon}$ is divisible by some prime other than l , whereas all powers of $\bar{\epsilon}[l]$ have order 1 or l . We can similarly deduce the vanishing of $H^0(G_{F_v^+}, (r_{\min} \circ \bar{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27}))$ and $H^0(G_{F_v^+}, (r_{\min} \circ \bar{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})(1))$, where \mathfrak{b}_{27} is the Lie algebra of any Borel subgroup $B_{27} \subset \mathrm{SL}_{27}$ containing the maximal torus of SL_{27} characterized by the property that it stabilizes each of the weight spaces of r_{\min} . Then $(r_{\min} \circ \bar{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})$ is a direct sum of characters

$$\bar{\epsilon}[l]^{\sum_{\alpha \in \Delta} n_{v,\alpha} \langle \lambda_1 - \lambda_2, \alpha^\vee \rangle},$$

for distinct weights $\lambda_1, \lambda_2 \in \Lambda_{\min}$. These exponents are by construction non-zero modulo l , so we win, and the same argument as for $\bar{\rho}_v(\mathfrak{g}/\mathfrak{b})(1)$ applies to show $H^0(G_{F_v^+}, (r_{\min} \circ \bar{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})(1))$ is zero as well.

The condition at the auxiliary prime w ensures absolute irreducibility of $r_{\min} \circ \bar{\rho}|_{G_{F(\zeta_p)}}$: P acts transitively on Λ_{\min} by Lemma 2.7(2), so any non-zero submodule of $r_{\min} \circ \bar{\rho}|_{G_{F(\zeta_p)}}$ has non-zero projection to each weight space (recall that $F(\zeta_p)$ is linearly disjoint from E over F); but for a place $w'|w$ of $F(\zeta_p)$, the image of $G_{F(\zeta_p)w'}$ in $T(\mathbf{F}_p)$ acts via distinct characters on the different weight spaces of r_{\min} . \square

3. LIFTING GALOIS REPRESENTATIONS

Let $\bar{\rho}: G_{F^+} \rightarrow N_G(T)(\mathbf{F}_p) \rtimes \mathrm{Out}(G) \subset {}^L G(\mathbf{F}_p)$ be a homomorphism constructed as in Proposition 2.8. We would like to lift $\bar{\rho}$ to a homomorphism

$$\rho: G_{F^+} \rightarrow {}^L G(\bar{\mathbf{Z}}_p)$$

that belongs to a compatible system of representations, all having Zariski-dense image and appearing in the cohomology of an algebraic variety. To achieve Zariski-dense monodromy for a lift ρ , we follow the approach of [Pat16]: ensuring local Steinberg-type ramification at one auxiliary prime and sufficiently general Hodge–Tate cocharacter suffices. To produce the lift, and to put it in a compatible system, we compare deformation rings for $\bar{\rho}$ and $r_{\min} \circ \bar{\rho}: G_{F^+} \rightarrow \mathcal{G}_{27}(\mathbf{F}_p)$. We control a suitable deformation ring for $r_{\min} \circ \bar{\rho}$ using the method of Khare–Wintenberger and the automorphy lifting results of [BLGTT14].

Let S be a finite set of primes of F^+ which split in F containing all those where $\bar{\rho}$ is ramified (by the construction of $\bar{\rho}$, the primes of F^+ at which $\bar{\rho}$ is ramified split in F). We will enlarge the set S as necessary. Let F_S be the maximal extension (in \bar{F}^+) of F unramified outside (places above) S , and set $G_S = \mathrm{Gal}(F_S/F^+)$. We will be deforming G_S -representations. For each $v \in S$, fix an extension \bar{v} of v to F , and fix a member of the $G_{F,S}$ -conjugacy class of homomorphisms $G_{F_{\bar{v}}} \rightarrow G_{F,S}$. Via the inclusion $G_{F,S} \subset G_S$, these choices specify what we mean by restricting $\bar{\rho}$ (or its lifts) to $G_{F_{\bar{v}}}$. We do not review in detail the mechanics of the deformation

theory in this setting, but instead refer the reader to [Pat16, §9.2] (for ${}^L G$) and [CHT08, §2] (for \mathcal{G}_{27}). We now specify local deformation conditions to define two global deformation functors, one for $\bar{\rho}$ and one for $r_{\min} \circ \bar{\rho}$.

For $v \in S$, define the following local deformation conditions \mathcal{P}_v on lifts of $\bar{\rho}|_{G_{F_{\bar{v}}}}$:

- For v above the auxiliary primes $q \in S_{\text{reg}}$ (see Proposition 2.8), we impose the Steinberg deformation condition as in [Pat16, §4.3], with respect to the Borel subgroup B of G specified by our based root datum. Note that by construction the order of $\bar{c}: G_{F_{\bar{v}}} \rightarrow \mathbf{F}_p^\times$ is greater than $h_G - 1$ (since $l > h_G$ is the order of q modulo p).
- For $v|p$, we take an ordinary deformation condition as in [Pat16, §4.1]. To be precise, fix the following lift χ_T of $\bar{\rho}|_{I_{F_{\bar{v}}}}$ to $T(\mathbf{Z}_p)$:

$$(1) \quad \chi_T = \prod_{\alpha \in \Delta} \alpha^\vee \circ (\epsilon^{\tilde{n}_{v,\alpha}} \cdot \chi^{-n_{v,\alpha}}),$$

where the $\tilde{n}_{v,\alpha}$ are sufficiently general positive (positive ensures, as in [Pat16, Lemma 4.8], that our characteristic zero lifts are de Rham) integers congruent to $n_{v,\alpha}$ modulo $l - 1$, and χ is the Teichmüller lift of $\bar{c} \cdot \bar{c}[l]^{-1}$.

