REDUCTIONS OF 2-DIMENSIONAL SEMI-STABLE REPRESENTATIONS WITH LARGE $\mathcal{L}$-INVARIANT

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ABSTRACT. We determine reductions of 2-dimensional, irreducible, semi-stable, and non-crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with Hodge–Tate weights $0 < k-1$ and with $\mathcal{L}$-invariant whose p-adic norm is sufficiently large, depending on k. Our main result provides the first systematic examples of the reductions for $k \geq p$.

CONTENTS

1. INTRODUCTION

Let p be a prime number and $\overline{\mathbb{Q}}_p$ be an algebraic closure of the p-adic numbers $\mathbb{Q}_p$. The goal of this article is to determine the reductions of certain 2-dimensional p-adic representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ that are semi-stable and not crystalline in the sense of Fontaine ([12]). Examples of such representations arise from local p-adic representations associated with eigenforms with $\Gamma_0(p)$-level.

1.1. **Main result.** Write v_p for the p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized so that $v_p(p) = 1$. Choose $\varpi \in \overline{\mathbb{Q}}_p$ such that $\varpi^2 = p$. Then, for each integer $k \geq 2$ and each $\mathcal{L} \in \overline{\mathbb{Q}}_p$, there is a 2-dimensional filtered (φ, N)-module $D_{k,\mathcal{L}} = \overline{\mathbb{Q}}_p e_1 \oplus \overline{\mathbb{Q}}_p e_2$ where, in the basis $\{e_1, e_2\}$, we have:

$$\varphi = \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{k-2} \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathrm{Fil}^i D_{k,\mathcal{L}} = \begin{cases} D_{k,\mathcal{L}} & \text{if } i \leq 0; \\ \overline{\mathbb{Q}}_p \cdot (e_1 + \mathcal{L} e_2) & \text{if } 1 \leq i \leq k-1; \\ \{0\} & \text{if } k \leq i. \end{cases} \tag{1.1}$$

Each $D_{k,\mathcal{L}}$ is weakly-admissible, so a theorem of Colmez and Fontaine implies there is a unique 2-dimensional $\overline{\mathbb{Q}}_p$-linear representation $V_{k,\mathcal{L}}$ of $G_{\mathbb{Q}_p}$ such that $D_{k,\mathcal{L}} = D^*_{\mathrm{st}}(V_{k,\mathcal{L}})$. Up to a twist by

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a crystalline character, the representations $V_{k,\mathcal{L}}$ enumerate all $\overline{\mathbb{Q}}_p$-linear 2-dimensional semi-stable and non-crystalline representations of $G_{\mathbb{Q}_p}$. They are irreducible except if $k = 2$.

We aim to determine the semi-simple mod p reductions $\overline{V}_{k,\mathcal{L}}$ of $V_{k,\mathcal{L}}$. Twenty years ago, Breuil and Mézard determined $\overline{V}_{k,\mathcal{L}}$ for even $k < p$ and any $\mathcal{L}$ ([6, Théorème 4.2.4.7]). Guerberoff and Park did the same, recently, for odd $k < p$ ([14, Theorem 5.0.5]). The reader who takes a moment to examine the cited theorems should be left with an impression of the complicated dependence of $\overline{V}_{k,\mathcal{L}}$ on $\mathcal{L}$, and that is just for $k < p$.

Prior results are limited by their ambition to determine $\overline{V}_{k,\mathcal{L}}$ for all $\mathcal{L}$. Here, we focus on determining $\overline{V}_{k,\mathcal{L}}$ for any k while restricting to $\mathcal{L}$ that place $V_{k,\mathcal{L}}$ in a p-adic neighborhood of a crystalline representation (see Section 1.2). Write $\mathbb{Q}_{p^2}$ for the unramified quadratic extension of $\mathbb{Q}_p$, χ for its quadratic character modulo p, and ω_2 for a niveau 2 fundamental character on $G_{\mathbb{Q}_{p^2}}$.

Theorem 1.1. *Assume $k > 2$, $p \neq 2$ and if $p = 3$, then $k \geq 4$. Then, if*

$$v_p(\mathcal{L}) < 2 - \frac{k}{2} - v_p((k-2)!),$$

then $\overline{V}_{k,\mathcal{L}} \cong \operatorname{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$.

To be accurate, our method proves Theorem 1.1 when $k \geq 5$ or $p = 3$ and $k = 4$. The theorem holds for small k by the work of Breuil–Mézard and Guerberoff–Park. It is unclear if the theorem holds for $p = k = 3$. Regardless, that case could be included with a weaker bound (see Remark 4.8). Our exclusion of $p = 2$ is more fundamental (see Remark 1.3).

Remark 1.2. The bound in Theorem 1.1 is optimal for $k < p$ by the results of Breuil–Mézard and Guerberoff–Park. We do not know to what extent the bound is optimal, in general. A global example discussed in Section 1.3 shows that the $v_p((k-2)!)$-term cannot be entirely removed (at least when $p = 3$).

Theorem 1.1 is a natural analog of widely-studied theorems that determine reductions of 2-dimensional, irreducible, crystalline representations of $G_{\mathbb{Q}_p}$. For instance, Buzzard and Gee ([8]) developed a strategy to determine reductions of certain crystalline representations, with unbounded Hodge–Tate weights, using the p-adic local Langlands correspondence. We do not know whether such an approach for semi-stable, but non-crystalline, representations has been tried or, even, if such an approach is feasible.

Another approach in the crystalline case is via integral p-adic Hodge theory. Berger, Li, and Zhu and Berger proved local constancy results for reductions of crystalline representations using Wach modules ([4, 3]). Recently, the first two named authors of this article improved the Berger–Li–Zhu result using Kisin modules ([2]). Those are what we will use here, also. One incentive to write the prior article was as training to conduct the current research.

1.2. **Overview of strategy.** We now describe our strategy, first re-contextualizing Theorem 1.1 through the lens of local constancy of reductions as in [4, 3, 2].

The parametrization of semi-stable and non-crystalline representations by $\mathcal{L} \in \overline{\mathbb{Q}}_p$ extends to a $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$-parametrization with a crystalline representation at ∞. Namely, for $\mathcal{L} \neq 0$ we consider $D_{k,\mathcal{L}}$ with basis $\{e_1', e_2'\} = \{e_1, \mathcal{L}e_2\}$ in which case, rather than (1.1), we have

$$(1.2) \qquad \varphi = \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{k-2} \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ \mathcal{L}^{-1} & 0 \end{pmatrix} \qquad \mathrm{Fil}^i\, D_{k,\mathcal{L}} = \begin{cases} D_{k,\mathcal{L}} & \text{if } i \leq 0; \\ \overline{\mathbb{Q}}_p \cdot (e_1' + e_2') & \text{if } 1 \leq i \leq k-1; \\ \{0\} & \text{if } k \leq i. \end{cases}$$

Thus, $D_{k,\mathcal{L}} \to D_{k,\infty}$ as $\mathcal{L}^{-1} \to 0$, where $D_{k,\infty}$ is the filtered (φ, N)-module with the same φ and filtration as (1.2) but with $N = 0$. In fact, $D_{k,\infty} \cong D^*_{\mathrm{crys}}(V_{k,\infty})$ where $V_{k,\infty}$ is a 2-dimensional *crystalline* representation of $G_{\mathbb{Q}_p}$ whose Frobenius trace is $a_p = \varpi^{k-2} + \varpi^k$. Replacing the filtered (φ, N)-modules with Galois representations, we have $V_{k,\mathcal{L}} \to V_{k,\infty}$ as $\mathcal{L}^{-1} \to 0$ (see the description in [9, Section 4.5-4.6] in terms of the space of trianguline representations, for instance). Thus, $\overline{V}_{k,\mathcal{L}} \cong \overline{V}_{k,\infty}$ for $\mathcal{L}^{-1} \to 0$. Furthermore, $v_p(a_p) = \frac{k-2}{2}$ and so $\lfloor \frac{k-1}{p} \rfloor < v_p(a_p)$, except if $p = 2$ or k is small, and so $\overline{V}_{k,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$ by [2, Corollary 5.2.3]. We have reduced the theorem to the question: at which point as $\mathcal{L}^{-1} \to 0$, do we have $\overline{V}_{k,\mathcal{L}} \cong \overline{V}_{k,\infty}$?

We recall the relationship between reductions and Kisin modules, now. To ease notations, assume for the remainder of this subsection that k is even and $\mathcal{L} \in \mathbb{Q}_p$, so $V_{k,\mathcal{L}}$ and $V_{k,\infty}$ are defined over $\mathbb{Q}_p$. Let $\mathfrak{S} = \mathbb{Z}_p[\![u]\!]$, and write $\varphi : \mathfrak{S} \to \mathfrak{S}$ for the Frobenius map $\varphi(u) = u^p$. Then, consider the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi, \leq k-1}$ of φ-modules over $\mathfrak{S}$ with height $\leq k-1$ ([15]). Objects in this category, which are called Kisin modules, are finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semilinear operator $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the linearization $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is annihilated by $E(u)^{k-1}$, where $E(u) = u + p$. When $\mathfrak{M}$ satisfies the *monodromy condition*, Kisin's theory constructs a canonical semi-stable representation $V_{\mathfrak{M}}$ such that $D^*_{\mathrm{st}}(V_{\mathfrak{M}}) \cong \mathfrak{M}/u\mathfrak{M}[1/p]$, for a certain filtration and monodromy on the right-hand side. Furthermore, $\overline{V}_{\mathfrak{M}}$ is determined by $\mathfrak{M}/p\mathfrak{M}[u^{-1}]$ as a φ-module over $\mathbb{F}_p(\!(u)\!)$.

The challenge in calculating $\overline{V}_{\mathfrak{M}}$ this way is determining $\mathfrak{M}$ from $V_{\mathfrak{M}}$ or, equivalently, $D^*_{\mathrm{st}}(V_{\mathfrak{M}})$. That task was carried out for $V_{k,\infty}$ in [2, Theorem 5.2.1]. The heart of this article is a two-step argument to do the same for $V_{k,\mathcal{L}}$ as $\mathcal{L}^{-1} \to 0$. The presence of non-trivial monodromy makes our task significantly more delicate than the crystalline case.

First, we make use of a category intermediate between filtered (φ, N)-modules and Kisin modules. Namely, write $\mathrm{Mod}_{S_{\mathbb{Q}_p}}^{\varphi, \leq k-1}$ for the category of φ-modules over $S_{\mathbb{Q}_p} = \mathbb{Z}_p[\![u, \frac{E^p}{p}]\!][\frac{1}{p}]$ with height $\leq k-1$. This category is close to certain filtered (φ, N)-modules considered by Breuil ([5]). Adapting Breuil's work, we explicitly construct a canonical object $\mathcal{M}_{k,\mathcal{L}} \in \mathrm{Mod}_{S_{\mathbb{Q}_p}}^{\varphi, \leq k-1}$ such that if $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi, \leq k-1}$ and $\mathcal{M}_{k,\mathcal{L}} \cong \mathfrak{M} \otimes_{\mathfrak{S}} S_{\mathbb{Q}_p}$, then $V_{\mathfrak{M}} \cong V_{k,\mathcal{L}}$. Explicit means, for any (non-zero) $\mathcal{L}$, we determine a basis of $\mathcal{M}_{k,\mathcal{L}}$ and an exact formula for φ in that basis. This is where we overcome the difficulty of non-trivial monodromy on $D_{k,\mathcal{L}}$.

