Midterm: Due 10-25 in class  
Math 513

This midterm is to be done at home. You may consult the textbooks for the course, course notes, and old homework assignments. You may not work with your friend or consult with google. All the work should be your own. Please show all work as you may receive partial credit. Each question is worth 20 points.

**Question 1.** (a) Show that the vectors \( \alpha_1 = (1, 1, 0, 0), \alpha_2 = (0, 0, 1, 1), \alpha_3 = (1, 0, 0, 4), \alpha_4 = (0, 0, 0, 2) \) form a basis for \( \mathbb{R}^4 \).

(b) Find the coordinates of each standard basis vector \( e_1, e_2, e_3, e_4 \) in the ordered basis \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \).

(c) Let \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) be a linear transformation such that the matrix of \( T \) with respect to \( B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \) is

\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}.
\]

Find \( T(e_1) \) and \( T(e_2) \).

**Proof.** (a) The row reduced form of

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 4 & 2
\end{pmatrix}
\]

is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

so the vectors are linearly independent and thus a basis.

(b) By row reduction, we get that the coordinates of standard basis vector are

\[
e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}.
\]
(c) By part (b), $e_1 = \alpha_3 - 2\alpha_4$. By definition of $T$, then
\[ T(e_1) = T(\alpha_3) - 2T(\alpha_4) = \alpha_2 + \alpha_4 - 2\alpha_1 - 2\alpha_3. \]
Similarly, $e_2 = \alpha_1 - \alpha_3 + 2\alpha_4$. Thus,
\[ T(e_2) = T(\alpha_1) - T(\alpha_3) + 2T(\alpha_4) = 2\alpha_1 + 2\alpha_3. \]

Question 2. Let $V$ be a finite dimensional vector space over $F$.

(a) Let $W_1, W_2$ be subspaces of $V$. Show that $T : W_1 \oplus W_2 \to V$ defined by $T(w_1, w_2) = w_1 + w_2$ is a linear transformation. Describe the nullspace and the image of $T$.

(b) We say a subspace $W_2$ of $V$ is a complement to $W_1$ if the map $T$ from part (a) is an isomorphism. Let $W_1$ be any non-trivial subspace of $V$. Show that there exists a complement to $W_1$.

Proof. (a) We first show that $T$ is a linear transformation. Let $(w_1, w_2), (w'_1, w'_2) \in W_1 \oplus W_2$. Then
\[ T((w_1, w_2) + (w'_1, w'_2)) = w_1 + w'_1 + w_2 + w'_2 = T(w_1, w_2) + T(w'_1, w'_2). \]
Also,
\[ T(c(w_1, w_2)) = cw_1 + cw_2 = cT(w_1, w_2). \]
Thus, $T$ is a linear transformation.

The image of $T$ is the set of vectors $V$ of the form $w_1 + w_2$. In other words, the image is the sum $W_1 + W_2$ of the subspaces.

A pair $(w_1, w_2)$ is in the kernel of $T$ if $w_1 + w_2 = 0$. Thus,
\[ N_T = \{(w_1, w_2) \mid w_1 = -w_2\} = \{(w, -w) \mid w \in W_1 \cap W_2\}. \]
Note that the null space is isomorphic to the intersection of $W_1 \cap W_2$.

(b) From part (a), we see that the map $T$ is injective if and only if $W_1 \cap W_2 = \{0\}$. Let $n = \dim V$ and $k = \dim W_1$. Choose a basis $\alpha_1, \alpha_2, \ldots, \alpha_k$ of $W_1$. We can complete the basis to a basis $\alpha_1, \ldots, \alpha_n$ of $V$. Set $W_2 = \text{Span}(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n)$.

Since $\dim W_2 = n - k$ and $\dim W_1 = k$, $\dim W_1 \oplus W_2 = n$. It suffices then to show that the map $T$ is injective. Let $\beta \in W_1 \cap W_2$, then
\[ \beta = \sum_{i=1}^{k} c_i \alpha_i = \sum_{j=k+1}^{n} c_j \alpha_j. \]

2
Thus,
\[ \sum_{i=1}^{k} c_i \alpha_i - \sum_{j=k+1}^{n} c_j \alpha_j = 0. \]
But since \( \{\alpha_i\} \) are linearly independent, we have \( c_i = c_j = 0 \) for all \( i, j \). Thus, \( \beta = 0 \) and so \( W_1 \cap W_2 = \{0\} \).

\[ \square \]

**Question 3.** Let \( V \) be a finite dimensional vector space over \( F \).

(a) Let \( f_1, f_2, \ldots, f_k \) be linear functionals on \( V \). Define a map \( f : V \to F^k \) by \( f(\alpha) = (f_1(\alpha), f_2(\alpha), \ldots, f_k(\alpha)) \). Prove that \( f \) is a linear transformation.