- For all other primes $v \in S$, the inertial image $\bar{\rho}(I_{F_{\bar{v}}})$ has order prime to p (indeed, $\bar{\rho}$ lands in the prime-to- p group $N_G(T)(\mathbf{F}_p) \rtimes \text{Out}(G)$), and we take the minimal deformation condition of [Pat16, §4.4].

Lemma 3.1. For all $v \in S$, let \mathcal{P}_v be the local condition just defined. Let $\mathcal{P} = \{\mathcal{P}_v\}_{v \in S}$, and let $\text{Lift}_p^{\mathcal{P}}$ be the associated global lifting functor associated to this collection of local conditions (see [Pat16, §9.2]).

- (1) For all $v \in S$ not above p , the local lifting ring associated to the condition \mathcal{P}_v just defined is formally smooth, and the associated deformation functor has tangent space L_v of dimension $\dim H^0(G_{F_{\bar{v}}}, \bar{\rho}(\mathfrak{g}))$. For $v|p$, the same holds, except the tangent space has dimension $\dim H^0(G_{F_{\bar{v}}}, \bar{\rho}(\mathfrak{g})) + \dim(G/B)$.
- (2) The associated deformation functor $\text{Def}_p^{\mathcal{P}}$ is representable. Let $R_p^{\mathcal{P}}$ be the representing object. For some integer δ , $R_p^{\mathcal{P}}$ has a presentation as the quotient of a power series ring over \mathbf{Z}_p in δ variables by an ideal generated by (at most) δ relations.

Proof. The local claims follow from [Pat16, §4.1, 4.3, 4.4]. The global claims follow from [Pat16, Proposition 9.2]), using that:

- the centralizer of $\bar{\rho}$ in \mathfrak{g} is trivial;
- for all complex conjugations c , $\bar{\rho}(c)$ is a split Cartan involution of G ;
- the local lifting rings have dimensions as computed in the first part of the lemma.

(We remark that the integer δ is the common dimension of the Selmer and dual Selmer groups associated to the global deformation functor.) \square

Next we define an analogous deformation ring for $r_{\min} \circ \bar{\rho}: G_S \rightarrow \mathcal{G}_{27}(\mathbf{F}_p)$. Recall the character $\nu: \mathcal{G}_{27} \rightarrow \mathbf{G}_m$. The composition $\nu \circ (r_{\min} \circ \bar{\rho})$ is the non-trivial character of $\text{Gal}(F/F^+)$, and we fix $\mu: G_{F^+} \rightarrow \mathbf{Z}_p^\times$ equal to its Teichmüller lift (we will consider lifts with this fixed character). To define a global deformation problem in the sense of [BLGGT14, §1.5] (see [CHT08] for more details), we must, for each $v \in S$, choose an irreducible component \mathcal{C}_v of the (generic fiber) lifting ring (in the case v not above p) $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^\square[1/p]$ or (in the case $v|p$) $\lim_K R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}, \{H\}, K\text{-ss}}}^\square[1/p]$, where here

we follow the notation of [BLGGT14]: H is a collection (indexed by embeddings $F_{\bar{v}} \rightarrow \mathbf{Q}_p$, but for us $F_{\bar{v}} = \mathbf{Q}_p$) of multi-sets of Hodge numbers, K varies over finite extensions of $F_{\bar{v}}$, and the lifting ring in question is the one constructed by Kisin ([Kis08]), whose characteristic zero points parametrize potentially semistable deformations, semistable over K , with the prescribed Hodge numbers; to be precise, it is the maximal reduced p -torsion-free quotient of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}$ whose $\overline{\mathbf{Q}}_p$ -points satisfy these properties (see [BLGGT14, §1.4] for an overview). In both cases $v|p$ and $v \nmid p$, we then associate the lifting ring $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{C}_v}$ given by the maximal reduced p -torsion-free quotient of $(v \nmid p)$ $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}$ or $(v|p)$ $\lim_K R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}, \{H\}, K\text{-ss}}}^{\square}$ that is, after inverting p , supported on the component \mathcal{C}_v . Namely, we take:

- For v above the auxiliary primes $q \in S_{\text{reg}}$, recall that $\bar{\rho}|_{G_{F_{\bar{v}}}}$ is unramified with (arithmetic) Frobenius Fr_v mapping to $\rho_G^{\vee}(q)$. Let $\varphi: \text{PGL}_2 \rightarrow G$ be the principal homomorphism associated to our fixed pinning of G , so $\bar{\rho}(\text{Fr}_v)$ equals $\varphi(\text{diag}(q, 1))$. The composite $r_{\min} \circ \varphi$ decomposes as $S^{16} \oplus S^8 \oplus S^0$ (see [Gro00, §7]), where we write S^i for the i^{th} symmetric power of the standard representation of SL_2 . Consider the lift $\rho_{\bar{v}}$ of $r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}$ given by $r_{16} \oplus r_8 \oplus r_0$ where r_i is the tame unipotent representation on the S^i component given by the matrices

$$(r_i(\text{Fr}_v))_{a,b} = \begin{cases} q^{i-2a+2} & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

and on a topological generator τ_v of tame inertia by

$$(r_i(\tau_v))_{a,b} = \begin{cases} p & \text{if } b = a + 1, \\ 0 & \text{if } b \neq a + 1. \end{cases}$$

A quick calculation shows that, for all finite extensions $K/F_{\bar{v}}$, we have an equality $H^0(G_K, \text{ad}(\rho_{\bar{v}})(1)) = 0$, so $\rho_{\bar{v}}$ is a robustly smooth point of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}[1/p]$, in the sense of [BLGGT14, §1.3]. In particular, $\rho_{\bar{v}}$ lies on a unique irreducible component of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}[1/p]$, and we take \mathcal{C}_v to be this component.

