Homework: Due 9-26
Math 513

If parts of a problem are specified, only those parts are assigned. If no parts are specified, you should do all the parts.

Problems from textbook: §3.4 - 1 (a), 1(c), 6 (a)-(b), 7 (a)-(b), 12 (a)-(c) §3.6 - 1 §3.7 - 4, 7

Question 1. §3.4- 12(c)

Proof. We want to construct a basis \{β₁, ..., βₙ\} for \(V\) such that \(S(β_j) = β_{j+1}\) for \(1 ≤ j < n\) and \(S(β_n) = 0\). Since \(S^{n-1} ≠ 0\), there exists a vector \(β_1\) such that \(S^{n-1}(β_1) ≠ 0\). Consider the collection \(Σ = \{β_1, S(β_1), S^2(β_1), ..., S^{n-1}(β_1)\}\). Set \(β_j = S^{j-1}(β_1)\) so \(Σ = \{β_1, ..., β_n\}\). Clearly, \(S(β_j) = β_{j+1}\) for \(1 ≤ j < n\) and \(S(β_n) = S^n(β_1) = 0\).

It suffices to show the \(Σ\) is a basis. Since \(V\) is \(n\)-dimensional, it suffices to show that the set is linearly independent. Consider a linear relation

\[\sum_{i=1}^{n} c_i β_i = 0.\]

Assume that \(c_N\) is the first non-zero coefficient. Then by applying \(S^{n-N}\), we get

\[\sum_{i=1}^{n} c_i S^{n-N}(β_i) = \sum_{i=1}^{n} c_i S^{n-N+i-1}(β_1) = \sum_{i=1}^{N} c_i S^{n-N+i-1}(β_1)\]

since \(S^n = 0\). Since \(c_i = 0\) for \(i < N\), this becomes \(c_N S^{n-1}(β_1) = 0\). Since \(S^{n-1}(β_1)\) is non-zero, \(c_N = 0\). This is contradiction so all the coefficients are zero. \(\square\)

Question 2. §3.6 - 1

Proof. (a) The dual space of \(F^n\) is spanned by the functional \(f_i\) given by \((x_1, ..., x_n) \mapsto x_i\). An arbitrary linear functional is of the form \(f = \sum c_i f_i\). Note that the element \(e_i - e_{i+1} \in W\). If \(f\) is in \(W^0\), then \(f(e_i) = f(e_{i+1})\) which implies \(c_i = c_{i+1}\). Since this holds for all \(i\), we get that \(f = c \sum f_i\) for some scalar \(c\). It is clear that any such \(f\) lies in \(W^0\).

(b) Let \(V\) be the subspace of \((F^n)^*\) consisting of \(f = \sum c_i f_i\) satisfying \(\sum c_i = 0\). Any such \(f\) defines an element of \(W^*\) just by restricting to the subspace \(W\). Since \(W^0\) is
1-dimensional from part (a), we deduce that $W$ is $n - 1$ dimensional. The space $V$ is also $n - 1$-dimensional since it is cut about by the single condition $\sum c_i = 0$.

It suffices then to show that the restriction map $V \to W^*$ is injective. That is, if $f, g \in V$, and $f(w) = g(w)$ for all $w \in W$, then $f = g$. If $f(w) = g(w)$ for all $w \in W$, then $f - g \in W^0$. If we write $g = \sum d_i f_i$, this says that for some $c$, $c_i - d_i = c$ for all $i$. But since $f, g \in V$ so is $f - g$ and so $n * c = 0$ and so $c = 0$. Thus, $c_i = d_i$ and $f = g$. \qed

**Question 3. §3.7 - 4**

**Proof.** Since $T(\alpha) = c\alpha$, then $\alpha$ is in the kernel of $T - cI$ where $I$ is the identity. Thus, rank of $T - cI$ is less than dimension of $V$. Then the rank of $(T - cI)^t = T^t - cI^t = T^t - cI$ is also less than dimension of $V$. Thus, the kernel of $T^t - cI$ is non-trivial. If $f \in V^*$ is in the kernel of $T^t - cI$, then

$$T^t(f) = cf.$$ 

An alternative would be by constructing a basis which includes $\alpha$ and then explicitly construction $f$.

Some of you argued by using that $T^t(f(\alpha)) = f(T(\alpha)) = cf(\alpha)$. However, this is not the same as $T^t f = cf$ as it doesn’t tell you what $f$ does to any other vector in $V$. \qed

**Question 4.** (a) Let $V, W, Z$ be vector spaces over $F$. Let $T : V \to W$, $S : W \to Z$ be linear transformations. Prove that

$$(S \circ T)^t = T^t \circ S^t.$$ 

(b) Using part (a), deduce that if $A$ is an $n \times m$ matrix and $B$ is an $m \times k$ matrix, then

$$(AB)^t = B^t A^t.$$ 

(c) Let $A, B$ be $n \times n$ matrices. Prove that the trace (sum of diagonal entries) of $AB - BA$ is 0. (Hint: First prove that for any $n \times n$-matrix $M$, the trace of $M$ is equal to the trace of $M^t$.)