(b) Describe the null space of \( f \) from part (a).

(c) Let \( V \) be an \( n \)-dimensional vector space. If \( f_1, f_2, \ldots, f_k \) are linearly independent in \( V^* \), what is the dimension of the null space of \( f \)? Prove your answer.

**Proof.** (a) To check \( f \) is a linear transformation, we observe that
\[ f(\alpha + \beta) = (f_1(\alpha + \beta), \ldots, f_k(\alpha + \beta)) = (f_1(\alpha), \ldots, f_k(\alpha)) + (f_1(\beta), \ldots, f_k(\beta)) \]
using that each \( f_i \) is linear. Similarly, \( f(c \alpha) = cf(\alpha) \).

(b) The null space of \( f \) is
\[ \{ \alpha \in V \mid f_i(\alpha) = 0 \text{ for all } 1 \leq i \leq k \}. \]
If \( N_{f_i} \) is the null space of \( f_i \), this is the intersection \( \bigcap_{i=1}^{k} N_{f_i} \).

(c) Recall the natural isomorphism \( L : V \to V^{**} \), where \( L_\alpha \) is the linear function given by evaluation at \( \alpha \).

Let \( W \) be the subspace spanned by \( f_1, \ldots, f_k \). By assumption \( \dim W = k \). Let \( W^0 \subset V^{**} \) be the annihilator of \( W \). \( L_\alpha \in W^0 \) if and only if \( f_i(\alpha) = 0 \) for all \( 1 \leq i \leq k \). Thus, \( W^0 \) is the image under \( L \) of the nullspace of \( f \). Thus, the nullspace has the same dimension as \( W^0 \).

By a result from class \( \dim W^0 + \dim W = \dim V^* \) so the nullspace has dimension \( \dim V - k \).

\[ \square \]

**Question 4.** Let \( V = F^4 \).

(a) Give an example of a non-zero alternating 2-form on \( V \).
(b) Let $S^2(V)$ be the vector space of symmetric 2-forms on $V$ (i.e., symmetric bilinear forms on $V \oplus V$). Find a basis for $S^2(V)$.

(c) Let $T : V \to V$ be a linear transformation such that $T(e_1) = e_1, T(e_2) = e_2, T(e_3) = 0, T(e_4) = 0$. If $w$ is a symmetric bilinear form on $V$, define $S^2(T)(w)$ to be the symmetric form given by $S^2(T)(w)(\alpha_1, \alpha_2) = w(T(\alpha_1), T(\alpha_2))$. Then $S^2(T)$ defines a linear transformation from $S^2(V)$ to $S^2(V)$ (you don’t need to prove this). Determine the rank of $S^2(T)$.

Proof. (a) If $f, g \in (F^4)^*$, I will write $f \otimes g$ for the bilinear form on $F^4$ which sends $(v, w)$ to $f(v)g(w)$.

Take any non-trivial $f, g$ where $f$ not a scalar multiple of $g$, then an example of non-zero alternating form is

$$f \otimes g - g \otimes f.$$

Let’s show that it is non-zero. Since $g$ is non-trivial, we can choose $\beta$ such that $g(\beta) = 1$. If $f \otimes g - g \otimes f = 0$, then we would get that for all $\alpha$

$$f(\alpha) = g(\alpha)f(\beta).$$

Since $\beta$ is fixed, this would imply that $f$ is a scalar multiple of $g$ which we assumed was not the case. The alternating condition is clear from the definition.

(b) Let $e_1^*, e_2^*, e_3^*, e_4^*$ be the dual basis to the standard basis on $F^4$. Recall that a bilinear form $w$ is symmetric if $w(v_1, v_2) = w(v_2, v_1)$ for all $v_1, v_2 \in V$. Let $\tau$ be the transposition with swaps the factors so $\tau(w)(v_1, v_2) = w(v_2, v_1)$. Also, we have a basis for the 16 dimensional space of bilinear forms given by $w_{ij} = e_i^* \otimes e_j^*$ as proved in class.