- For $v|p$, let the set H of Hodge–Tate weights be

$$\{h_{\lambda} = \sum_{\alpha \in \Delta} \tilde{n}_{v,\alpha} \langle \lambda, \alpha^{\vee} \rangle\}_{\lambda \in \Lambda_{\min}},$$

where Λ_{\min} is the set of weights of r_{\min} as before. Borel subgroups of SL_{27} containing the maximal torus T_{27} (the unique torus stabilizing the weight spaces in r_{\min}) are in bijection with orderings of the set Λ_{\min} ; let B_{27} be the Borel defined by the ordering $\lambda > \lambda' \iff h_{\lambda} > h_{\lambda'}$. The ordinary deformation ring (again following the notation of [Pat16, §4.1]) associated to the Borel B_{27} and the lift $\chi_{T_{27}}: I_{F_{\bar{v}}} \rightarrow T_{27}(\mathbf{Z}_p)$ of $r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}$ given by

$$\chi_{T_{27}} = r_{\min} \left(\prod_{\alpha \in \Delta} \alpha^{\vee} (\epsilon^{\tilde{n}_{v,\alpha}} \chi^{-n_{v,\alpha}}) \right)$$

is formally smooth and receives by the universal property a canonical surjection from $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}, H, K\text{-ss}}}^{\square}$ for $K = F_{\bar{v}}(\chi)$ (recall that χ is the Teichmüller

lift of $\bar{\epsilon} \cdot (\bar{\epsilon}[l])^{-1}$, and after inverting p it induces an isomorphism from a unique irreducible component of the source (the equality of dimensions follows from [Kis08] and [Pat16, §4.1]). We take \mathcal{C}_v to be this component.

- For all other $v \in S$, we take the irreducible component of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}$ parametrizing minimal deformations in the sense of [Pat16, §4.4]; using the standard argument, this is a power series ring and induces a unique irreducible component \mathcal{C}_v of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}[1/p]$.

In each case, we write $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{C}_v}$ for the associated local lifting ring. Recall ([BLGGT14, §1.3]) that for v not above p , $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{C}_v}$ is the maximal quotient of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}$ that is reduced, p -torsion-free, and after inverting p is supported on the component \mathcal{C}_v . For $v|p$, $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{C}_v}$ is in general constructed similarly, but for us it is simply the formally smooth ordinary deformation ring produced by [Pat16, §4.1].

Lemma 3.2. For all $v \in S$, the representation r_{\min} induces a map $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{C}_v} \rightarrow R_{\bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{P}_v}$

Proof. For $v|p$, this follows directly from the definitions once we check that $r_{\min}(B) \subset B_{27}$. Let $\lambda = \sum_{\alpha \in \Delta} \tilde{n}_{v,\alpha} \alpha$. Since all $\tilde{n}_{v,\alpha}$ are positive, B is the locus of $g \in G$ where $\lim_{t \rightarrow 0} \text{Ad}(\lambda(t))g$ exists. By construction, B_{27} is the locus where $\lim_{t \rightarrow 0} \text{Ad}(r_{\min} \circ \lambda(t))g$ exists. The claim follows.

For $v|q$, we write $R_{\bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{P}_v}$ for the Steinberg lifting ring; recall that under our hypotheses it is a power series ring over \mathbf{Z}_p (in $\dim(\mathfrak{g})$ variables). We have surjections

$$R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square} \rightarrow R_{\bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square} \rightarrow R_{\bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{P}_v}.$$

Recall that $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{C}_v}$ is the maximal reduced p -torsion-free quotient of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}$ whose $\overline{\mathbf{Q}}_p$ -points lie on \mathcal{C}_v . Since $R_{\bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{P}_v}$ is a reduced, p -torsion-free quotient of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}$, it suffices to show that every $\overline{\mathbf{Q}}_p$ -point of $R_{\bar{\rho}|_{G_{F_{\bar{v}}}}}^{\mathcal{P}_v}$ lies on the same irreducible component of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}[1/p]$ as the representation $\rho_{\bar{v}}$ (defined above) that characterizes the component \mathcal{C}_v . This claim follows from [BLGGT14, Lemma 1.3.5].

For the other ramified primes $v \in S$, the lemma is evident. \square

Now consider the global deformation problem (in the sense of [BLGGT14, §1.5]; see [CHT08] for details)

$$\mathcal{S} = (F/F^+, S, \{\tilde{v}\}_{v \in S}, \mathbf{Z}_p, r_{\min} \circ \bar{\rho}, \mu, \{\mathcal{C}_v\}_{v \in S}).$$

Since $r_{\min} \circ \bar{\rho}|_{G_F}$ is absolutely irreducible, this deformation functor is pro-represented by some $R_{r_{\min} \circ \bar{\rho}}^{\mathcal{S}}$.