The second step is to descend $\mathcal{M}_{k,\mathcal{L}}$ from $S_{\mathbb{Q}_p}$ to $\mathfrak{S}$ when $\mathcal{L}^{-1} \to 0$, thus producing an $\mathfrak{M}$ for $V_{k,\mathcal{L}}$. Here, we view $S_{\mathbb{Q}_p}$ as subring of R_2, where R_2 is the ring of p-adic rigid analytic functions on $|u| \leq$

$p^{-1/2}$ (using that $p \neq 2$). Section 4 of [2] presents a row reduction algorithm for semilinear operators that, under certain conditions, can descend from R_2 to $\mathfrak{S}$. Specifically, the main theorem in *loc. cit.* gives a sufficient condition to descend $\mathcal{M}_{k,\mathcal{L}} \otimes_{S_{\mathbb{Q}_p}} R_2$ to $\mathfrak{S}$. Saving the details for later, we use the explicit calculation of $\mathcal{M}_{k,\mathcal{L}}$ to check those conditions are met when $v_p(\mathcal{L}) < 2 - \frac{k}{2} + v_p((k-2)!)$.

Remark 1.3. We exclude $p = 2$ twice. The second time, when we embed $S_{\mathbb{Q}_p}$ into R_2 is likely technical. However, we also exclude $p = 2$ when referencing the calculation of $\overline{V}_{k,\infty}$ in [2], and that seems crucial: our strategy is based not just on knowing $\overline{V}_{k,\infty}$, but also how to construct a Kisin module for $V_{k,\infty}$. Including $p = 2$, here would necessarily require calculating $\overline{V}_{k,\infty}$ when $p = 2$ as well. We note the formula $\overline{V}_{k,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$ should still be true, but we cannot justify it.

1.3. **Global context.** We end this introduction with a discussion of the global situation. Suppose $N \geq 1$ and $f = \sum a_n(f)q^n$ is a cuspidal (normalized) eigenform of level $\Gamma_1(N)$, weight $k \geq 2$, and nebentype character ψ_f. Eichler–Shimura and Deligne famously associated to f a 2-dimensional, irreducible, continuous representation V_f of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We normalize V_f so that for $\ell \nmid Np$ the restriction $V_f|_{D_\ell}$ to D_ℓ, a decomposition group at ℓ, is unramified and the characteristic polynomial of a geometric Frobenius element is $t^2 - a_\ell(f)t + \psi_f(\ell)\ell^{k-1}$. The representation $V_f|_{D_p}$ is semi-stable when $p^2 \nmid N$ and the conductor of ψ_f is prime-to-p; it is crystalline when $p \nmid N$ ([19]).

We assume now that $V_f|_{D_p}$ is semi-stable and non-crystalline, in which case we define the $\mathcal{L}$-invariant of f to be the unique $\mathcal{L}_f \in \overline{\mathbb{Q}}_p$ such that $V_f|_{D_p} \cong V_{k,\mathcal{L}_f}$. The $\mathcal{L}$-invariant defined this way is called the Fontaine–Mazur $\mathcal{L}$-invariant ([13]). It is a local quantity, but it famously arises in global situations. Examining how it arises allows us to provide global examples where Theorem 1.1 applies and to connect $\mathcal{L}$-invariants to global phenomena on p-adic families.

Theorem 1.1 determines $(\overline{V_f}|_{D_p})^{\mathrm{ss}}$ in arbitrary weights $k \geq p$ as long as $v_p(\mathcal{L}_f)$ is sufficiently negative, but it is not immediately obvious that eigenforms exist with $v_p(\mathcal{L}_f)$ so negative. Using the presence of $\mathcal{L}$-invariants in the exceptional zero phenomena for p-adic L-functions, Pollack has written computer code that calculates $\mathcal{L}$-invariants, form-by-form, in level $\Gamma_0(N)$ where p divides N exactly once.[1] In Table 1, we partially list the p-adic valuations Pollack's code found when $p = 3$ and $N = 51 = 3 \cdot 17$. The bound in Theorem 1.1 is $v_3(\mathcal{L}_f) < 0$ in weight $k = 4$ and $v_p(\mathcal{L}_f) < -2$ in

TABLE 1. 3-adic valuations of some $\mathcal{L}$-invariants.

k	$v_3(\mathcal{L}_f)$ for newforms $f \in S_k(\Gamma_0(51))$
4	$-2, -1, 0, 0, \ldots$
6	$-3, -2, -2, -2, -1, \ldots$
8	$-3, -3, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, -1, \ldots$

weight $k = 6$, so Table 1 provides examples where Theorem 1.1 applies in those weights (though not in weight 8). We can also test the boundary case when $k = 6$. In fact, since Pollack's code works

[1]Pollack's code, which requires MAGMA to run, can be found in a github repository https://github.com/rpollack9974/L-invariants.

form-by-form, we can say that among the eigenforms h with $v_3(\mathcal{L}_h) = -2$, two of them satisfy $\overline{V}_h|_{D_p} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi) \otimes \omega$ where ω is the mod 3 cyclotomic character. Thus, Theorem 1.1 does not extend to $v_p(\mathcal{L}_f) = -2$ when $p = 3$ and $k = 6$.

The $\mathcal{L}$-invariants also arise, globally, from p-adic families. Namely, f lives in a p-adic family of eigenforms parametrized by weights $k \in \mathbb{Z}_p$ and $\mathcal{L}_f = -2\,\mathrm{dlog}\, a_p(k) = -2\frac{a_p'(k)}{a_p(f)}$ ([10, Corollarie 0.7]). This appearance reveals an obstruction to the "radius" of the largest "constant slope" family through f. Indeed, for $p \neq 2$, [1, Theorem 4.3] implies $v_p(\mathcal{L}_f^{-1}) \leq m(f)$ where $m(f)$ is the least positive integer such that f lives in a p-adic family of eigenforms f' with $v_p(a_p(f')) = v_p(a_p(f))$ and weight $k' \equiv k \bmod (p-1)p^{m(f)}$. So, ruling out exceptions to Theorem 1.1, we have $v_p(\mathcal{L}_f) < 2 - \frac{k}{2} - v_p((k-2)!)$ implies

- $(\overline{V}_f|_{D_p})^{\mathrm{ss}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$, and
- $m(f) > \frac{k}{2} - 2 + v_p((k-2)!) \approx \frac{k-2}{2} + \frac{k}{p-1}$.

To connect these, if $k \not\equiv 1 \bmod p+1$, then $\overline{V}_f|_{D_p}$ is irreducible. On the other hand, condition (2) generically implies $m(f) > \frac{k-2}{2} = v_p(a_p(f))$. The fact that $m(f) > v_p(a_p(f))$ occurs in a situation where $\overline{V}_f|_{D_p}$ is irreducible is not a coincidence. It follows a pattern of counter-examples to a conjecture of Gouvêa and Mazur, which is related to the $m(f)$, found by Buzzard and Calegari ([7]). See [1, Section 9] for more discussion.

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2. Theoretical background

In this section, we recall filtered (φ, N)-modules and Breuil and Kisin modules. We explain, in theory, how to calculate a finite height φ-module, over a ring larger than $\mathfrak{S}$, associated with a filtered (φ, N)-module (Theorem 2.7). In Section 3, we carry this out in practice in a special case.

2.1. **Notations.** Let k be a finite field and $W(k)$ be the Witt vectors over k. Set $K_0 = W(k)[1/p]$ and assume K/K_0 is a totally ramified extension of degree e. Let Λ_K be the ring of integers of K, $\pi \in \Lambda_K$ a uniformizer and $E = E(u) \in W(k)[u]$ its Eisenstein polynomial. Choosing $\pi_0 = \pi$ and $\pi_1, \pi_2, \ldots$ a sequence in $\overline{K}$ such that $\pi_{i+1}^p = \pi_i$, we let G_∞ be the absolute Galois group of $\varinjlim K(\pi_i)$. Let $\mathcal{O} \subseteq K_0[\![u]\!]$ be the rigid analytic functions on $|u| < 1$ and $\mathfrak{S} = W(k)[\![u]\!] \subseteq \mathcal{O}$. The action of φ on $K_0[\![u]\!]$, by the Frobenius on K_0 and $\varphi(u) = u^p$, preserves $\mathfrak{S} \subseteq \mathcal{O} \subseteq K_0[\![u]\!]$.

We also choose $F/\mathbb{Q}_p$ a finite extension, which will play the role of linear coefficients. In Section 2.4, we assume F contains a subfield isomorphic the Galois closure of K. We write $\Lambda \subseteq F$ for the ring of integers and $\mathbb{F}$ for the residue field. Define $\mathfrak{S}_\Lambda = \mathfrak{S} \otimes_{\mathbb{Z}_p} \Lambda$ and $\mathcal{O}_F = \mathcal{O} \otimes_{K_0} F$. Extending φ linearly, we have φ-stable subrings of $\mathfrak{S}_\Lambda \subseteq S_F \subseteq (K_0 \otimes_{\mathbb{Q}_p} F)[\![u]\!]$, where $S_F = \mathfrak{S}[\![\frac{E^p}{p}]\!] \otimes_{\mathbb{Q}_p} F$.

2.2. **Kisin modules.** Let $R \subseteq (K_0 \otimes_{\mathbb{Q}_p} F)[\![u]\!]$ be a φ-stable subring containing E. A φ-module over R is a finite free R-module M equipped with an injective φ-semilinear operator $\varphi_M : M \to M$. Let Mod_R^φ be the category of φ-modules over R with morphisms being R-linear maps that commute with φ. For a φ-module M, write $\varphi^* M = R \otimes_{\varphi,R} M$, so $1 \otimes \varphi_M$ defines an R-linear map $\varphi^* M \to M$ called the linearization of φ. For $h \geq 0$, an element $M \in \mathrm{Mod}_R^\varphi$ has (E)-height $\leq h$ if its linearization has cokernel annihilated by E^h. The subcategory of φ-modules over R with height $\leq h$ is denoted $\mathrm{Mod}_R^{\varphi,\leq h}$. A *Kisin module* over $\mathfrak{S}_\Lambda$ with height $\leq h$ is an object in $\mathrm{Mod}_{\mathfrak{S}_\Lambda}^{\varphi,\leq h}$.

Let $\mathrm{MF}_F^{\varphi,N}$ be the category of *positive* filtered (φ, N, K, F)-modules, which we shorten to just filtered (φ, N)-modules over F (see [6, Section 3.1.1]). For $D \in \mathrm{MF}_F^{\varphi,N}$ set $D_K = K \otimes_{K_0} D$; here, positive means $\mathrm{Fil}^0 D_K = D_K$. Let $\mathrm{Rep}_F^{\mathrm{st},h}$ be the category of F-linear semi-stable representations V of G_K whose Hodge-Tate weights lie in $\{0, \dots, h\}$. Then, there exists a fully faithful, contravariant, functor

$$D_{\mathrm{st}}^* : \mathrm{Rep}_F^{\mathrm{st},h} \to \mathrm{MF}_F^{\varphi,N}$$

whose image is the subcategory of weakly-admissible filtered (φ, N)-modules over F (see [12, 11] and [6, Corollaire 3.1.1.3]). For $V \in \mathrm{Rep}_F^{\mathrm{st},h}$ and $T \subseteq V$ a G_∞-stable and Λ-linear lattice there exists, by [17, Theorem 5.4.1], a canonical Kisin module $\mathfrak{M} = \mathfrak{M}(T)$ over $\mathfrak{S}_\Lambda$ with height $\leq h$. Any choice of $\mathfrak{M}$ determines the semi-simple mod p reduction $\overline{V} = (T/\mathfrak{m}_F T)^{\mathrm{ss}}$ ([2, Corollary 2.3.2]).

