Potentially crystalline deformation rings and Iwahori local models

In joint work with Daniel Le, Bao V. Le Hung and Stefano Morra, we compute potentially crystalline deformation rings for three dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. As an application, we deduce the weight part of Serre's conjecture for forms of $U(3)$ which are compact at infinity and split at places dividing p as conjectured by [Her09] for residual representations which are semisimple and generic at all primes above p. We also exhibit the geometric Breuil-Mézard conjecture. The method involves a detailed study of the moduli space of Kisin modules with descent datum. This builds on work of Breuil [Bre12], Breuil-Mézard [BM14], Caruso-David-Mézard [CDM], Caraiani-Emerton-Gee-Savitt [CEGS]. I also discussed joint work with Ana Caraiani which relates moduli of Kisin modules with descent data to Iwahori local models.

0.1. **Potentially crystalline deformation rings.** Fix a residual representation $\overline{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_3\left(\overline{\mathbb{F}}_p\right)$. Let $\tau = \omega^a \oplus \omega^b \oplus \omega^c$ where ω is the mod p cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We assume τ is *generic*, i.e., $p-4 \geq |a-b|, |b-c|, |a-c| \geq 3$. We want to describe the framed potentially crystalline deformation ring $R_{\overline{\rho}}^{(2,1,0),\tau,\square}$ with Hodge-Tate weights $(2,1,0)$ and Galois type τ.

We consider the moduli space $X^{(2,1,0),\tau}$ of Kisin modules with p-adic Hodge type $(2,1,0)$ and descent data of type τ constructed in [CL] building on work of [CEGS] in the Barsotti-Tate case for GL_2. There is local model diagram relating $X^{(2,1,0),\tau}$ to the local model M^{loc} for $(\mathrm{GL}_3, \mu = (2,1,0), \text{Iwahori level})$. This induces a stratification of $X^{(2,1,0),\tau} = \bigcup_{w \in \mathrm{Adm}(2,1,0)} X_w^{(2,1,0),\tau}$ indexed by the $(2,1,0)$-admissible set. A mod p Kisin module in $X_w^{(2,1,0),\tau}(\overline{\mathbb{F}}_p)$ is said to have *shape* (or *genre*) w. This generalizes the notion of *genre* for rank 2 Breuil/Kisin modules which was crucial in [Bre12, CDM].

We can now outline our strategy for computing the deformation ring.

(1) Classify all Kisin modules of shape $w \in \mathrm{Adm}(2,1,0)$ over $\overline{\mathbb{F}}_p$.
(2) For $\overline{\mathfrak{M}} \in X_w^{(2,1,0),\tau}(\overline{\mathbb{F}}_p)$, construct the universal deformation space with height conditions. This amounts to constructing local coordinates for the local model.
(3) Impose monodromy condition on the universal family.

Previously, the finer properties of local Galois deformation rings were known for the most part only for Fontaine-Laffaille and potentially Barsotti-Tate deformation rings. Steps (1) and (2) generalize techniques of [Bre12, CDM, EGS15] used to compute potentially Barsotti-Tate deformation rings for GL_2. To extend these techniques to three dimensions, a key point was the correct notion of shape as discussed above. In addition, a more systematic approach to the p-adic convergence algorithm employed by [Bre12, CDM] was necessary for Step (2).

Step (3) requires a genuinely new method not present in the potentially Barsotti-Tate case where the link between moduli of finite flat groups schemes and Galois representations is stronger than in our situation (and hence, there is no monodromy condition). Let us briefly comment on the details of Step (3). Kisin [Kis06] gave a characterization of when a torsion-free Kisin module $\mathfrak{M}$ comes from a crystalline representation in terms poles of a monodromy operator $N_{\mathcal{M}}$. While one cannot compute $N_{\mathcal{M}}$ completely, one can approximate it using the genericity condition on τ. Vanishing of the poles corresponds to the vanishing of the sum of an explicit polynomial equation and an error term which is divisible by p^3. This suffices for determining the deformation ring and its special fiber.

As an illustration, we have the following theorem relating components of the deformation ring and predicted local weights. Here $\mathrm{JH}(\overline{\sigma}(\tau))$ denotes the Jordan-Hölder factors mod p of the principal series representation corresponding to τ under inertial local Langlands and $W^?(\overline{\rho})$ denotes the conjectural set of local weights predicted by [Her09].

Theorem 1 (LLLM). *Let $\overline{\rho}$ be a semisimple representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *and let τ be a* generic *tame inertial type. If $|W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))| < 6$, then the irreducible components of* Spec $R_{\overline{\rho}}^{(2,1,0),\tau,\square}$ mod p *are in bijection with $W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))$.*

We expect the same result in the case of six common weights which is the maximal number possible, but there is one case remaining. Theorem 1 should be thought of as an instance of the geometric Breuil-Mézard conjecture of [EG14] (the intrinsic multiplicity of each local weight turns out to be one in this case). The bijection matches components labelled by Fontaine-Laffaille weights with the special fibers of Fontaine-Laffaille deformation rings.