Lemma 3.3. The representation r_{\min} induces a surjection $R_{r_{\min} \circ \bar{\rho}}^{\mathcal{S}} \rightarrow R_{\bar{\rho}}^{\mathcal{P}}$.

Proof. There is an induced map $R_{r_{\min} \circ \bar{\rho}}^{\mathcal{S}} \rightarrow R_{\bar{\rho}}^{\mathcal{P}}$ by Lemma 3.2. It is a surjection because the G_{F^+} -module $\bar{\rho}(\mathfrak{g})$ is a direct summand of $\text{ad}(r_{\min} \circ \bar{\rho})$ (indeed, these representations factor through representations of a finite prime-to- p group), so the

associated map on tangent spaces is injective; dually, the map on co-tangent spaces is surjective, and we conclude by Nakayama's lemma. \square

Finally, we can invoke the main results of [BLGGT14] to deduce that $R_{\bar{\rho}}^{\mathcal{P}}$ has a $\overline{\mathbf{Q}}_p$ -point ρ such that $r_{\min} \circ \rho$ is potentially automorphic:

Theorem 3.4. *For sufficiently general choice of lifts $\tilde{n}_{v,\alpha}$ as in Equation 1, the representation $\bar{\rho}: G_{F^+} \rightarrow {}^L G(\mathbf{F}_p)$ constructed in Proposition 2.8 admits a geometric lift $\rho: G_{F^+} \rightarrow {}^L G(\overline{\mathbf{Z}}_p)$ such that:*

- (1) *The Zariski closure of the image of ρ is ${}^L G$.*
- (2) *The composite $r_{\min} \circ \rho|_{G_F}$ is potentially automorphic in the sense of [BLGGT14].*
- (3) *The composite $r_{\min} \circ \rho|_{G_F}$ belongs to a compatible system of l -adic representations: there exist a number field M and a strictly pure — in the sense of [BLGGT14, §5.1] — compatible system*

$$r_\lambda: G_F \rightarrow \mathrm{GL}_{27}(\overline{M}_\lambda)$$

indexed over all finite places λ of M . In particular, the restriction of r_λ to I_v for v above the auxiliary primes $q \in S_{\mathrm{reg}}$ is unipotent with Jordan blocks of size 1, 9, and 17 as long as v has residue characteristic different from λ .

Proof. By the proof of [BLGGT14, Theorem 4.3.1] (see especially the last paragraph), $R_{r_{\min} \circ \bar{\rho}}^{\mathcal{S}}$ is a finite \mathbf{Z}_p -module. Lemma 3.3 then implies that $R_{\bar{\rho}}^{\mathcal{P}}$ is a finite \mathbf{Z}_p -module. We have already seen in Lemma 3.1 that it has dimension at least one, so we conclude that $R_{\bar{\rho}}^{\mathcal{P}}(\overline{\mathbf{Z}}_p)$ is non-empty. Let ρ be an element of $R_{\bar{\rho}}^{\mathcal{P}}(\overline{\mathbf{Z}}_p)$. Then:

- By [BLGGT14, Theorem 4.5.1], the composite $r_{\min} \circ \rho|_{G_F}$ is potentially automorphic (in the sense of [BLGGT14]).
- The Zariski closure G_ρ of the image of $\rho|_{G_F}$ is G : by [Pat16, Lemma 7.8], it suffices to show
 - G_ρ is reductive;
 - G_ρ contains a regular unipotent element of G ; and
 - for some $v|p$, $\rho|_{G_{F_{\bar{v}}}}$ is B -ordinary, and, for all simple roots α , $\alpha \circ \rho|_{I_K} = \epsilon^{r_\alpha}$ for some finite extension $K/F_{\bar{v}}$ and for distinct integers r_α .

Reductivity is immediate since ρ is irreducible. The third condition follows by taking the integers $\{\tilde{n}_{v,\alpha}\}_{\alpha \in \Delta}$ in the definition of the local condition \mathcal{P}_v (see the discussion preceding Lemma 3.1) to be sufficiently general. Finally, to show that the image of ρ contains a regular unipotent element, we check that for $v|q$, the tame inertia in $\rho|_{G_{F_{\bar{v}}}}$ acts by a regular unipotent. This would follow from the corresponding claim that $r_{\min} \circ \rho(I_{F_{\bar{v}}})$ contains a unipotent element with Jordan blocks of dimension 17, 9, and 1. Let π be the automorphic representation of $\mathrm{GL}_{27}(\mathbf{A}_{F'})$, for a suitable finite extension F'/F , witnessing the potential automorphy of $r_{\min} \circ \rho$, and let v' be a place of F' above v . By local-global compatibility at $l \neq p$ (Proved in general by [Car12], but known for odd dimensional representations by previous work of [HT01, TY07, Shi11]), the (Frobenius semi-simple) Weil–Deligne representation associated to $r_{\min} \circ \rho|_{G_{F_{v'}}}$ is isomorphic to the image of $\pi_{v'}$ under the local Langlands correspondence. It follows (eg, using [BLGGT14, Lemma 1.3.2(1)]) that $r_{\min} \circ \rho|_{G_{F_{\bar{v}}}}$ lies on a unique irreducible component

of $R_{r_{\min} \circ \rho|_{G_{F_{\bar{v}}}}}^{\square}$. By construction, $r_{\min} \circ \rho|_{G_{F_{\bar{v}}}}$ and $r_{\min} \circ \rho_{\bar{v}}$ lie on the same irreducible component \mathcal{C}_v of $R_{r_{\min} \circ \bar{\rho}|_{G_{F_{\bar{v}}}}}^{\square}$, and since they both lie on a unique component, [BLGGT14, Lemma 1.3.4(2)] implies their inertial restrictions are isomorphic. The result follows.