One category that intervenes in determining an $\mathfrak{M}$ associated with V is the category of (φ, N_∇)-modules over $\mathcal{O}_F$ ([15]). Let $\lambda = \prod_{n\geq 0} \varphi^n(E(u)/E(0)) \in \mathcal{O}_F$. An object $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$ is a finite height φ-module over $\mathcal{O}_F$ equipped with a differential operator N_∇ lying over $-u\lambda\frac{d}{du}$ on $\mathcal{O}_F$ and satisfying $N_\nabla \varphi = p\frac{E(u)}{E(0)}\varphi N_\nabla$. By [15, Theorem 1.2.15], we have quasi-inverse equivalences of categories

$$\mathrm{MF}_F^{\varphi,N} \underset{\underline{\mathcal{M}}_{\mathcal{O}_F}}{\overset{\underline{D}_{\mathcal{O}_F}}{\leftrightarrows}} \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}. \tag{2.1}$$

For $s > 0$, write $\mathcal{O}_s$ for the $\mathcal{O}_F$-algebra of rigid analytic functions converging on $|u| < p^{-s}$.

Proposition 2.1. *Suppose* $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\Lambda}^{\varphi,\leq h}$, $V \in \mathrm{Rep}_F^{\mathrm{st},h}$, *and* s *is such that* $1/pe < s < 1/e$ *and* $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_s \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{\mathrm{st}}^*(V)) \otimes_{\mathcal{O}_F} \mathcal{O}_s$ *in* $\mathrm{Mod}_{\mathcal{O}_s}^{\varphi,\leq h}$. *Then,* $\mathfrak{M} = \mathfrak{M}(T)$ *for some* $T \subseteq V$ *as above.*

Proof. Since $s < 1/e$, π lies in the disc $|u| < p^{-s}$. Since $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_s \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{\mathrm{st}}^*(V)) \otimes_{\mathcal{O}_F} \mathcal{O}_s$, [2, Corollary 2.2.5] implies that $\mathcal{M}_{\mathcal{O}_F} := \mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_F$ is canonically an object in $\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$. Then, [17, Theorem 5.4.1] implies that there exists a $V' \in \mathrm{Rep}_F^{\mathrm{st},h}$ such that $\mathfrak{M} = \mathfrak{M}(T)$ for a lattice $T \subseteq V'$ for some T. We claim that $V \cong V'$. Indeed, since $1/pe < s < 1/e$, the definition of $\underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F})$ in

[15, Section 1.2.5-7] depends only the finite height φ-module $\mathcal{M}_{\mathcal{O}_F} \otimes_{\mathcal{O}_F} \mathcal{O}_s$ over $\mathcal{O}_s$. Thus, we have

$$D^*_{\mathrm{st}}(V') \cong \underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) \cong \underline{D}_{\mathcal{O}_F}(\underline{\mathcal{M}}_{\mathcal{O}_F}(D^*_{\mathrm{st}}(V))) \cong D^*_{\mathrm{st}}(V).$$

Since D^*_{st} is fully faithful, we have $V \cong V'$, completing the proof. □

Remark 2.2. To be accurate, the equivalence (2.1) is constructed in [15] only when $F = \mathbb{Q}_p$. We use multiple references with the same technical limitation. We pause to detail one approach to resolving the issue. Later, we omit details for other functors.

First, we may define the functors $\underline{D}_{\mathcal{O}_F}$ and $\underline{\mathcal{M}}_{\mathcal{O}_F}$ using the same formulas as (2.1), or, equivalently, we can define them by forcing the diagram

$$\begin{array}{ccc} \mathrm{MF}_F^{\varphi,N} & \underset{\underline{\mathcal{M}}_{\mathcal{O}_F}}{\overset{\underline{D}_{\mathcal{O}_F}}{\rightleftarrows}} & \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla} \\ {\scriptstyle\text{forget}}\downarrow & & \downarrow{\scriptstyle\text{forget}} \\ \mathrm{MF}_{\mathbb{Q}_p}^{\varphi,N} & \underset{\underline{\mathcal{M}}_{\mathcal{O}}}{\overset{\underline{D}_{\mathcal{O}}}{\rightleftarrows}} & \mathrm{Mod}_{\mathcal{O}}^{\varphi,N\nabla} \end{array}$$

to commute. Here, the vertical arrows are the natural forgetful functors and the bottom arrows are as in [15], where they proved to be quasi-inverses. If $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla}$, we thus have a natural isomorphism $\alpha : \underline{\mathcal{M}}_{\mathcal{O}_F}(\underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F})) \cong \mathcal{M}_{\mathcal{O}_F}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi,N\nabla}$. Since multiplication by $x \in F$ defines an endomorphism of $\mathcal{M}_{\mathcal{O}_F}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi,N\nabla}$ and α is natural, we see α is an isomorphism in $\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla}$. Thus, $\underline{\mathcal{M}}_{\mathcal{O}_F}$ is a left quasi-inverse to $\underline{D}_{\mathcal{O}_F}$. Proving $\underline{D}_{\mathcal{O}_F}$ is a right quasi-inverse to $\underline{\mathcal{M}}_{\mathcal{O}_F}$ is analogous.

2.3. **Breuil modules.** Let S_{Br} be the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by E. Breuil ([5]) classically identified $\mathrm{MF}_{\mathbb{Q}_p}^{\varphi,N}$ with a category of filtered (φ, N)-modules over $S_{\mathrm{Br}}[\frac{1}{p}]$. We recall this, replacing S_{Br} with a simpler ring.

One can the extend Frobenius φ to $K_0[\![u]\!]$ via $\varphi(u) = u^p$. We define $N = -u\frac{d}{du}$ on $K_0[\![u]\!]$. Let $\widehat{S}_E$ be the E-completion of $S_{\mathrm{Br}}[\frac{1}{p}]$. For a subring $R \subseteq \widehat{S}_E$ and $j \geq 0$, set $\mathrm{Fil}^j R = R \cap E^j \hat{S}_E$. In particular, we can take $R = S := W(k)[\![u, \frac{E^p}{p}]\!]$. As a subring of $K_0[\![u]\!]$, S is closed under φ and N. We define $S_\Lambda = S \otimes_{\mathbb{Z}_p} \Lambda$ and $S_F = S \otimes_{\mathbb{Z}_p} F$, extending φ, N, and $\mathrm{Fil}^\bullet$ linearly. Thus, $\mathrm{Fil}^j S_F = E^j S_F$. This is one advantage S enjoys over S_{Br}. Note as well: S_F is an $\mathcal{O}_F$-algebra and $\varphi(E) = p\mathfrak{c}$ with $\mathfrak{c} \in S^\times$. In particular, $\varphi(\lambda) \in S^\times \subseteq S_F^\times$.

The category $\mathrm{MF}_{S_F}^{\varphi,N}$ of filtered (φ, N)-modules over S_F, or *Breuil modules* over S_F, are objects $(\mathcal{D}, \varphi_{\mathcal{D}}) \in \mathrm{Mod}_{S_F}^{\varphi}$ such that the linearization of $\varphi_{\mathcal{D}}$ is an isomorphism, and $\mathcal{D}$ is equipped with:

- a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{D}$ by S_F-submodules such that $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and $\mathrm{Fil}^i S_F \cdot \mathrm{Fil}^j \mathcal{D} \subseteq \mathrm{Fil}^{i+j} \mathcal{D}$ for all $i, j \geq 0$;
- an operator $N_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ that acts as a derivation over N, and
  - $N_{\mathcal{D}}\varphi_{\mathcal{D}} = p\varphi_{\mathcal{D}}N_{\mathcal{D}}$, and
  - $N_{\mathcal{D}}(\mathrm{Fil}^i \mathcal{D}) \subseteq \mathrm{Fil}^{i-1} \mathcal{D}$ for all $i \geq 1$.

A morphism in $\mathrm{MF}_{S_F}^{\varphi,N}$ is an S_F-linear map equivariant for φ, N, and $\mathrm{Fil}^\bullet$.

We define a functor $\underline{\mathcal{D}} : \mathrm{MF}_F^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$ as follows:

- $\mathcal{D} := \underline{\mathcal{D}}(D) = S_F \otimes_{K_0 \otimes_{\mathbb{Q}_p} F} D$ as an S_F-module;
- $\varphi_{\mathcal{D}} = \varphi \otimes \varphi_D$;
- $N_{\mathcal{D}} = N \otimes 1 + 1 \otimes N_D$;
- $\mathrm{Fil}^0(\mathcal{D}) = \mathcal{D}$ and

$$\mathrm{Fil}^i(\mathcal{D}) = \{x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{i-1}\mathcal{D} \text{ and } (\mathrm{ev}_\pi \otimes 1)(x) \in \mathrm{Fil}^i D_K\}$$

for $i \geq 1$.

Here, $\mathrm{ev}_\pi : S_F \to F \otimes_{\mathbb{Q}_p} K$ is the scalar extension of $\mathrm{ev}_\pi : W(k)[u] \twoheadrightarrow \Lambda_K$, the evaluation at π map.

Theorem 2.3 (Breuil)**.** *The functor $\underline{\mathcal{D}} : \mathrm{MF}_F^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$ is an equivalence of categories.*

Breuil proves in [5, Section 6] that $\underline{\mathcal{D}}$ is an equivalence of categories when $F = \mathbb{Q}_p$ and S is replaced by S_{Br}. That one can replace S_{Br} by S is known to some, but there does not appear to be a reference. The only step in the proof of Breuil that requires honestly new justification is the following analogue of [5, Proposition 6.2.1.1]. (This version is even easier to prove.)

Lemma 2.4. *Let $\mathcal{D} \in \mathrm{MF}_{S_F}^{\varphi,N}$ and $D = \mathcal{D}/u\mathcal{D}$. Then, there exists a unique $F \otimes_{\mathbb{Q}_p} K_0$-linear φ-equivariant section $s : D \to \mathcal{D}$ of the reduction map.*

Proof. First, suppose $F = \mathbb{Q}_p$ and let $(\widehat{e}_1, \ldots, \widehat{e}_d)$ be an $S[\frac{1}{p}]$-basis of $\mathcal{D}$. Write $\varphi_{\mathcal{D}}(\widehat{e}_1, \ldots, \widehat{e}_d) = (\widehat{e}_1, \ldots, \widehat{e}_d)X$ and set $X_0 = X \bmod u$. Then, $X \in p^k \mathrm{Mat}_d(S)$, $X_0^{-1} \in p^\ell \mathrm{Mat}_d(W(k))$, and $XX_0^{-1} \in I + up^m \mathrm{Mat}_d(S)$ for some $k, \ell, m \in \mathbb{Z}$. As in the proof of [5, Proposition 6.2.1.1], we need to show

$$Y_n := X\varphi(X) \cdots \varphi^n(X)\varphi^n(X_0^{-1}) \cdots \varphi(X_0^{-1})X_0^{-1}$$

converges in $\mathrm{Mat}_d(S[\frac{1}{p}])$ as $n \to \infty$. But, in the notation above,

$$Y_n - Y_{n-1} \in \varphi^n(u)p^{n(k+\ell)+m} \mathrm{Mat}_d(S).$$

Since $\varphi^n(u)p^{nr} \to 0$ in $S[\frac{1}{p}]$ for any fixed r, we see that $Y_n - Y_{n-1} \to 0$ in $\mathrm{Mat}_d(S[\frac{1}{p}])$, as needed.

If $F \neq \mathbb{Q}_p$, the proof already given implies there exists a unique K_0-linear φ-equivariant section $s : D \to \mathcal{D}$. If $x \in F^\times$ then $x^{-1}sx$ also K_0-linear and φ-equivariant and thus s is F-linear. □

Proof of Theorem 2.3. Define $\underline{D}_{S_F} : \mathrm{MF}_{S_F}^{\varphi,N} \to \mathrm{MF}_F^{\varphi,N}$ as follows. Set $D = \underline{D}_{S_F}(\mathcal{D}) = \mathcal{D}/u\mathcal{D}$ with its induced action of φ and N. For s in Lemma 2.4, $(\mathrm{ev}_\pi \otimes 1) \circ s : D \to \mathcal{D}/E\mathcal{D}$ induces a canonical isomorphism $D_K \cong \mathcal{D}/E\mathcal{D}$. The filtration $\mathrm{Fil}^i(D_K)$ is the pullback of the filtration on $\mathcal{D}/E\mathcal{D}$ defined as the image $\mathrm{Fil}^i(\mathcal{D}) \to \mathcal{D}/E\mathcal{D}$. The arguments in [5], with Lemma 2.4 replacing Proposition 6.2.1.1 of *loc. cit.*, show $\underline{D}_{S[\frac{1}{p}]}$ and $\underline{\mathcal{D}}$ are quasi-inverses when $F = \mathbb{Q}_p$. In general, see Remark 2.2. □

2.4. **Comparison.** We now assume that F contains a subfield isomorphic to the Galois closure of K (see Lemma 2.5). In practice, as in Sections 3 and 4, we take $K = \mathbb{Q}_p$ so this is no hindrance.

