A Harder-Narasimhan theory for Kisin modules

Brandon Levin and Carl Wang Erickson

October 2, 2015

Modularity

Theorem (Wiles, Taylor-Wiles, BCDT)

Any elliptic curve $E/\mathbb{Q}$ is modular.

A key input into Wiles' method and subsequent improvements is a good understanding of Galois deformation spaces at $\ell = p$.

Local deformation problem

Let K be a finite extension of $\mathbb{Q}_p$, and let Γ_K be the absolute Galois group of K. Fix

$$\overline{\rho} : \Gamma_K \to \mathrm{GL}_n(\mathbb{F}_p).$$

There is a universal deformation space $D_{\overline{\rho}}$ represented by a quotient of a power-series ring $R_{\overline{\rho}}$ over $\mathbb{Z}_p$ (when $\overline{\rho}$ is absolutely irreducible).

Flat deformation spaces

If E is an elliptic curve over K with good reduction, then $E[p^n]$ is a finite flat group scheme over $\mathcal{O}_K$ for all n. The representation of Γ_K on the p^n-torsion points is called **flat**.

The **flat deformation space** $D_{\overline{\rho}}^{\mathrm{fl}}$ is the subspace of $D_{\overline{\rho}}$ of representations that come from finite flat group schemes over $\mathcal{O}_K$.

Connected components

What are the connected components of $D_{\bar{\rho}}^{\mathrm{fl}}[1/p]$?

- (Ramakrishna) When $n = 2$ and $K = \mathbb{Q}_p$, then $D_{\bar{\rho}}^{\mathrm{fl}}[1/p]$ is connected.
- When $n = 2$, we have full description of connected components for any K by work of Kisin, Imai, Gee, and Hellmann.
- When $n > 2$, the question is open in general (unless K is mildly ramified).

Kisin varieties

Theorem (Kisin)

There is a projective variety $X_{\overline{\rho}}$ over $\mathbb{F}_p$ such that $X_{\overline{\rho}}(\mathbb{F})$ is the set of finite flat group schemes $\mathcal{G}$ over $\mathcal{O}_K$ such that $\mathcal{G}(\overline{K}) \cong \overline{\rho} \otimes_{\mathbb{F}_p} \mathbb{F}$.

Application

Connected components of $D^{\mathrm{fl}}_{\overline{\rho}}[1/p]$ are related to the connected components of $X_{\overline{\rho}}$.

Kisin modules

Definition

Assume $K/\mathbb{Q}_p$ is totally ramified of degree e and $\mathbb{F}$ is a finite field. Let $\varphi : \mathbb{F}[[u]] \to \mathbb{F}[[u]]$ be the homomorphism sending $u \mapsto u^p$. A *Kisin module* of rank n and height $\leq h$ over $\mathbb{F}$ is a finite free $\mathbb{F}[[u]]$-module $\mathfrak{M}_{\mathbb{F}}$ with a semilinear map

$$\phi_{\mathfrak{M}_{\mathbb{F}}} : \mathfrak{M}_{\mathbb{F}} \to \mathfrak{M}_{\mathbb{F}}$$

such that the cokernel (of the linearization) is killed by u^{eh}.

Theorem (Kisin)

The category of Kisin modules over $\mathbb{F}$ of height ≤ 1 is anti-equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ with an $\mathbb{F}$-action.

Slope function

- The *generic fiber* of $(\mathfrak{M}, \phi)$ is $(\mathfrak{M}[1/u], \phi_{\mathfrak{M}}[1/u])$. (This is an étale $\mathbb{F}((u))$-module).
- The *degree* of $(\mathfrak{M}, \phi_{\mathfrak{M}})$ is $\frac{1}{e} \dim_{\mathbb{F}} \mathrm{coker}(\phi_{\mathfrak{M}})$.
- The *slope* is $\mu(\mathfrak{M}) := \deg(\mathfrak{M}) / \operatorname{rank}(\mathfrak{M})$.

This was inspired by Fargues' (2010) theory of Harder-Narasimhan filtrations for finite flat group schemes.

HN-theory for Kisin modules

Examples

Let $(\mathfrak{M}, \phi_{\mathfrak{M}})$ have height ≤ 1 and let $\mathcal{G}_{\mathfrak{M}}$ be corresponding finite flat group scheme over $\mathcal{O}_K$.