The claim that $r_{\min} \circ \rho$ can be put in a compatible system, follows from [BLGGT14, Theorem 5.5.1], and the claim concerning the restriction to I_v for $v|q$ follows as above from local-global compatibility at $l \neq p$. \square

In the next section, we will show that in fact all members of the compatible system $\{r_{\lambda}\}_{\lambda}$ have algebraic monodromy group equal to G .

4. CONTROLLING THE IMAGE IN THE COMPATIBLE SYSTEM

Theorem 3.4 provides the existence of a compatible system $\{r_{\lambda}\}$ of G_F representations with the property that the geometric monodromy group at one prime is precisely E_6 . Our goal in this section is to use known properties of compatible systems ([LP92]) together with the additional properties our compatible system satisfies at the auxiliary primes S_{reg} to ensure that the monodromy group is E_6 at *all* primes λ .

Let M denote the coefficient field of our compatible system.

Lemma 4.1. The monodromy group G for each prime λ of M has the following properties:

- (1) The component group of G is trivial.
- (2) The rank of G is 6.
- (3) The formal character of the torus $\chi : T \rightarrow \text{GL}_{27}$ is the formal character of the torus of E_6 under the minuscule representation.
- (4) If $G = G^{\circ}$ acts irreducibly, then G is equal to E_6 .
- (5) There exists a unipotent element in the image with Jordan blocks of size 1, 9, and 17.

Proof. The first three properties involve quantities which are constant in a compatible system, c.f. Propositions 6.12 and 6.14 of [LP92]. The fourth claim follows from Theorem 5.6 of [LP92], noting that E_6 does not occur in the explicit list of groups which gives rise to any of the basic similarity relations of §5 of *ibid*. This is enough to deduce that the Lie algebra of the monodromy representation must be \mathfrak{e}_6 , from which it follows that G is E_6 (acting in the natural way). The nilpotent operator of the Weil–Deligne representation of r_{λ} at the auxiliary prime $v|q$ for $q \in S_{\text{reg}}$ decomposes (by Theorem 3.4 part 3) into Jordan blocks of size 1, 9, and 17, *assuming* that the residual characteristic of q is different from that of λ . Yet S_{reg} was chosen (for this purpose!) to consist of two primes, so this holds for at least one prime v . \square

We now show that these conditions are sufficient to imply — purely by representation theoretic methods — that G acts irreducibly, which will prove the claims in Theorem 1.1 concerning the monodromy groups of $\{r_{\lambda}\}$. In light of the existence of the unipotent element whose existence is guaranteed by Lemma 4.1 part 5, it suffices to show that G cannot act faithfully on a direct sum of representations of

dimension

$$27 = 26 + 1 = 18 + 9 = 17 + 10 = 17 + 9 + 1$$

unless $G = E_6$ and the representation is irreducible.

4.1. The Formal Character of \mathfrak{e}_6 . The root lattice Φ of E_6 consists of 72 roots; it may be given as $\Phi^+ \cup \Phi^-$, where the positive roots Φ^+ are given explicitly in \mathbf{R}^6 by the $2\binom{5}{2} = 20$ vectors $e_i \pm e_j$ for $2 \leq i < j \leq 6$, and the $2^4 = \binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$ vectors

$$\left(\frac{\sqrt{3}}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2} \right)$$

where there are an even number of minus signs. If $2\rho = \sum_{\alpha \in \Phi^+} \alpha$, then $\rho = (4\sqrt{3}/2, 4, 3, 2, 1, 0)$. The root lattice is not self-dual, but has discriminant 3. A weight μ corresponding to a choice of minuscule representation is given by

$$\mu = \frac{1}{3}(2\sqrt{3}, 0, 0, 0, 0, 0).$$

The 27 weights Σ of the corresponding minuscule representation may be obtained from μ from the orbit of the Weyl group; all 27 such weights may be obtained by applying at most 2 reflections in the roots of Φ to μ . We have the following: (cf. [Lur01]) Of the $\binom{27}{3} = 2925$ collections of 3 vectors in Σ , exactly 45 such triples generate a subspace of dimension 2, and they all consist of a triple of weights (μ, μ', μ'') with $\mu + \mu' + \mu'' = 0$. If Λ is the weight lattice, then Σ injects into $V = \Lambda/2\Lambda$, which acquires the structure of a quadratic space via the map $q(\mu) = \frac{1}{2}\langle \mu, \mu \rangle$ (note that Λ is an even lattice). The pairing $\langle x, y \rangle = q(x+y) - q(x) - q(y)$ is preserved by the Weyl group W_G , which may be identified with the corresponding orthogonal group. The lattice V also admits a Hermitian structure corresponding to q . With respect to this structure, the quadratic space V has Arf invariant 1, and the elements $\{\mu, \mu', \mu''\}$ above lie inside a maximal isotropic subspace $U \subset V$ of dimension 2. The stabilizer of U (and of a triple) is a subgroup of W_G of index 45, which correspondingly acts transitively on the set of 45 triples. The stabilizer is also isomorphic to the Weyl group of F_4 . An explicit example of a triple is given by