In the prior sections, we have described equivalences

$$\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla} \xrightarrow{\cong} \mathrm{MF}_F^{\varphi,N} \xrightarrow{\cong} \mathrm{MF}_{S_F}^{\varphi,N}. \tag{2.2}$$

An analogue of [18, Corollary 3.2.3] allows for a description of the composition that, unfortunately, is not practical for calculations. Below, though, we explain how to determine $\underline{\mathcal{M}}_{\mathcal{O}_F}(D) \otimes_{\mathcal{O}_F} S_F$ as a φ-module over S_F from D, up to determining $\mathcal{D} = \underline{\mathcal{D}}(D)$. A key technical point, which follows from the next lemma, is that filtrations on Breuil modules over S_F are always free, in contrast to the filtrations on objects in $\mathrm{MF}_F^{\varphi,N}$ (cf. [6, Exemple 3.1.1.4]).

Lemma 2.5. *Suppose that $\mathcal{N}$ is a finite free S_F-module and $\mathcal{H} \subseteq \mathcal{N}$ is an S_F-submodule such that $E^j\mathcal{N} \subseteq \mathcal{H}$ for some $j \geq 0$. Then, $\mathcal{H}$ is finite free over S_F.*

Proof. We may assume $j = 1$. Indeed, consider the nested sequence $\mathcal{H}_i = \mathcal{H} + E^i\mathcal{N}$ of S_F-modules, which satisfy $E\mathcal{H}_i \subseteq \mathcal{H}_{i+1} \subseteq \mathcal{H}_i$. By the $j = 1$ case we deduce $\mathcal{H}_1 \subseteq \mathcal{N}$ is free, and then $\mathcal{H}_2$, and so on until $\mathcal{H}_j = \mathcal{H}$ is free. We may also assume $\mathcal{N} \cong S_F$. Indeed, if $0 \to \mathcal{N}'' \to \mathcal{N} \xrightarrow{f} \mathcal{N}' \to 0$ is an exact sequence of finite free S_F-modules, then $\mathcal{H}' = f(\mathcal{H})$ and $\mathcal{H}'' = \ker(f) \cap \mathcal{H}$ satisfy $E\mathcal{N}'' \subseteq \mathcal{H}''$ and $E\mathcal{N}' \subseteq \mathcal{H}'$. So, if both $\mathcal{H}''$ and $\mathcal{H}'$ are free, then $\mathcal{H} \cong \mathcal{H}'' \oplus \mathcal{H}'$ is free as well.

We have reduced to proving: if $I \subseteq S_F$ is an ideal containing E, then I is free. Since F contains a subfield isomorphic to the Galois closure of K, we may decompose $S_F = \prod_{\sigma \in \mathrm{Hom}(K_0,F)} S_{F,\sigma}$ where $S_{F,\sigma} = \Lambda[\![u, \frac{\sigma(E)^p}{p}]\!][\frac{1}{p}]$ is a domain. The ideal I decomposes as a product of ideals I_σ such that $\sigma(E)S_{F,\sigma} \subseteq I_\sigma$. Since $\sigma(E)$ is non-zero, it suffices to show each I_σ is principal. Write $\mathrm{Hom}_\sigma(K, F)$ for the embeddings $\tau : K \to F$ lifting σ. Then, we have a canonical isomorphism

$$S_{F,\sigma}/\sigma(E)S_{F,\sigma} \cong K \otimes_{K_0,\sigma} F \cong F^{\mathrm{Hom}_\sigma(K,F)}.$$

So, $I_\sigma/\sigma(E)S_{F,\sigma} \cong F^T$ for some subset $T \subseteq \mathrm{Hom}_\sigma(K, F)$. But, $J_T = \prod_{\tau \in T}(u - \tau(\pi)) \cdot S_F$ also contains $\sigma(E)S_{F,\sigma}$ and $J_T/\sigma(E)S_{F,\sigma} \cong F^T$. Thus $I_\sigma = J_T$ is principal, completing the proof. ☐

We now consider an *ad hoc* category of “Breuil modules without monodromy”. Let $\mathrm{MF}_{S_F}^{\varphi,h}$ denote the category whose objects are $(\mathcal{D}, \varphi_{\mathcal{D}}) \in \mathrm{Mod}_{S_F}^{\varphi}$ such that the linearization of $\varphi_{\mathcal{D}}$ is an isomorphism, and $\mathcal{D}$ is equipped with a finite *free* S_F-submodule $\mathrm{Fil}^h \mathcal{D} \subseteq \mathcal{D}$ such that $\mathrm{Fil}^h S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^h \mathcal{D}$. By Lemma 2.5 there is a natural forgetful functor $\mathrm{MF}_{S_F}^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,h}$.

Now define $\underline{\mathcal{D}}' : \mathrm{Mod}_{S_F}^{\varphi,\leq h} \to \mathrm{MF}_{S_F}^{\varphi,h}$ by declaring $\underline{\mathcal{D}}'(\mathcal{M}) = S_F \otimes_{\varphi,S_F} \mathcal{M}$ as an S_F-module, and

- $\varphi_{\underline{\mathcal{D}}'(\mathcal{M})} = \varphi \otimes \varphi_{\mathcal{M}}$, and
- $\mathrm{Fil}^h \underline{\mathcal{D}}'(\mathcal{M}) = \{x \in \underline{\mathcal{D}}'(\mathcal{M}) \mid (1 \otimes \varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^h S_F \cdot \mathcal{M}\}$.

Since $E^h\underline{\mathcal{D}}'(\mathcal{M}) \subseteq \mathrm{Fil}^h \underline{\mathcal{D}}'(\mathcal{M})$, Lemma 2.5 implies $\mathrm{Fil}^h \underline{\mathcal{D}}'(\mathcal{M})$ is finite free over S_F.

Proposition 2.6. *The functor $\underline{\mathcal{D}}'$ is an equivalence.*

Proof. We first show $\underline{\mathcal{D}}'$ is fully faithful. Suppose $\mathcal{M}$ and $\mathcal{M}'$ are in $\mathrm{Mod}_{S_F}^{\varphi,\leq h}$. Write $\mathcal{D} := \underline{\mathcal{D}}'(\mathcal{M})$ and $\mathcal{D}' := \underline{\mathcal{D}}'(\mathcal{M}')$. Choose a basis $(e_1, \ldots, e_d)$ of $\mathcal{M}$ and write $\varphi_{\mathcal{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$ with $A \in \mathrm{Mat}_d(S_F)$. Since $\mathcal{M}$ has height $\leq h$, there exists a matrix $B \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^hI_d$. By assumption, $\mathrm{Fil}^h \mathcal{D}$ has basis $(\alpha_1, \ldots, \alpha_d) = (\widetilde{e}_1, \ldots, \widetilde{e}_d)B$ where $\widetilde{e}_i = 1 \otimes e_i \in \mathcal{D}$ compose a basis of $\mathcal{D}$. Similarly, we get A', B' and $\widetilde{e}_i'$ from a basis $(e_1', \ldots, e_{d'}')$ of $\mathcal{M}'$.

Now suppose $f : \mathcal{D} \to \mathcal{D}'$ is a morphism in $\mathrm{MF}_{S_F}^{\varphi,h}$. We write $f(\widetilde{e}_1, \ldots, \widetilde{e}_d) = (\widetilde{e}_1', \ldots, \widetilde{e}_{d'}')X$ for $X \in \mathrm{Mat}_d(S_F)$. Since f is φ-equivariant, we have $X\varphi(A) = \varphi(A')\varphi(X)$, and, since $f(\mathrm{Fil}^h \mathcal{D}) \subseteq$

$\mathrm{Fil}^h \mathcal{D}'$, we have $XB = B'Y$ for some $Y \in \mathrm{Mat}_d(S_F)$. Using $AB = BA = E^h I_d$ and $A'B' = B'A' = E^h I_{d'}$, we see $\varphi(Y)\varphi(E^h) = X\varphi(E^h)$, and so $X = \varphi(Y)$ because $\varphi(E) \in S_F^\times$. It follows that $YA = A'\varphi(Y)$. Define $\mathfrak{f} : \mathcal{M} \to \mathcal{M}'$ by $\mathfrak{f}(e_1, \dots, e_d) = (e'_1, \dots e'_{d'})Y$. Then, $\mathfrak{f}$ is φ-equivariant and $f = \underline{\mathcal{D}}'(\mathfrak{f})$ since $X = \varphi(Y)$. This shows $\underline{\mathcal{D}}'$ is full, and since Y determines X, we also see $\underline{\mathcal{D}}'$ is faithful.

Now we prove $\underline{\mathcal{D}}'$ is essentially surjective. Given a $\mathcal{D} \in \mathrm{MF}_{S_F}^{\varphi,h}$, choose bases $(e_1, \dots, e_d)$ of $\mathcal{D}$ and $(\alpha_1, \dots, \alpha_d)$ of $\mathrm{Fil}^h \mathcal{D}$. Write $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)B$ and $\varphi_{\mathcal{D}}(e_1, \dots, e_d) = (e_1, \dots, e_d)X$ with $\det(X) \in S_F^\times$. Since $E^h\mathcal{D} \subseteq \mathrm{Fil}^h \mathcal{D}$, there exists $A \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^h I_d$. Since $\varphi(E) = p\mathfrak{c} \in S_F^\times$, we see $X\varphi(B) \in \mathrm{GL}_d(S_F)$, whereas $\varphi_{\mathcal{D}}(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)X\varphi(B)$. Thus $(f_1, \dots, f_d) = (e_1, \dots, e_d)X\varphi(B)p^{-h}\mathfrak{c}^{-h}$ is a basis of $\mathcal{D}$ and $\varphi_{\mathcal{D}}(\alpha_1, \dots, \alpha_d) = (f_1, \dots, f_d)p^h\mathfrak{c}^h$. Finally, $(\alpha_1, \dots, \alpha_d) = (f_1, \dots, f_d)B'$ where $B' = YB$ and $Y = (X\varphi(B)p^{-h}\mathfrak{c}^{-h})^{-1}$, so there exists an A' such that $A'B' = B'A' = E^h I_d$. Now define $\mathcal{M} = \bigoplus_{i=1}^d S_F\mathfrak{f}_i$ and set $\varphi_{\mathcal{M}}(\mathfrak{f}_1, \dots, \mathfrak{f}_d) = (\mathfrak{f}_1, \dots, \mathfrak{f}_d)A'$. Then, $\mathcal{M} \in \mathrm{Mod}_{S_F}^{\varphi,\leq h}$ and $\underline{\mathcal{D}}'(\mathcal{M}) = \mathcal{D}$ (set $f_i = 1 \otimes \mathfrak{f}_i$). □

We now reach the main theorem of this section, which provides a mechanism to calculate a finite height φ-module over S_F explicitly from $D \in \mathrm{MF}_F^{\varphi,N}$. We write $\varphi(E) = p\mathfrak{c}$ with $\mathfrak{c} \in S^\times$ as above.