- If $\mathfrak{M}$ has slope 1, then $\mathcal{G}_{\mathfrak{M}}$ is *étale*.
- If $\mathfrak{M}$ has slope 0, then $\mathcal{G}_{\mathfrak{M}}$ is *multiplicative*, i.e., Cartier dual to étale.

Remark

The HN-filtration generalizes the connected-etale sequence for finite flat group schemes.

$$1 \to \mathcal{G}^0 \to \mathcal{G} \to \mathcal{G}^{\mathrm{et}} \to 1$$

HN-theory for Kisin modules

Theorem (L.-W. E.)

The function μ defines an HN-theory on the category of Kisin modules. In particular, any $\mathfrak{M}$ has a canonical HN-filtration

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \ldots \subset \mathfrak{M}_k = \mathfrak{M}$$

by strict subobjects such that $\mathfrak{M}_{i+1}/\mathfrak{M}_i$ is semi-stable and $\mu(\mathfrak{M}_i/\mathfrak{M}_{i-1}) < \mu(\mathfrak{M}_{i+1}/\mathfrak{M}_i)$.

Definition

The HN-polygon is the concave polygon with breakpoints given by $(\mathrm{rank}(\mathfrak{M}_i), \deg(\mathfrak{M}_i))$. In particular, it starts at $(0,0)$ and ends at $(\mathrm{rank}(\mathfrak{M}), \deg(\mathfrak{M}))$.

Kisin varieties

Definition

For $\nu = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{Z}$ and $a_{i+1} \geq a_i$, a Kisin module $(\mathfrak{M}, \phi_{\mathfrak{M}})$ over $\mathbb{F}$ of rank n has **Hodge type** ν if there exists a basis $\{e_i\}$ of $\mathfrak{M}$ such that $u^{a_i} e_i$ generates the image of $\phi_{\mathfrak{M}}$.

Definition

Let $(\mathcal{M}_{\overline{\rho}}, \phi)$ be the étale $\mathbb{F}_p((u))$-module of rank n attached to $\overline{\rho}$. The closed *Kisin variety* has points given by

$$X^{\nu}_{\overline{\rho}} = \{\mathfrak{M}[1/u] \cong \mathcal{M}_{\overline{\rho}} \mid \mathfrak{M} \text{ has Hodge type} \leq \nu\}.$$

It is a projective scheme over $\mathbb{F}_p$.

Stratification

Theorem (L.-W. E.)

There is a stratification

$$\bigcup_P X_{\overline{\rho}}^{\nu,P} = X_{\overline{\rho}}^{\nu}$$

by locally closed subschemes indexed by concave polygons P such that the points of $X_{\overline{\rho}}^{\nu,P}$ are the Kisin modules with HN-polygon P.

Remark

For any point in $X_{\overline{\rho}}^{\nu}$, the HN-polygon lies above the Hodge polygon ν with the same endpoints. Hence, there are a finite number of such strata.

Pictures

Explanation

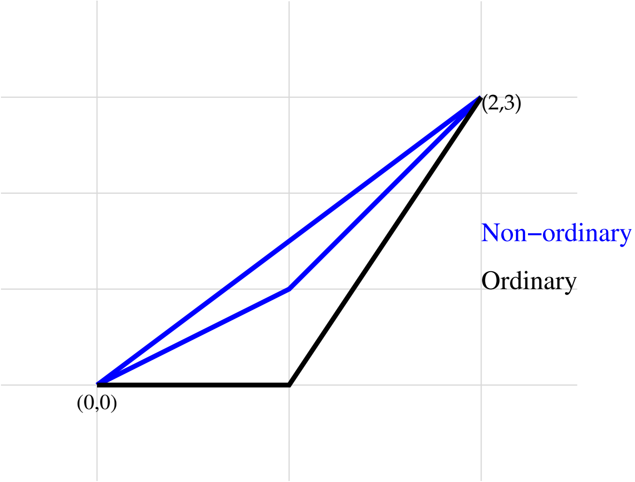
In the following slides, for different Hodge polygons ν, we draw the set of possible HN-polygons.