A straightforward computation shows that

$$\tau(w_{ij}) = w_{ji}.$$

Thus, the bases elements $w_{11}, w_{22}, w_{33}, w_{44}$ are all symmetric. For any $i \neq j$, we can form the symmetric bilinear form $S_{ij} = w_{ij} + w_{ji}$. There are six choices of such pairs. This gives a bases for the sapce $S^2(V)$. To see this, let

$$w = \sum_{i,j} a_{ij} w_{ij}$$

be an arbitrary bilinear form. Then $\tau(w) = \sum_{i,j} a_{ij} w_{ji}$. Swapping $i, j$, this becomes $\tau(w) = \sum_{i,j} a_{ji} w_{ij}$. Thus, $\tau(w) = w$ if and only if $a_{ji} = a_{ij}$ for all $i, j$. It is clear then that $w$ is in the span of the ten elements given before.
(c) To determine the rank of $S^2(T)$, we look at its action on the bases vectors. First observe for any $v \in V$, $T(v)$ is in the span of $e_1$ and $e_2$ since it kills the $e_3$ and $e_4$ components. Note that $w_{34}(v_1, v_2) = 0$ if $v_1$ has no $e_3$ component. Thus, $w_{34}(T(v), T(w)) = 0$ for all $v, w$. Similarly, $w_{41}(T(v), T(w)) = 0$ as well as $w_{33}(T(v), T(w))$ and $w_{44}(T(v), T(w))$. Thus, $S^2(T)$ is zero on all the bases elements of $S^2(V)$ except for $w_{11}, w_{22}$ and $w_{12} + w_{21}$. In fact, since $T$ is the identity on subspace spanned by $e_1, e_2$,
\[ S^2(T)(w_{11}) = w_{11}, S^2(T)(w_{22}) = w_{22}, S^2(T)(w_{12} + w_{21}) = w_{12} + w_{21}, \]
The rank of $S^2(T)$ is 3.

\[ \square \]

**Question 5.** Let $V$ be a finite dimensional vector space of $F$ equipped with a basis $B = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$. Let $T : V \rightarrow V$ be a linear operator such that $[T]_B = A$ for an $n \times n$ matrix $A$.

(a) Let $A'$ be an $n \times n$ matrix obtained from $A$ by multiplying the $i$th column by a scalar $c$. Let $T' : V \rightarrow V$ be a linear transformation such that $[T']_B = A'$. Using the definition of determinant given in class prove that $\det(T') = c \det(T)$.

(b) Let $A'$ be an $n \times n$ matrix obtained from $A$ by swapping the $i$ and $j$th columns. Let $T' : V \rightarrow V$ be a linear transformation such that $[T']_B = A'$. Using the definition of determinant given in class prove that $\det(T') = -\det(T)$.

**Proof.** (a) Let $w$ be a non-zero alternating $n$-form on $V$. We can compute the determinant of $T'$ by
\[ w(T'(\alpha_1), T'(\alpha_2), \ldots, T'(\alpha_n)) = (\det T')w(\alpha_1, \ldots, \alpha_n) \]
By the definition of $T'$, we have
\[ w(T'(\alpha_1), T'(\alpha_2), \ldots, T'(\alpha_n)) = w(T(\alpha_1), \ldots, cT(\alpha_i), \ldots, T(\alpha_n)) \]
which by multi-linearity is
\[ c \ast w(T(\alpha_1), \ldots, T(\alpha_n)) = (c \det T)w(\alpha_1, \ldots, \alpha_n). \]
Thus, $\det T' = c \det T$.

(b) Again, let $w$ be a non-zero alternating $n$-form. Consider $w(T'(\alpha_1), T'(\alpha_2), \ldots, T'(\alpha_n))$. Let $(ij)$ denote the transposition with $i < j$. Since $w$ is alternating,
\[ w(T'(\alpha_1), \ldots, T'(\alpha_n)) = -(ij)w(T'(\alpha_1), \ldots, T'(\alpha_n)) \]
which is equal to

\[-w(T'(\alpha_1), \ldots, T'(\alpha_{i-1}), T'(\alpha_j), T'(\alpha_{i+1}), \ldots, T'(\alpha_{j-1}), T'(\alpha_i), T'(\alpha_{j+1}), \ldots, T'(\alpha_n)).\]

Since $T'$ is the same as $T$ except with $i$ and $j$th columns swapped, this is equal to

\[-w(T(\alpha_1), \ldots, T(\alpha_n)) = (-\det T)w(\alpha_1, \ldots, \alpha_n).\]