Theorem 2.7. *Suppose $D \in \mathrm{MF}_F^{\varphi,N}$. Write $\mathcal{D}' \in \mathrm{MF}_{S_F}^{\varphi,h}$ for the image of $\underline{\mathcal{D}}(D)$ under the natural forgetful functor and $\mathcal{M} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D) \otimes_{\mathcal{O}_F} S_F$. Then, there is an natural isomorphism $\underline{\mathcal{D}}'(\mathcal{M}) \cong \mathcal{D}'$.*

In particular, $\mathcal{M}$ is recovered from D via the following steps:

(1) Select S_F-bases $(e_1, \dots, e_d)$ of $\mathcal{D} = \underline{\mathcal{D}}(D)$ and $(\alpha_1, \dots, \alpha_d)$ of $\mathrm{Fil}^h \mathcal{D}$.
(2) Write $\varphi_{\mathcal{D}}(e_1, \dots, e_d) = (e_1, \dots, e_d)X$ and $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)B$ with $X, B \in \mathrm{Mat}_d(S_F)$.
(3) Then, $\mathcal{M}$ has an S_F-basis $(\mathfrak{f}_1, \dots, \mathfrak{f}_d)$ in which $\varphi_{\mathcal{M}}(\mathfrak{f}_1, \dots, \mathfrak{f}_d) = (\mathfrak{f}_1, \dots, \mathfrak{f}_d)A$, where

$$A = E^h B^{-1} X\varphi(B)p^{-h}\mathfrak{c}^{-h}.$$

Proof. To start, once the isomorphism $\underline{\mathcal{D}}'(\mathcal{M}) \cong \mathcal{D}'$ is justified, the "in particular" follows by tracing through the second half of the proof of Proposition 2.6.

For $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$ we define $\mathcal{D} = \underline{\mathcal{D}}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) = S_F \otimes_{\varphi,\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}$, which is a finite free S_F-module, and equip it with the following structure of a Breuil module over S_F:

- $\varphi_{\mathcal{D}} = \varphi \otimes \varphi_{\mathcal{M}}$;
- $N_{\mathcal{D}} = N \otimes 1 + \frac{p}{\varphi(\lambda)} \otimes N_\nabla$;
- $\mathrm{Fil}^i(\mathcal{D}) = \{x \in \mathcal{D} \mid (1 \otimes \varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^i S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}\}$.

Following the proof of [17, Proposition 3.2.1], replacing S by S_{Br} and adding linear F-coefficients, we see $\underline{\mathcal{D}}_{\mathcal{O}_F} : \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla} \to \mathrm{MF}_{S_F}^{\varphi,N}$ defines a functor. Moreover, if $\mathcal{M}_{\mathcal{O}_F}$ has height $\leq h$, then

$$\underline{\mathcal{D}}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) \cong \underline{\mathcal{D}}'(\mathcal{M}_{\mathcal{O}_F} \otimes_{\mathcal{O}_F} S_F)$$

in the category $\mathrm{MF}^{\varphi,h}_{S_F}$. Thus, it remains to show that $\underline{\mathcal{D}}_{\mathcal{O}_F}$ makes the diagram of functors

$$\begin{array}{ccc} \mathrm{MF}^{\varphi,N}_{F} & \xrightarrow{\mathcal{D}} & \mathrm{MF}^{\varphi,N}_{S_F} \\ \uparrow{\scriptstyle \underline{\mathcal{D}}_{\mathcal{O}_F}} & \nearrow{\scriptstyle \underline{\mathcal{D}}_{\mathcal{O}_F}} & \\ \mathrm{Mod}^{\varphi,N\nabla}_{\mathcal{O}_F} & & \end{array} \tag{2.3}$$

commute as well. (In particular, $\underline{\mathcal{D}}_{\mathcal{O}_F}$ is an equivalence.) It is enough to check this when $F = \mathbb{Q}_p$ (by Remark 2.2). In that case, if S is replaced by S_{Br}, this is the statement of [17, Corollary 3.2.3]. The proof in *loc. cit.* goes through here with only one adjustment. Namely, the isomorphism $S_{\mathrm{Br}}[\frac{1}{p}]\otimes_{K_0}\underline{D}_{\mathcal{O}}(\mathcal{M}_{\mathcal{O}}) \cong S_{\mathrm{Br}}[\frac{1}{p}]\otimes_{\varphi,\mathcal{O}}\mathcal{M}_{\mathcal{O}}$ implicit in the first two displayed equations of *loc. cit.* needs to have S_{Br} replaced by S. To make this adjustment, consider the map $\xi : \mathcal{O}\otimes_{K_0}\underline{D}(\mathcal{M}_{\mathcal{O}}) \to \mathcal{M}_{\mathcal{O}}$ constructed in [15, Lemma 1.2.6]. Thus ξ is a φ-equivariant injection with cokernel annihilated by λ^h for some $h \geq 0$. From the diagram in the middle of the proof of *loc. cit.* we have ξ factors

$$\begin{array}{ccc} \mathcal{O}\otimes_{K_0}\underline{D}(\mathcal{M}_{\mathcal{O}}) & \xrightarrow{\xi} & \mathcal{M}_{\mathcal{O}} \\ \downarrow & \nearrow{\scriptstyle 1\otimes\varphi} & \\ \mathcal{O}\otimes_{\varphi,\mathcal{O}}\mathcal{M}_{\mathcal{O}}. & & \end{array} \tag{2.4}$$

We deduce the vertical arrow in (2.4) has cokernel annihilated by $\varphi(\lambda)^h$. Since $\varphi(\lambda) \in S^{\times}$, we have

$$S[1/p]\otimes_{K_0}\underline{D}(\mathcal{M}_{\mathcal{O}}) \overset{1\otimes\xi}{\cong} S[1/p]\otimes_{\varphi,\mathcal{O}}\mathcal{M}_{\mathcal{O}}.$$

This completes the proof. □

Remark 2.8. The above proof makes it clear to see that for $D \in \mathrm{MF}^{\varphi,N}_{F}$ and $\mathcal{D} = \underline{\mathcal{D}}(D) \in MF^{\varphi,N}_{S_F}$, the map ev_π induces an isomorphism $\mathrm{Fil}^{i+1}\mathcal{D}/E\,\mathrm{Fil}^i\mathcal{D} \cong \mathrm{Fil}^{i+1} D_K$. Indeed, since $\mathrm{ev}_\pi(\mathrm{Fil}^{i+1}\mathcal{D}) = \mathrm{Fil}^{i+1} D_K$, it suffices to show that $E\mathcal{D}\cap\mathrm{Fil}^{i+1}\mathcal{D} = E\,\mathrm{Fil}^i\mathcal{D}$. Pick $y = Ex \in \mathrm{Fil}^{i+1}\mathcal{D}$ with $x \in \mathcal{D}$. The proof of the theorem, especially the fact that (2.3) commutes, shows that

$$\mathrm{Fil}^{i+1}(\mathcal{D}) = \{x \in \mathcal{D} \mid (1\otimes\varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^{i+1} S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}\}.$$

Thus, we see that $(1\otimes\varphi_{\mathcal{M}})(Ex) = E(1\otimes\varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^{i+1} S_F\otimes_{\mathcal{O}_F}\mathcal{M}_{\mathcal{O}_F}$. Since $\mathrm{Fil}^n S_F = E^n S_F$, it is clear that $(1\otimes\varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^i S_F\otimes_{\mathcal{O}_F}\mathcal{M}_{\mathcal{O}_F}$ and hence $x \in \mathrm{Fil}^i\mathcal{D}$ as required. (Compare with the end of the proof of [18, Proposition 3.2.1].)

Example 2.9. Suppose $K = \mathbb{Q}_p$ and V is crystalline. By [16], $D = D^*_{\mathrm{st}}(V)$ admits a strongly divisible lattice $(M, \mathrm{Fil}^i M, \varphi_i)$. More precisely, there exists an F-basis $(e_1,\dots,e_d)$ of D and integers $0 = n_0 \leq n_1 \leq \cdots \leq n_h \leq d$ such that $\mathrm{Fil}^i D := \bigoplus_{j\geq n_i} Fe_j$, and $\varphi(e_1,\dots,e_d) = (e_1,\dots,e_d)XP$ where $X \in \mathrm{GL}_d(\Lambda)$ and P is a diagonal matrix whose ii-th entry is p^{s_i} where $s_i = \max\{j \mid n_j \leq i\} = \max\{j \mid e_i \in \mathrm{Fil}^j D\}$. Since $N = 0$ on D, we easily compute that $\mathrm{Fil}^h\mathcal{D}$ admits a basis $(e_1,\dots,e_d)B$ where B is the diagonal matrix with (i,i)-th entry is E^{h-s_i} (cf. Section 3.1 below). By the steps outlined in Theorem 2.7 , using the basis $1\otimes e_i \in \mathcal{D}$ we see the matrix of φ on $\mathcal{M}$ is given by $A = E^h B^{-1} X P \varphi(B) p^{-h} \mathfrak{c}^{-h}$, where $A = \Lambda X C$, and Λ is a diagonal matrix with (i,i)-th entry is E^{s_i} and C is a diagonal matrix with (i,i)-th entry is $\mathfrak{c}^{-s_i}$.

3. An explicit determination of a Breuil module

In this section, we assume $K = \mathbb{Q}_p$. We choose $\pi = -p$, so $E(u) = u + p$. We keep $F/\mathbb{Q}_p$ as a linear coefficient field and recall Λ is its ring of integers. In Section 3.2, we explain the definition of the filtered (φ, N)-module $D_{h+1,\mathcal{L}} \in \mathrm{MF}_F^{\varphi,N}$, for $h \geq 1$ and $\mathcal{L} \in F$, discussed in the introduction. Let $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F \in \mathrm{Mod}_{S_F}^{\varphi,\leq h}$. The ultimate goal (Theorem 3.7) is to describe the matrix of φ in a certain trivialization $\mathcal{M}_{h+1,\mathcal{L}} \cong S_F^{\oplus 2}$, at least if $\mathcal{L} \neq 0$. We begin by describing the the Breuil module $\mathcal{D}_{h+1,\mathcal{L}} = \underline{\mathcal{D}}(D_{h+1,\mathcal{L}})$.

3.1. The filtration on some rank 2 Breuil modules.

In order to minimize notation, in this subsection, we let $D \in \mathrm{MF}_F^{\varphi,N}$ be any 2-dimensional filtered (φ, N)-module with Hodge–Tate weights $0 < h$. We also choose any basis $\{f_1, f_2\}$ for D such that $\mathrm{Fil}^h D = F f_2$. We write $N_D((f_1, f_2)) = (f_1, f_2)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_2(F)$. (Compare with Lemma 3.6.)

Set $\mathcal{D} = \underline{\mathcal{D}}(D) = S_F \otimes_F D$. For $f \in D$ we write $\widehat{f} = 1 \otimes f \in \mathcal{D}$. In particular, $\mathcal{D}$ is a free S_F-module with basis $\{\widehat{f}_1, \widehat{f}_2\}$. Recall that $\mathrm{Fil}^i \mathcal{D}$ is defined by $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and, for $i \geq 1$,

$$\mathrm{Fil}^i \mathcal{D} = \{x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{i-1} \mathcal{D} \text{ and } \mathrm{ev}_\pi(x) \in \mathrm{Fil}^i D\}.$$

When $i = 1$, the condition $N_{\mathcal{D}}(x) \in \mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ is a tautology. So, $\mathrm{Fil}^1 \mathcal{D} = S_F \widehat{f}_2 + S_F E \widehat{f}_1$.