$$\begin{pmatrix} \mu \\ \mu' \\ \mu'' \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & +3 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & -3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where $\mu' = \sigma_\alpha \sigma_\beta \mu$, $\mu'' = \sigma_\gamma \sigma_\delta \mu$, and

$$\alpha = (\sqrt{3}/2, -1/2, -1/2, 1/2, 1/2, 1/2)$$

$$\beta = (\sqrt{3}/2, -1/2, 1/2, -1/2, -1/2, -1/2)$$

$$\gamma = (\sqrt{3}/2, 1/2, -1/2, -1/2, 1/2, 1/2)$$

$$\delta = (\sqrt{3}/2, 1/2, -1/2, 1/2, -1/2, -1/2).$$

We derive the following consequence:

Lemma 4.2. The restriction of G to any sub-representation of dimension ≥ 4 must factor through a quotient of rank at least 3, and the restriction of G to any sub-representation of dimension 26 must have rank 6.

Proof. Given four distinct weights of a sub-representation on which the action of G factors through a quotient of rank at most 2, any three of them must consist of a triple (μ, μ', μ'') which sum to zero, which cannot hold for more than one such triple. One can also prove this by a direct explicit computation. The second claim follows from the fact that the sum of all 27 weights in Σ is zero, and none of the weights in Σ is zero. \square

We now note the following:

Lemma 4.3. Suppose that \mathfrak{g} is a reductive Lie algebra with a faithful irreducible representation of dimension d for $d \in \{17, 26, 18\}$. Then $\mathfrak{g} = \mathfrak{h}$ or $\mathfrak{h} \oplus \mathfrak{t}$, where \mathfrak{h} is semi-simple and \mathfrak{t} is a rank one torus. Furthermore, assuming that \mathfrak{h} is simple when $d = 18$, then \mathfrak{h} is one of the following:

\mathfrak{h}	d	$\text{rank}(\mathfrak{h})$
\mathfrak{sl}_2	17	1
\mathfrak{so}_{17}	17	8
\mathfrak{sl}_{17}	17	16
\mathfrak{sl}_2	26	1
\mathfrak{f}_4	26	4
$\mathfrak{so}_{13} \times \mathfrak{sl}_2$	26	7
$\mathfrak{sl}_{13} \times \mathfrak{sl}_2$	26	13
\mathfrak{sp}_{26}	26	13
\mathfrak{so}_{26}	26	13
\mathfrak{sl}_{26}	26	25
\mathfrak{sl}_2	18	1
\mathfrak{sp}_{18}	18	9
\mathfrak{so}_{18}	18	9
\mathfrak{sl}_{18}	18	17

Proof. It suffices to classify all small (of dimension at most 27) representations of the simple Lie groups; these may be computed using the Weyl character formula. \square

Let us now return to the possible cases in which our 27 dimensional Galois representation is reducible, and consider the corresponding monodromy groups. Suppose there is a constituent of dimension 17. The rank must be bounded by 6. From Lemma 4.3, it follows that the Lie algebra of the monodromy group on this summand must be \mathfrak{sl}_2 or $\mathfrak{sl}_2 \times \mathfrak{t}$. But the rank of these algebras is at most 2, which violates Corollary 4.2. Suppose there is a constituent of dimension 26. Then by Corollary 4.2, the rank of the Lie algebra of this representation is exactly 6, and hence the rank of \mathfrak{h} with $\mathfrak{g} = \mathfrak{h}$ or $\mathfrak{h} \times \mathfrak{t}$ is 6 or 5. Since there are no such groups of this rank with irreducible representations of dimension 26 by Lemma 4.3, we once more derive a contradiction. Hence the only remaining possibility is that the 27 dimensional representation decomposes into two irreducible pieces of dimensions 9 and 18, corresponding to a decomposition of weights $\Sigma = \Sigma_9 \cup \Sigma_{18}$. On the other hand, we know that the 18 dimensional representation must have a unipotent element with Jordan blocks of size 1 + 17. The tensor product of two Jordan blocks of size m and n with $n \leq m$ decomposes into blocks of size $n + m - 1, n + m - 3, \dots, n - m + 1$. In particular, it must be the case that the 18-dimensional representation does not factor into a product of smaller dimensional representations, since otherwise there could not be a Jordan block of a unipotent element of

size as large as 17. Hence the monodromy group on this representation must have a simple Lie algebra (up to a torus). Again, we deduce from Lemma 4.3 and using rank considerations that $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sl}_2 \times \mathfrak{t}$, once more contradicting Corollary 4.2.

We conclude:

Corollary 4.4. Let $\{r_\lambda: G_F \rightarrow \mathrm{GL}_{27}(\overline{M}_\lambda)\}_\lambda$ be the compatible system produced in Theorem 3.4. Then for all λ , the Zariski closure of the image of r_λ is isomorphic to G .