0.2. **Application: Weight part of Serre's conjecture.** Serre's original modularity conjecture asserted that every odd irreducible continuous representation $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ arises from a modular form. Serre [Ser87] gave a precise recipe for the minimal possible prime to p level and weight of such a modular form. The recipe for the minimal weight is given in terms of the restriction of $\overline{r}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (or even the inertia subgroup). The weight recipe and subsequent generalizations are often referred to as the weight part of Serre's conjecture (or even just Serre weight conjectures). There has much progress in recent years in formulating generalizations the weight part of Serre's conjecture (see [Her09, GHS]) and in proving generalizations of the conjecture for 2-dimensional Galois representations over totally real fields. However, thus far, there are only a few theoretical results in the case of semisimple rank > 1: [GG12, BLGG14] study modularity of 'obvious' weights and [EGH13] proves Herzig's conjecture in our setting under the hypothesis that $\overline{r}$ is irreducible at all places above p. We reprove the results of [EGH13] and extend them to the case where $\overline{r}$ is semisimple at all places above p.

Recall the setup for algebraic modular forms. Let F be an imaginary CM field with totally real subfield F^+ such that all primes of F^+ above p split in F. Let G is unitary group over F^+ which is isomorphic to $U(3)$ at each infinite place and split at each place above p. Let $\mathcal{G}$ be a reductive model over $\mathcal{O}_{F^+}[1/N]$ with $(N, p) = 1$.

Definition 2. *A* (*global*) Serre weight *is an irreducible $\overline{\mathbb{F}}_p$-representation F_λ of $\mathcal{G}(\mathcal{O}_{F^+,p})$.*

For each place $w \mid p$, let k_w denote the residue field. A (global) Serre weight is equivalent to a collection $(F_{\lambda_w})_{w|p}$ of irreducible representation of $\mathrm{GL}_3(k_w)$ which is conjugate self-dual, i.e., $(F^*_{\lambda_w})^c \cong F_{\lambda_{w^c}}$. For any Serre weight F_λ and any compact open $U \subset G(\mathbb{A}^{f,p}_{F^+})$, there is an associated space of mod p algebraic modular forms $S(U, F_\lambda)$. Let $\overline{r} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation.

Definition 3. *We say $\overline{r}$ is* modular of weight F_λ *if there exists some* (*nice*) *open compact U unramified above p such that*

$$S(U, F_\lambda)_{\overline{m}} \neq 0$$

where $\overline{m}$ is the maximal ideal of Hecke algebra associated to $\overline{r}$.

We say $\overline{r}$ is modular *if $\overline{r}$ is modular of some Serre weight $(F_{\lambda_w})_{w|p}$. Let $W(\overline{r})$ denote the set of modular Serre weights.*

The weight part of Serre's conjectures predicts, in particular, that the set of modular Serre weights should be determined by the restrictions $\overline{r}|_{G_{F_w}}$ for all primes $w \mid p$. For $\overline{r}|_{G_{F_w}}$ semisimple and $F_w = \mathbb{Q}_p$, [Her09] gives a recipe for a collection $W^?(\overline{r}|_{G_{F_w}})$ of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_3(k_w)$.

Theorem 4 (LLLM). *Let $\overline{r}: G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ be an irreducible modular representation. Suppose p splits completely in F and for all places $w \mid p$ of F suppose $\overline{r}|_{G_{F_w}}$ is semisimple and generic. Suppose further that $\overline{r}$ satisfies the Taylor-Wiles conditions. Then*

$$W(\overline{r}) = (W^?(\overline{r}|_{G_{F_w}}))_{w|p}.$$

The weight elimination direction $W(\overline{r}) \subset (W^?(\overline{r}|_{G_{F_w}}))_{w|p}$ was already completed (or forthcoming) in [EGH13, HLM, MP]. The other inclusion (weight existence) is an application of our results on deformation rings following roughly the strategy outlined in [GHS, §3-4].

Briefly, the patching techniques of Gee-Kisin [GK14], Emerton-Gee [EG14] allow one to construct a patched module $M_\infty(\sigma(\tau))$ over the product of the local deformation rings (adjoin some patching variables). Furthermore, for each $\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau))$, there is a subquotient $M_\infty(\overline{\sigma})$ of $M_\infty(\sigma(\tau)) \mod p$. We show that the generic fiber of the local deformation ring is connected and so $M_\infty(\sigma(\tau))$ has full support. Knowing this and the number of components in the special fiber, we deduce that $M_\infty(\overline{\sigma}) \neq 0$ for $\overline{\sigma} \in W^?(\overline{r}|_{G_{F_w}})$ by an inductive procedure involving a careful choice of tame types.

We expect our methods to carry over also to the case where p is unramified (but not necessarily split in F^+). We also aim to address the question of cyclicity of patched modules and the analogue of Breuil's lattice conjecture in future work.

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