Proposition 3.1. *There exists $x_1, \ldots, x_{h-1} \in F$ such that, if $0 \leq i \leq h$, then*

$$\mathrm{Fil}^i \mathcal{D} = S_F \cdot \big(\widehat{f}_2 + (\sum_{j=1}^{i-1} x_j E^j)\widehat{f}_1\big) + S_F \cdot E^i \widehat{f}_1.$$

Proof. Assume by induction on $0 \leq i < h$, that there exists $x_1, \ldots, x_{i-1} \in F$ such that for each $0 \leq j \leq i$ we have $\mathrm{Fil}^j \mathcal{D} = S_F \cdot \widehat{f}_2^{(j)} + S_F \cdot \widehat{f}_1$, where $\widehat{f}_2^{(j)} = \widehat{f}_2 + (\sum_{m=1}^{j-1} x_m E^m)\widehat{f}_1$. Setting $\widehat{f}_2^{(0)} = \widehat{f}_2^{(1)} = \widehat{f}_2$ handles the case $i = 0$ and $i = 1$. So, suppose $1 \leq i < h$.

For the $(i+1)$-th case, we first define $x_i \in F$. By induction, $N_{\mathcal{D}}(\widehat{f}_2^{(i)}) \in \mathrm{Fil}^{i-1} \mathcal{D} = S_F \widehat{f}_2^{(i-1)} + S_F E^{i-1} \widehat{f}_1$. Since $\widehat{f}_2^{(i-1)} = \widehat{f}_2^{(i)} - x_{i-1} E^{i-1} \widehat{f}_1$, we can write

$$N_{\mathcal{D}}(\widehat{f}_2^{(i)}) = d_i \widehat{f}_2^{(i)} + b_i E^{i-1} \widehat{f}_1$$

for some $d_i, b_i \in S_F$ (cf. Lemma 3.2 below). Set $x_i = b_i(\pi)/i\pi$, and then set $\widehat{f}_2^{(i+1)} = \widehat{f}_2^{(i)} + x_i E^i \widehat{f}_1$. Since $2 \leq i+1 \leq h$, we have $\mathrm{Fil}^{i+1} D = F f_2$. Thus, $\mathrm{ev}_\pi(\widehat{f}_2^{(i+1)}) = \widehat{f}_2 \in \mathrm{Fil}^{i+1} D$. Further,

$$\begin{aligned} N_{\mathcal{D}}(\widehat{f}_2^{(i+1)}) &= N_{\mathcal{D}}(\widehat{f}_2^{(i)}) - x_i i u E^{i-1} \widehat{f}_1 + x_i E^i N_{\mathcal{D}}(\widehat{f}_1) \\ &= d_i \widehat{f}_2^{(i)} + (b_i - x_i i u) E^{i-1} \widehat{f}_1 + x_i E^i N_{\mathcal{D}}(\widehat{f}_1). \end{aligned} \tag{3.1}$$

Note, the last summand in (3.1) lies in $\mathrm{Fil}^i S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^i \mathcal{D}$, while the first lies in $\mathrm{Fil}^i \mathcal{D}$. By definition we have $\mathrm{ev}_\pi(b_i - x_i i u) = 0$ and so the the middle summand also lies in $\mathrm{Fil}^i S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^i \mathcal{D}$. Thus $\widehat{f}_2^{(i+1)} \in \mathrm{Fil}^{i+1} \mathcal{D}$.

For a moment, define $F^{i+1}\mathcal{D} = S_F \widehat{f}_2^{(i+1)} + S_F E^{i+1}\widehat{f}_1 \subseteq \operatorname{Fil}^{i+1}\mathcal{D}$. We want to show equality. Since $E\widehat{f}_2^{(i)} = E\widehat{f}_2^{(i+1)} - x_i E^{i+1}\widehat{f}_1$, we in fact have

$$E\operatorname{Fil}^i\mathcal{D} \subseteq F^{i+1}\mathcal{D} \subseteq \operatorname{Fil}^{i+1}\mathcal{D}.$$

Since ev_π gives an isomorphism $\operatorname{Fil}^{i+1}\mathcal{D}/E\operatorname{Fil}^i\mathcal{D} \cong Ff_2$ by Remark 2.8, and $\mathrm{ev}_\pi(F^{i+1}\mathcal{D}) \neq 0$, we conclude the natural map $F^{i+1}\mathcal{D}/E\operatorname{Fil}^i\mathcal{D} \to \operatorname{Fil}^{i+1}\mathcal{D}/E\operatorname{Fil}^i\mathcal{D}$ is an isomorphism. Thus, $F^{i+1}\mathcal{D} = \operatorname{Fil}^{i+1}\mathcal{D}$. □

The proof of Proposition 3.1 allows for explicit control of the scalars x_j in terms of the monodromy matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For the next two results, we explain this by re-examining the proof.

Lemma 3.2. *For $1 \leq i \leq h-1$, let $d_i, b_i \in S_F$ be such that $N_{\mathcal{D}}(\widehat{f}_2^{(i)}) = d_i\widehat{f}_2^{(i)} + b_i E^{i-1}\widehat{f}_1$. Then, $d_1 = d$, $b_1 = b$, $x_1 = \frac{b}{\pi}$ and for $1 \leq i < h-1$*

$$\begin{aligned} d_{i+1} &= d_i + cx_iE^i \\ b_{i+1} &= x_i(a - cz_i - d_i) + (b_i - x_i iu)/E \\ x_{i+1} &= \frac{b_{i+1}(\pi)}{(i+1)\pi} \end{aligned}$$

where $z_i = \sum_{j=1}^{i} x_j E^j$.

Proof. The values of d_1, b_1, and x_1 follow immediately from $\widehat{f}_2^{(1)} = \widehat{f}_2$ and $N_{\mathcal{D}}(\widehat{f}_2) = b\widehat{f}_1 + d\widehat{f}_2$. Next, by (3.1) and because $N_{\mathcal{D}}(\widehat{f}_1) = a\widehat{f}_1 + c\widehat{f}_2$, we have

$$N_{\mathcal{D}}(\widehat{f}_2^{(i+1)}) = d_i\widehat{f}_2^{(i)} + (b_i - x_i iu)E^{i-1}\widehat{f}_1 + x_iE^i(a\widehat{f}_1 + c\widehat{f}_2). \tag{3.2}$$

We can write $\widehat{f}_2^{(i)} = \widehat{f}_2^{(i+1)} - x_iE^i\widehat{f}_1$ and, separately, $\widehat{f}_2 = \widehat{f}_2^{(i+1)} - z_i\widehat{f}_1$. Thus (3.2) becomes

$$N_{\mathcal{D}}(\widehat{f}_2^{(i+1)}) = (d_i + cx_iE^i)\widehat{f}_2^{(i+1)} + \big(-d_ix_iE^i + (b_i - x_i iu)E^{i-1} + x_iE^i(a - cz_i)\big)\widehat{f}_1.$$

Factoring E^i out of the $\widehat{f}_1$-coefficient, the result is clear. □

Example 3.3. Below, in Lemma 4.4, we will need an explicit calculation of the x_i and z. This can be done using the recursive formulas above. The calculations we need, both of which are straightforward, are:

$$\begin{aligned} x_2 &= \frac{b}{2\pi^2}(a - d - 1) \\ z_2(0) &= \frac{b}{2}(a - d - 3). \end{aligned}$$

(See Example 3.9, also.)

Lemma 3.4. *Assume that $a - d \in \Lambda$ and $bc \in \Lambda$. Then, for $1 \leq i \leq h-1$, we have*

$$v_p(x_i) + v_p(i!) + i \geq v_p(b).$$

Remark 3.5. The lemma is consistent with $b = 0$ since $x_i = 0$, for all i, in that case.

Proof of Lemma 3.4. Given $v \in \mathbb{R}$ we write

$$A_v = \{\sum_{j\geq 0} y_j E^j \in F[u] \mid v_p(y_j) + v_p(j!) + j \geq v\}.$$

Note that A_v is a subgroup of $F[u]$. Since $v_p((j+k)!) \geq v_p(j!) + v_p(k!)$ for all non-negative integers j, k (because binomial coefficients are integers), we have $A_v A_w \subseteq A_{v+w}$, as well. In particular, A_0 is a ring containing Λ as a subring and each A_v is an A_0-module.

The lemma is equivalent to $x_i E^i \in A_{v_p(b)}$ for all $1 \leq i \leq h-1$, but to show $x_i E^i \in A_{v_p(b)}$ it suffices to show $b_i E^{i-1} \in A_{v_p(b)}$. Indeed, $b_i E^{i-1} \in b_i(\pi) E^{i-1} + E^i F[u]$, and so if $b_i E^{i-1} \in A_v$ (for any v) then $v_p(b_i(\pi)) + v_p((i-1)!) + i - 1 \geq v$. Since $b_i(\pi) = x_i i\pi$, by definition, we would clearly have $v_p(x_i) + v_p(i!) + i \geq v$ as well.

We have reduced to showing $b_i E^{i-1} \in A_{v_p(b)}$ for $1 \leq i \leq h-1$. For $i = 1$, by Lemma 3.2, we have $b_1 = b$ and so the claim is clear. Now assume that $b_j E^{j-1} \in A_{v_p(b)}$ for all $j \leq i$. By the previous paragraph we have $x_j E^j \in A_{v_p(b)}$ for all $j \leq i$, and so $z_j \in A_{v_p(b)}$ for all $j \leq i$ (including z_0, which we define to be 0). By Lemma 3.2, we have

$$\begin{aligned} b_{i+1}E^i &= (a - cz_i - d_i)x_i E^i + (b_i - x_i i u)E^{i-1} \\ &= (a - d - c(z_i + z_{i-1}))x_i E^i + b_i E^{i-1} - x_i i\pi E^{i-1} - x_i i E^i. \end{aligned} \tag{3.3}$$

It is clear by induction that the final three summands are in $A_{v_p(b)}$. For the first summand, we know $z_i + z_{i-1} \in A_{v_p(b)}$. Since $v_p(c) + v_p(b) \geq 0$ and $a - d \in \Lambda$, we see $a - d - c(z_i + z_{i-1}) \in A_0$. Since $x_i E^i \in A_{v_p(b)}$, by induction, the first summand also lies in $A_{v_p(b)}$. Thus, $b_{i+1}E^i \in A_{v_p(b)}$. ☐

3.2. **Explicit filtered (φ, N)-modules.** Now assume F contains an element ϖ such that $\varpi^2 = p$. For $\mathcal{L} \in F$ and $h \geq 1$, we define $D_{h+1,\mathcal{L}} = Fe_1 \oplus Fe_2 \in \mathrm{MF}_F^{\varphi,N}$ where, in the basis $\{e_1, e_2\}$,

$$\varphi = \begin{pmatrix} \varpi^{h+1} & 0 \\ 0 & \varpi^{h-1} \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathrm{Fil}^i D_{h+1,\mathcal{L}} = \begin{cases} D_{h+1,\mathcal{L}} & \text{if } i \leq 0; \\ F \cdot (e_1 + \mathcal{L}e_2) & \text{if } 1 \leq i \leq h; \\ \{0\} & \text{if } h < i. \end{cases}$$

See [6, Exemple 3.1.2.2(iv)]. It is useful make a change of a basis. Set $a_p = \varpi^{h-1} + \varpi^{h+1}$.

Lemma 3.6. *If $\mathcal{L} \neq 0$, then $\{f_1, f_2\} = \{-\varphi(e_1 + \mathcal{L}e_2), e_1 + \mathcal{L}e_2\}$ is a basis of $D_{h+1,\mathcal{L}}$ in which*

$$\varphi = \begin{pmatrix} a_p & -1 \\ p^h & 0 \end{pmatrix} \qquad N = \frac{p}{\mathcal{L}(1-p)} \begin{pmatrix} 1 & \varpi^{-h-1} \\ \varpi^{h+1} & -1 \end{pmatrix} \qquad \mathrm{Fil}^i D_{h+1,\mathcal{L}} = \begin{cases} D_{h+1,\mathcal{L}} & \text{if } i \leq 0; \\ Ff_2 & \text{if } 1 \leq i \leq h; \\ \{0\} & \text{if } h < i. \end{cases}$$

Proof. If $\mathcal{L} \neq 0$, then $e_1 + \mathcal{L}e_2$ is *not* an eigenvector of φ, so $\{f_1, f_2\}$ is a basis. We leave calculating the matrices for the reader. ☐

Now let $\mathcal{D}_{h+1,\mathcal{L}} = \underline{\mathcal{D}}(D_{h+1,\mathcal{L}})$ and $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F \in \mathrm{Mod}_{S_F}^{\varphi,\leq h}$. Recall that $\mathfrak{c} = \varphi(E)/p \in S_F^\times$. Let $\lambda_- = \prod_{n\geq 0} \varphi^{2n+1}(E)/p$ and $\lambda_{++} = \varphi(\lambda_-)$.