5. E_6 MOTIVES OVER CM FIELDS

We conclude by showing that r_λ is a sub-representation of the cohomology of some smooth projective variety over F , thus completing the proof of Theorem 1.1. Note that, according to the Tate conjecture, we expect that such a compatible family should be cut out by correspondences over F , and thus arise from a motive M over F . We do not have any idea how to prove this. On the other hand, we do know that the compatible family r_λ becomes automorphic over a CM extension $H/F/\mathbf{Q}$, and (since we are in highly regular weight), using standard methods combined with the work of Shin [Shi11], one can associate a motive M over H whose associated p -adic Galois representations $\{r_\lambda|_{G_H}\}$ have monodromy group E_6 . More precisely, we should say that one *expects* to be able to associate such a motive where the correspondences cutting out M arise from Hecke operators. In practice, we take a shortcut and deduce from [Shi11] the weaker claim that the Galois representations over G_H (and thus over G_F by restriction of scalars) came from cohomology. We apologize for the omission and leave it as an exercise to the more responsible reader to write down the correct argument.

To set up all the required notation would be quite cumbersome, so we will simply use the notation of [Shi11], giving precise references to where the relevant terms are defined. We hope that a reader with a copy of [Shi11] at hand can easily follow this argument. Fix an isomorphism $\iota_l: \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$; it is implicit in all of the constructions of [Shi11]. By Theorem 3.4, there is a CM extension H/F and a cuspidal automorphic representation Π^0 of $\mathrm{GL}_{27}(\mathbf{A}_H)$ such that

- $(\Pi^0)^\vee \cong (\Pi^0)^c$.
- $R_l(\Pi^0) \cong r_\lambda|_{G_H}$, in the notation of [Shi11, Theorem 7.5].
- $[H^+ : \mathbf{Q}] \geq 2$, and H contains a quadratic imaginary field (we can simply enlarge an initial choice of H to ensure these conditions).

Set $n = 27$, for ease of reference to [Shi11]; we will recall the construction of $R_l(\Pi^0)$ and see as a result that after some further base-change that there is an explicit description of this Galois representation in the cohomology of a unitary similitude group Shimura variety. We begin with two reductions. Let E be an imaginary quadratic field not contained in H satisfying the four bulleted conditions in Step (II) of the proof of [Shi11, Theorem 7.5]. Replace H by HE and Π^0 by $\mathrm{BC}_{HE/H}(\Pi^0)$. Then *having made this replacement* the triple (E, H, Π^0) satisfies the six bulleted conditions at the beginning of Step (I) of the proof of [Shi11, Theorem 7.5]. Let H' be an imaginary quadratic extension of H^+ satisfying the three bulleted conditions (defining the set denoted $\mathcal{F}(H)$ — but note our H is Shin's F) in Step (I) of [Shi11, Theorem 7.5]. Then replace H by HH' and Π^0 by $\mathrm{BC}_{HH'/H}(\Pi^0)$. Again having made this replacement, Proposition 7.4 of [Shi11] now applies to the triple (E, H, Π^0) . There is a Hecke character ψ of $\mathbf{A}_E^\times/E^\times$ such that, setting $\Pi = \psi \otimes \Pi^0$

(an automorphic representation of the group $\mathbb{G}_n(\mathbf{A}) \cong \mathrm{GL}_1(\mathbf{A}_E) \times \mathrm{GL}_n(\mathbf{A}_H)$ of [Shi11, §3.1]), we have (in the notation of [Shi11, Corollary 6.8])

$$R_l(\Pi^0) := R'_l(\Pi) := \tilde{R}'_l(\Pi) \otimes \mathrm{rec}_{l, \iota}(\psi^c)|_{G_H},$$

where (see [Shi11, 5.5, 6.23], and note that the group G no longer denotes E_6 , but rather the unitary similitude group defined in [Shi11, §5.1!])

$$C_G \cdot \tilde{R}'_l(\Pi) = \sum_{\pi^\infty \in \mathcal{R}_l(\Pi)} R_{\xi, l}^{n-1}(\pi^\infty)^{\mathrm{ss}}.$$

In our case, the integer $C_G = \tau(G) \ker^1(\mathbf{Q}, G)$ (defined in [Shi11, Theorem 6.1]) is 2: this is explained in [Tay12, p. 411–412]. Moreover, $\tilde{R}'_l(\Pi)$ is irreducible (since the image of $r_\lambda|_{\Gamma_H}$ is Zariski-dense in E_6), and $\mathcal{R}_l(\Pi)$ in fact contains at least two elements: following [Tay12, p. 413], there will be two automorphic representations of $G(\mathbf{A}_\mathbf{Q})$, differing by a twist but having isomorphic base-changes to $\mathrm{GL}_n(\mathbf{A}_H)$, that contribute to $\mathcal{R}_l(\Pi)$ (these are denoted $\tilde{\pi}$ and $\tilde{\pi} \otimes (\delta_{A/\mathbf{Q}} \circ \nu)$ in [Tay12, p. 413]). We fix one such $\tilde{\pi}^\infty \in \mathcal{R}_l(\Pi)$. It follows that $\tilde{R}'_l(\Pi) \cong R_{\xi, l}^{n-1}(\tilde{\pi}^\infty)$ (no semisimplification necessary because these are irreducible representations).