- For any $\overline{\rho}$ of the appropriate dimension, the strata of $X_{\overline{\rho}}^{\nu}$ will be indexed by this finite set of polygons.
- The Hodge polygon ν appears in black.
- We color the polygons the same if they share the same segments in common with the Hodge polygon.
- Only strata with the same color can occur on the same connected component (i.e., the union of the strata with same color is open and closed in $X_{\overline{\rho}}^{\nu}$).

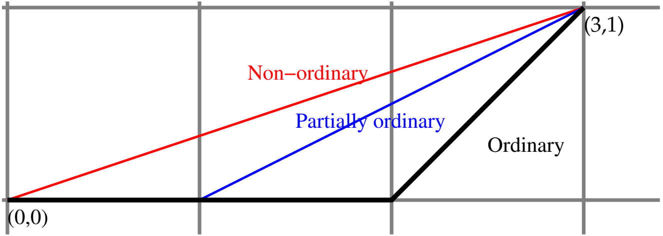
Remark

For any particular $\overline{\rho}$, many of the strata could be empty. For example, if $\overline{\rho}$ is irreducible, then only the constant slope stratum will appear.

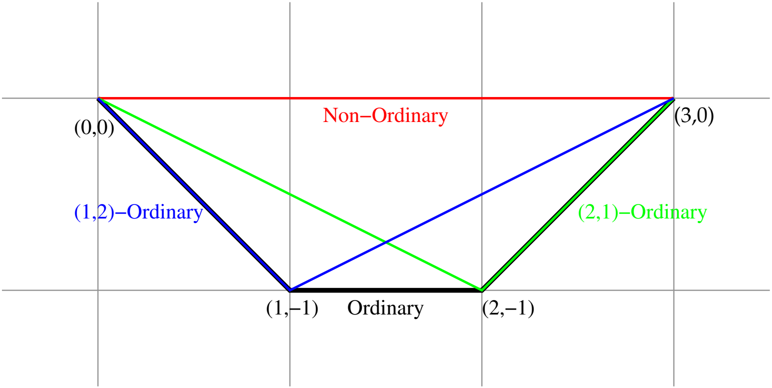
Components for GL_2, $\nu = (0, 3)$



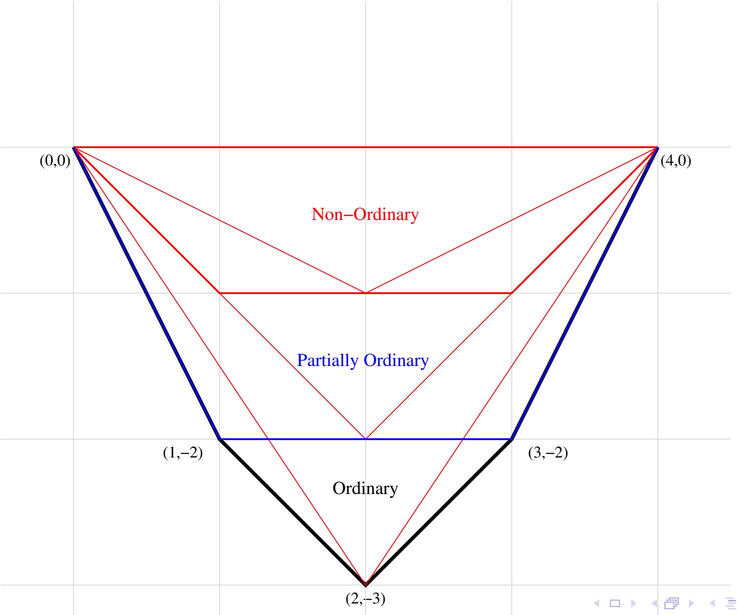
Components for GL_3, $\nu = (0, 0, 1)$



Components for GL_3, $\nu = (-1, 0, 1)$



Components for GSp_4, $\nu = (-2, -1, 1, 2)$



Tensor product theorem

Expected Theorem (L.-W. E.)

Let $\mathfrak{M}$ and $\mathfrak{N}$ be Kisin modules over $\mathbb{F}$. If $\mathfrak{M}$ and $\mathfrak{N}$ are semistable, then

$$\mathfrak{M} \otimes_{\mathbb{F}} \mathfrak{N}$$

is semistable of slope $\mu(\mathfrak{M}) + \mu(\mathfrak{N})$.

Application

Study Kisin varieties for reductive groups G and G-valued flat deformation rings.