Theorem 3.7. *If $\mathcal{L} \neq 0$, there exists a basis of $\mathcal{M}_{h+1,\mathcal{L}}$ in which the matrix of φ is given by*

$$A = \begin{pmatrix} (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h & -1 + \varphi(z)(a_p - p^h z) \\ E^h & E^h \varphi(z)\left(\frac{\lambda_{++}}{\lambda_-}\right)^h \end{pmatrix},$$

where $z = \sum_{j=1}^{h-1} x_j E^j \in F[E]$. Moreover, if $v_p(\mathcal{L}^{-1}) \geq -1$, then

$$v_p(x_j) \geq v_p(\mathcal{L}^{-1}) - \frac{h-1}{2} - v_p(j!) - j \tag{3.4}$$

for each $1 \leq j \leq h-1$.

Proof. Let $\{f_1, f_2\}$ be the basis as in Lemma 3.6. Set $\widehat{f_1} = 1 \otimes f_1$ and $\widehat{f_2} = 1 \otimes f_2$, elements of $\mathcal{D}_{h+1,\mathcal{L}}$, as before. Then, the matrix of φ in the basis $\{\widehat{f_1}, \widehat{f_2}\}$ of $\mathcal{D}_{h+1,\mathcal{L}}$ is $X = \left(\begin{smallmatrix} a_p & -1 \\ p^h & 0 \end{smallmatrix}\right)$. Moreover, Proposition 3.1 implies that $\mathrm{Fil}^h \mathcal{D}_{h+1,\mathcal{L}} = S_F \alpha_1 \oplus S_F \alpha_2$, where

$$(\alpha_1, \alpha_2) = (\widehat{f_1}, \widehat{f_2}) \begin{pmatrix} E^h & z \\ 0 & 1 \end{pmatrix} =: (\widehat{f_1}, \widehat{f_2}) B$$

for $z = \sum_{j=1}^{h-1} x_j E^j$ and some $x_j \in F$. Theorem 2.7 implies that $\mathcal{M}_{h+1,\mathcal{L}}$ has a basis in which the matrix of φ is given by

$$A' = E^h B^{-1} X \varphi(B) p^{-h} \mathfrak{c}^{-h} = \begin{pmatrix} a - p^h z & p^{-h}\mathfrak{c}^{-h}\left(-1 + \varphi(z)(a_p - p^h z)\right) \\ E^h p^h & p^{-h}\mathfrak{c}^{-h} E^h p^h \varphi(z) \end{pmatrix}. \tag{3.5}$$

Since λ_- and λ_{++} are units in S_F, we can replace A' by $CA'\varphi(C^{-1})$ for $C = \begin{pmatrix} p^h \lambda_-^h & 0 \\ 0 & \lambda_{++}^h \end{pmatrix}$. A short calculation shows $A = CA'\varphi(C^{-1})$, completing the general proof.

Finally, if $v_p(\mathcal{L}^{-1}) \geq -1$, then the matrix of N in Lemma 3.6 satisfies the hypotheses of Lemma 3.4. So, the estimates (3.4) follow from the b-entry of the monodromy matrix being

$$b = \frac{-p}{\varpi^{h+1}\mathcal{L}(1-p)} = \frac{-1}{\varpi^{h-1}\mathcal{L}(1-p)}.$$

This completes the proof. □

Remark 3.8. An analogous calculation in the crystalline case, where $z = 0$ (see Remark 3.5), was made in [2, Section 3]. The technique here, passing through the category $\mathrm{MF}_{S_F}^{\varphi,N}$, is different than *loc. cit.* The descriptions are the same, though. Compare with Example 2.9.

Example 3.9. We need one ad hoc calculation in Lemma 4.4 below. Let $h = 3$. By Example 3.3, the element z in Theorem 3.7 satisfies $z(0) = \frac{b}{2}(a - d - 3)$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the monodromy matrix in Lemma 3.6. For $p = 3$, plugging in the explicit matrix, we see $z(0) = \frac{-1}{2\mathcal{L}}\left(\frac{1}{\mathcal{L}} + 1\right)$.

4. Descent and reductions

The goal in this section is to prove the main theorem of this article. Given $h \geq 1$ and $\mathcal{L} \in F$ we write $V_{h+1,\mathcal{L}}$ for the unique two-dimensional representation of $G_{\mathbb{Q}_p}$ such that $D^*_{\mathrm{st}}(V_{h+1,\mathcal{L}}) \cong D_{h+1,\mathcal{L}}$ where $D_{h+1,\mathcal{L}}$ is as in Section 3.2. Write $\overline{V}$ for the semi-simple reduction modulo $\mathfrak{m}_F$ of V. Let $\mathbb{Q}_{p^2}$ be the unramified quadratic extension of $\mathbb{Q}_p$, χ the unramified quadratic character of $G_{\mathbb{Q}_{p^2}}$, and ω_2 a niveau 2 fundamental character of $\mathbb{Q}_{p^2}$. Note that $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$ has determinant ω^h, where ω is the cyclotomic character, and its restriction to inertia is $\omega_2^h \oplus \omega_2^{ph}$.

Theorem 4.1. *Assume $h \geq 2$, $p \neq 2$, and if $p = 3$ then $h \geq 3$. Then, if $\mathcal{L}$ satisfies*

$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} - 1 + v_p((h-1)!),$$

then $\overline{V}_{h+1,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$.

Remark 4.2. Our contribution toward Theorem 4.1 is limited to $h \geq 4$ and $p = h = 3$. We could also include $p = 3$ and $h = 2$ if we weaken the bound to $v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} + v_p((h-1)!)$. In fact, if we use the weaker bound then we do not need to give references to prior work, either. The proof and further details are given in Section 4.3 below.

We plan to take the matrix of φ acting on $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F$ as in Theorem 3.7 and replace it with a φ-conjugate defined over $\mathfrak{S}_\Lambda$ when $v_p(\mathcal{L}^{-1})$ satisfies the bound in theorem. This defines a Kisin module $\mathfrak{M}$ for $V_{h+1,\mathcal{L}}$ that allows us to calculate the reduction $\overline{V}_{h+1,\mathcal{L}}$. Throughout, we assume without further comment that:

$$p \neq 2 \text{ and } h \geq 2; \tag{4.1}$$
$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} - 1 + v_p((h-1)!).$$

We will clarify result-by-result where we need to limit to $h \geq 3$ or $h \geq 4$. Also, fix $z = \sum x_j E^j$ as in Theorem 3.7. Note that by (4.1) $v_p(\mathcal{L}^{-1}) \geq -1$ so the estimates (3.4) in Theorem 3.7 hold.

4.1. Preparing for descent.

Consider the ring

$$R_2 = \{f = \sum a_i u^i \in F[\![u]\!] \mid i + 2v_p(a_i) \to \infty \text{ as } i \to \infty\}.$$

Thus R_2 is the F-Banach algebra of series converging on $|u| \leq p^{-1/2}$. We equip R_2 with the valuation $v_{R_2}(\sum a_i u^i) = \inf_i\{i + 2v_p(a_i)\}$. The canonical map $\mathcal{O}_F \hookrightarrow R_2$ factors through S_F since $v_{R_2}(E^p/p) = p - 2 > 0$. Finally, given $v \in \mathbb{R}$, we define additive subgroups $H_v^\circ \subseteq H_v \subseteq R_2$ by

$$H_v = \{f \in R_2 \mid v_{R_2}(f) \geq v\}; \qquad H_v^\circ = \{f \in R_2 \mid v_{R_2}(f) > v\}.$$

Our first lemma, concerning some entries of the matrix in Theorem 3.7, is straightforward so we omit the proof (compare with [2, Lemma 5.1.1]).

Lemma 4.3. *Let $\lambda_- = \prod_{n\geq 0} \varphi^{2n+1}(E)/p$ and $\lambda_{++} = \varphi(\lambda_-)$ be as in Theorem 3.7. Then,*

(a) $\lambda_- \in 1 + H_{p-2}$ and $\lambda_{++} \in 1 + H_{p^2-2}$;

(b) $\lambda_-, \lambda_{++} \in R_2^\times$;
(c) $v_{R_2}(\lambda_-^{\pm 1}) = 0 = v_{R_2}(\lambda_{++}^{\pm 1})$.

We also prepare estimates for z. Note that by (4.1) the estimate (3.4) becomes

$$v_p(x_j) > v_p((h-1)!) - v_p(j!) - j - 1 \geq -j - 1. \tag{4.2}$$

Recall, we write $a_p = \varpi^{h-1} + \varpi^{h+1}$. Thus, $v_p(a_p) = \frac{h-1}{2}$.

Lemma 4.4. *For $z = \sum_{j=1}^{h-1} x_j E^j$ as above, and $\nu = -1 + \varphi(z)(a_p - p^h z)$, we have*

(a) $p^h z \in H^\circ_{h-1}$;
(b) $\varphi(z) \in H^\circ_{-2}$;
(c) $\nu \in -1 + H^\circ_{h-3}$;
(d) If $h \geq 3$, *then* $\nu \in R_2^\times$.

Furthermore, if $p = 3$ and $h = 3$, then $\varphi(z) \in H^\circ_{-1}$ and $\nu \in -1 + H^\circ_{h-2} = -1 + H^\circ_1$.

Proof. First, $v_{R_2}(E^j) = j$. By the ultrametric inequality and (4.2), we see

$$v_{R_2}(z) > \inf\{2(-j-1) + j \mid 1 \leq j \leq h-1\} = -1 - h.$$

Part (a) follows because $v_{R_2}(p^h) = 2h$. For (b), note $v_{R_2}(\varphi(E)^j) = 2j$. Thus, using (4.2),

$$v_{R_2}(\varphi(z)) > \inf\{2(-j-1) + 2j \mid 1 \leq j \leq h-1\} = -2.$$

Continuing, $\varphi(z)p^h z \in H^\circ_{h-3}$ by parts (a) and (b) and, since $v_{R_2}(a_p) = h-1$, we have $\varphi(z)a_p \in H^\circ_{h-3}$. This proves (c). Finally, part (d) follows from the geometric series and part (c).

Finally, suppose $p = h = 3$. By the argument for (c) above, it suffices to show $\varphi(z) \in H^\circ_{-1}$. We note $v_{R_2}(\varphi(E)^j - E(0)^j) \geq p + 2j - 2$ for any j. Thus, by (4.2)

$$v_{R_2}(\varphi(z) - \varphi(z)(0)) > p + 2j - 2 - 2(j+1) > p - 4 = -1. \tag{4.3}$$

But, by Example 3.9 we have $\varphi(z)(0) = z(0) = \frac{-1}{2\mathcal{L}}\left(\frac{1}{\mathcal{L}} + 1\right)$. Since $v_p(\mathcal{L}^{-1}) > 0$, (4.3) then implies so $v_{R_2}(\varphi(z)) > -1$ as we wanted. ☐

We now write $\mathcal{M}_2 = \mathcal{M}_{h+1,\mathcal{L}} \otimes_{S_F} R_2 \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} R_2$. Thus, $\mathcal{M}_2 \in \mathrm{Mod}_{R_2}^{\varphi,\leq h}$. We also introduce some notation. Given $A \in \mathrm{Mat}_d(R_2)$ and $C \in \mathrm{GL}_d(R_2)$ we write $C *_\varphi A = C \cdot A \cdot \varphi(C)^{-1}$. Thus, if $\{e_1, e_2\}$ is a basis of $\mathcal{M}_2$ and A is the matrix of $\varphi_{\mathcal{M}_2}$ in that basis, then $C *_\varphi A$ is the matrix of $\varphi_{\mathcal{M}_2}$ in the basis $\{e_1', e_2'\}$ is given by $(e_1', e_2') = (e_1, e_2)C^{-1}$.