Now recall the decomposition ([Shi11, 5.5])

$$H^{n-1}(\mathrm{Sh}, \mathcal{L}_\xi) = \bigoplus_{\pi^\infty} \pi^\infty \otimes R_{\xi, l}^{n-1}(\pi^\infty).$$

At some finite level U , we deduce that $\tilde{R}'_l(\Pi)$ is contained in $H^{n-1}(\mathrm{Sh}_U \times_H \overline{H}, \mathcal{L}_\xi)$ (it is even a direct summand cut out by Hecke operators). Finally, letting $\mathcal{A}_U \rightarrow \mathrm{Sh}_U$ denote the universal abelian scheme (arising from the PEL moduli problem), and letting $\mathcal{A}_U^{(m)}$ denote its m -fold fiber product over Sh_U (for any integer $m \geq 1$), then there are integers m_ξ and t_ξ (see [HT01, p. 98]) such that

$$H^{n-1}(\mathrm{Sh}_U \times_H \overline{H}; \mathcal{L}_\xi) \cong \varepsilon \cdot H^{n-1+m_\xi}(\mathcal{A}_U^{(m_\xi)} \times_H \overline{H}, \overline{\mathbf{Q}}_l)(t_\xi),$$

where ε is a suitable idempotent projector. We recall that our assumption $[H^+ : \mathbf{Q}] \geq 2$ implies that Sh_U , and therefore $\mathcal{A}_U^{(m_\xi)}$, are smooth projective varieties over H . Thus $r_\lambda|_{G_H} \otimes \mathrm{rec}_{l, \iota}(\psi^c)^{-1}|_{G_H}$ is a sub-representation of the cohomology of the smooth projective variety $\mathcal{A}_U^{(m_\xi)}$ over H . Possibly replacing H by a finite extension, we can find a product of CM abelian varieties A/H such that $\mathrm{rec}_{l, \iota}(\psi^c)|_{G_H}$ is a sub-representation of $H^i(A \times_H \overline{H}, \overline{\mathbf{Q}}_l)(j)$ for some integers i and j ([DMOS82, IV. Proposition D.1]). We conclude that $r_\lambda|_{G_H}$ is a sub-representation of $H^r(X \times_H \overline{H}, \overline{\mathbf{Q}}_l)(s)$ for some smooth projective variety X/H (namely, $X = A \times_H \mathcal{A}_U^{(m_\xi)}$). By Frobenius reciprocity and irreducibility of r_λ , r_λ (as G_F -representation) is a sub-representation of $\mathrm{Ind}_{G_H}^{G_F}(H^r(X \times_H \overline{H}, \overline{\mathbf{Q}}_l)(s))$. This induction is just the cohomology of X regarded as a variety over F (i.e., via $X \rightarrow \mathrm{Spec}(H) \rightarrow \mathrm{Spec}(F)$), so the proof of Theorem 1.1 is complete.

Remark 5.1. One may reasonably ask whether there exist E_6 motives (or strongly compatible systems) over \mathbf{Q} . We do not know the answer. There is, however, a technical obstruction for applying the methods of this paper. To use automorphic methods, the Hodge–Tate weights of $r_{\min} \circ \rho_\lambda$ must be distinct. However, there cannot exist such a compatible system over \mathbf{Q} (or any totally real field),

since, by Corollary 5.4.3 of [BLGGT14], this would imply that the trace of complex conjugation representation must be ± 1 , and there are no such involutions in $E_6(\mathbf{C}) \subset \mathrm{GL}_{27}(\mathbf{C})$.

6. COMPLEMENTS

We note the following application of Theorem 1.1 to the inverse Galois problem. Let us write $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ for the \mathbf{F}_l -points of the (simply connected) form of E_6 that we have been considering. The group $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ is the Schur cover of the simple Chevalley group of type E_6 (with Schur multiplier of order $(3, l - 1)$), which we denote below by $E_6(\mathbf{F}_l)$. The groups $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ and $E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ are known to occur as Galois groups over \mathbf{Q} for $p \equiv 4, 5, 6, 9, 16, 17 \pmod{19}$ (these are primes of order 9 in \mathbf{F}_{19}^\times) by [Mal88, Thm 2.3] as a consequence of the rigidity method. In contrast, we can prove that $E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ is a Galois group over \mathbf{Q} for a positive density of primes $\equiv 1 \pmod{19}$ (by taking $E = \mathbf{Q}(\zeta_{19})$ below).

Corollary 6.1. Let F be an imaginary quadratic field. Then, for a set S of primes l of positive density, there exists a number field L/\mathbf{Q} containing F with $\mathrm{Gal}(L/\mathbf{Q}) = {}^L G(\mathbf{F}_l) = E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ and $\mathrm{Gal}(L/F) = E_6^{\mathrm{sc}}(\mathbf{F}_l)$. Moreover, one may assume that all the primes in S split completely in any finite extension E/\mathbf{Q} .

Proof. We combine the previous result with Theorem 3.17 of [Lar95], to deduce that $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ is the Galois group of the kernel of ρ_λ over F for a relative density one set of primes l which split completely in the coefficient field M of the compatible system. In particular, the intersection of S with the set of primes split completely in E/\mathbf{Q} has positive density. Since the representation $\bar{r}_\lambda : G_F \rightarrow \mathrm{GL}_{27}(\mathbf{F}_l)$ is, by construction, conjugate self-dual, it extends to a representation of $G_{\mathbf{Q}}$ to $\mathcal{G}_{27}(\mathbf{F}_l)$ whose kernel therefore has Galois group ${}^L G(\mathbf{F}_l)$. \square

Remark 6.2. Note that Corollary 6.1 remains true if one replaces $E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ and $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ by $E_6(\mathbf{F}_l).2$ and $E_6(\mathbf{F}_l)$, for the obvious reason that the latter groups are quotients of the former groups.

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