For this problem set and all subsequent sets, if the solution requires putting a matrix in row reduced form, you do not need to show the steps as to how you obtained the row reduced form.

Problems from textbook: §2.2 - 1, 2 (If it is subspace, you do not need to prove it. If it is not, please explain why not), §2.2 - 7, 9, §2.3 - 2, 4, 6, 9

Question 1. Let $V$ be a vector space over $F$. Let $W_1, W_2, \ldots, W_n$ be subspaces of $V$. Define the sum of the subspaces to be

$$W = W_1 + W_2 + \cdots + W_n = \{w_1 + w_2 + \cdots + w_n \mid w_i \in W_i\}$$

1. Show that $W$ is a subspace of $V$.

2. Show that $W = \text{Span}(\bigcup_i W_i)$.

Proof. Let $w, w'$ be in $W$. We can write $w = w_1 + w_2 + \cdots + w_n$ and $w' = w'_1 + w'_2 + \cdots + w'_n$ where $w_i, w'_i \in W_i$. Then

$$w + w' = (w_1 + w'_1) + (w_2 + w'_2) + \cdots + (w_n + w'_n)$$

where $w_i + w'_i \in W_i$ since $W_i$ is a subspace. Thus, $w + w' \in W$. Similarly if $c$ is any scalar, then

$$cw = cw_1 + cw_2 + \cdots + cw_n \in W.$$ 

Hence, $W$ is a subspace.

Clearly $\bigcup_i W_i \subset W$, and since $W$ is a subspace, $\text{Span}(\bigcup_i W_i) \subset W$. For the opposite inclusion, note that any element $w \in W$ is of the form $\sum w_i$ which is a linear combination of elements in $\bigcup_i W_i$.

\[\square\]

Question 2. Let $V$ and $W$ be vector spaces over $F$. The direct sum of $V \oplus W = \{(v,w) \mid v \in V, w \in W\}$ has a natural vector space structure under component-wise addition and scalar multiplication.
1. If $V'$ is a subspace of $V$ and $W'$ is a subspace of $W$, then show that $V' \oplus W'$ is a subspace of $V \oplus W$.

2. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a collection of linearly independent vectors in $V$. Similarly, let $\{\beta_1, \ldots, \beta_\ell\}$ be a collection of linearly independent vectors in $W$. Prove that the collection $\{(\alpha_1, 0), \ldots, (\alpha_k, 0), (0, \beta_1), \ldots, (0, \beta_\ell)\}$ of vectors in $V \oplus W$ is linearly independent.

**Proof.** Let $(v_1, w_1) \in V' \oplus W'$ and $(v_2, w_2) \in V' \oplus W'$. Then $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \in V' \oplus W'$ since $V'$ and $W'$ are subspaces. The proof for closure under scalar multiplication is similar.

Let $\sum c_i (\alpha_i, 0) + \sum d_j (0, \beta_j) = 0$ be a linear relation among the vectors. Then,

$$(\sum c_i \alpha_i, \sum d_j \beta_j) = (0, 0).$$

Since the $\alpha_i$ are linearly independent, this implies that $c_i = 0$ for all $i$. Since the $\beta_j$ are linearly independent, this implies that $d_j = 0$ for all $j$. Thus, the vectors are linearly independent. \qed