Proposition 4.5. *Assume $h \geq 4$ or $p = h = 3$. Then, there exists a basis of $\mathcal{M}_2$ in which the matrix of $\varphi_{\mathcal{M}_2}$ is $\left(\begin{smallmatrix} G & -1 \\ E^h & 0 \end{smallmatrix}\right)$, where $G \in (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h + H^\circ_h$.*

Proof. By Theorem 3.7, there is a basis $\{e_1, e_2\}$ of $\mathcal{M}_2$ such that $\varphi_{\mathcal{M}_2}((e_1, e_2)) = (e_1, e_2)A$, where

$$A = \begin{pmatrix} (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h & -1 + \varphi(z)(a_p - p^h z) \\ E^h & E^h\varphi(z)\left(\frac{\lambda_{++}}{\lambda_-}\right)^h \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ E^h & \eta \end{pmatrix},$$

where ν is as in Lemma 4.4 and μ and η are defined by the equality. Assume for now just that $h \geq 3$. Then, by Lemma 4.4(d), $\nu \in R_2^\times$. Making a change of basis on $\mathcal{M}_2$, we replace A by (note that $\mu\eta = (1+\nu)E^h$)

$$A' = \begin{pmatrix} 1 & 0 \\ -\eta/\nu & 0 \end{pmatrix} *_\varphi A = \begin{pmatrix} \mu + \frac{\nu\varphi(\eta)}{\varphi(\nu)} & \nu \\ -E^h\nu^{-1} & 0 \end{pmatrix}.$$

Since $v_{R_2}(\nu+1) > 0$ by Lemma 4.4(c), we have $\nu(0) \in \Lambda^\times$. Thus $\nu_0 = \nu/\nu(0) \in 1 + (H_{h-3}^\circ \cap uR_2)$. By [2, Lemma 4.1.1], we have $\varphi^k(\nu_0) \in 1 + H_{h-3+m_k}$ where $m_k \to \infty$ as $k \to \infty$. Thus, the infinite product $\nu_+ = \prod_{n\geq 0} \varphi^{2n}(\nu_0)$ converges in R_2. Set $\nu_- = \varphi(\nu_+)$, so $\nu_\pm \in 1 + H_{h-3}^\circ \subseteq R_2^\times$. We now change basis on $\mathcal{M}_2$ again to get a matrix A'' for $\varphi_{\mathcal{M}_2}$ given by

$$A'' = \begin{pmatrix} \frac{-1}{\nu(0)}\frac{\nu_-}{\nu_+} & 0 \\ 0 & \frac{\nu_+}{\nu_-} \end{pmatrix} *_\varphi A' = \begin{pmatrix} G & -1 \\ E^h & 0 \end{pmatrix},$$

where

$$G = \left(\mu + \frac{\nu\varphi(\eta)}{\varphi(\nu)}\right)\frac{\nu_-^2}{\nu_+\nu_{++}} \tag{4.4}$$

and $\nu_{++} = \varphi(\nu_-)$.

To complete the argument, we justify $G \in \mu + H_h^\circ$. We already know $\nu_-^2/\nu_+\nu_{++} \in 1 + H_{h-3}^\circ$. The same is true for $\nu/\varphi(\nu)$. So,

$$v_{R_2}\left(\frac{\nu\varphi(\eta)}{\varphi(\nu)}\right) \geq v_{R_2}(\varphi(\eta)) \geq v_{R_2}(\varphi(E)^h\varphi^2(z)), \tag{4.5}$$

where we used Lemma 4.3 to remove λ_- and λ_{++} from the estimate. We note $v_{R_2}(\varphi(E)^h) = 2h$ and $v_{R_2}(\varphi^2(z)) \geq v_{R_2}(\varphi(z)) > -2$, by Lemma 4.4(b) (cf. [2, Lemma 4.1.1]). Thus from (4.5) we deduce that $v_{R_2}(\nu\varphi(\eta)/\varphi(\nu)) > 2h-2 = 2(h-1)$. We also note that $a_p - p^h z \in H_{h-1}$. Thus, $\mu \in H_{h-1}$ and so, returning to the definition (4.4) of μ and G, we see

$$G \in \left(\mu + H_{2(h-1)}^\circ\right)\cdot(1 + H_{h-3}^\circ) \subseteq \mu + H_{2h-4}^\circ + H_{2(h-1)}^\circ = \mu + H_{2h-4}^\circ.$$

Now, if $h \geq 4$, then $2h-4 \geq h$ and so $G \in \mu + H_h^\circ$. This completes the proof except if $p = h = 3$. In that case, Lemma 4.4 shows $\nu \in -1 + H_1^\circ$, rather than $-1 + H_0^\circ$, from which we deduce

$$G \in \left(\mu + H_4^\circ\right)\cdot(1 + H_1^\circ) \subseteq \mu + H_3^\circ = \mu + H_h^\circ$$

anyways. This completes the proof. □

4.2. **Descent.** To descend to $\mathfrak{S}_\Lambda$, we use the algorithm from [2, Section 4]. Write $T_{\leq d} : R_2 \to F[u]$ for the "truncation" operation $T_{\leq d}(\sum a_i u^i) = \sum_{i\leq d} a_i u^i$ and $T_{>d}(f) = f - T_{\leq d}(f)$. In the next two proofs, we will use the following principle: if $f \in R_2$ and $v_{R_2}(T_{\leq d}(f)) > d$ then $f \in \mathfrak{m}_F[u]$.

Proposition 4.6. *Suppose that $G \in R_2$ such that*

(a) $G \in H_{h-1}$;
(b) $T_{>h}(G) \in H_{h-1}^\circ$;
(c) $T_{\leq h}(G) \in \mathfrak{m}_F[u]$.

Then, given $A = \left(\begin{smallmatrix} G & -1 \\ E^h & 0 \end{smallmatrix}\right)$*, there exists* $C \in \mathrm{GL}_2(R_2)$ *and* $P \in \mathfrak{m}_F[u]$ *such that* $C *_\varphi A = \left(\begin{smallmatrix} P & -1 \\ E^h & 0 \end{smallmatrix}\right)$*.*

Proof. Since $E^h \in u^h + H_{h+1}$, the assumption (a) implies that

$$A \in \begin{pmatrix} 0 & -1 \\ u^h & 0 \end{pmatrix} + \begin{pmatrix} H_{h-1} & 0 \\ H_{h+1} & 0 \end{pmatrix}.$$

In the notation of [2, Section 4.3], set $a = 0$, $b = h$, $a' = \frac{h}{2} - \frac{p-1}{2}$ and $b' = \frac{h}{2} + \frac{p-1}{2}$, and $(c_0, c_h) = (-1, 1)$. Since $h - 1 - a' = \frac{h}{2} - 1 + \frac{p-1}{2} \geq 1$, we see A is γ-allowable with $\gamma = 1$ in the sense of [2, Definition 4.3.1]. The error of A, in the same definition, is $\varepsilon = v_{R_2}(T_{>h}(G)) - a'$. By [2, Theorem 4.3.7], with $R = R_2$ in *loc. cit.*, there exists $C \in \mathrm{GL}_2(R_2)$ such that $C|_{u=0} = I$ and

$$C *_\varphi A = \begin{pmatrix} P & -1 \\ f & 0 \end{pmatrix}$$

where P and f are polynomials of degree at most h and P satisfies, because of assumption (b),

$$v_{R_2}(P - T_{\leq h}(G)) \geq \varepsilon + a' + 1 = v_{R_2}(T_{>h}(G)) + 1 > h.$$

Since $P - T_{\leq h}(G)$ has degree at most h, we deduce $P - T_{\leq h}(G) \in \mathfrak{m}_F[u]$. By assumption (c), we have $P \in \mathfrak{m}_F[u]$ as well. Finally, $f = \det(C *_\varphi A) = rE^h$ for some $r \in R_2^\times$. But, f and E^h are both polynomials of degree at most h with the same constant term, since $C|_{u=0} = I$, and thus $f = E^h$. □

We now verify the G from Proposition 4.5 satisfies the hypothesis of Proposition 4.6.

Lemma 4.7. *Let* $G \in (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h + H_h^\circ$*. Then,*

(a) $G \in H_{h-1}$*,*
(b) $T_{>h}(G) \in H_{h-1}^\circ$*, and*
(c) $T_{\leq h}(G) \in \mathfrak{m}_F[u]$*.*

Proof. First, the conclusions depend only on $G \bmod H_h^\circ$, so we suppose $G = (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h$. Part (a) follows from Lemmas 4.3 and 4.4. For part (b), we first have, by Lemma 4.3(a), that $a_p\left(\frac{\lambda_-}{\lambda_{++}}\right)^h \in a_p + a_p H_{p-2}$. So, $T_{>0}\big(a_p\left(\frac{\lambda_-}{\lambda_{++}}\right)^h\big) \in H_{h+p-3} \subseteq H_h$. On the other hand, by Lemma 4.4(a) we have $p^h z \in H_{h-1}^\circ$. Thus we've shown in fact $T_{>0}(G) \in H_{h-1}^\circ$.

Finally, we consider part (c). Since $E = u + p$, any $f \in S_\Lambda$ can be written $f = \sum_{n=0}^{\infty} \alpha_n \frac{E^n}{p^{\lfloor \frac{n}{p} \rfloor}}$ with $\alpha_n \in \Lambda$. Let $f = \frac{\lambda_-}{\lambda_{++}} \in S_\Lambda$, in particular. Since $v_p(a_p) = \frac{h-1}{2} > \lfloor \frac{h}{p} \rfloor$ unless $p = h = 3$ (or $p = 2$, which we have excluded in (4.1)), we see immediately that $T_{\leq h}(a_p f^h) \in \mathfrak{m}_F[u]$ except when $h = p = 3$. When $h = p = 3$, though, $f^h = f^p$ is of the form $\alpha_0' + \sum_{n=1}^{\infty} p\alpha_n' \frac{E^n}{p^{\lfloor \frac{n}{p} \rfloor}}$ with $\alpha_i' \in \Lambda$; thus $T_{\leq h}(a_p f^h) \in \mathfrak{m}_F[u]$ in every case.

It remains to show that $T_{\leq h}(p^h z f^h) \in \mathfrak{m}_F[u]$ as well. Since $z = \sum_{j<h} x_j E^j$, it suffices to show $h + v_p(x_j) - \lfloor \frac{n-j}{p} \rfloor > 0$ for $n \leq h$ and $j < h$. This follows from (4.2). □

4.3. **Proof of Theorem 4.1.** Finally, we give the proof of the main theorem:

Assume that $h \geq 2$, $p \neq 2$, and if $p = 3$ then $h \geq 3$. Then, if $\mathcal{L}$ satisfies

$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} - 1 + v_p((h-1)!),$$

then $\overline{V}_{h+1,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^h \chi)$.

Proof of Theorem 4.1. First if $p \geq 5$ and $h < 4$ then $h < p - 1$. Thus the result follows from [6, Theorem 4.2.4.7(iii)] when h is odd and [14, Theorem 5.0.5] when h is even.

Now we assume that either $h \geq 4$ or $p = h = 3$. Then, applying Proposition 4.5, Lemma 4.7, and Proposition 4.6, we deduce that there exists a basis of $\mathcal{M}_2$ in which the matrix of $\varphi_{\mathcal{M}_2}$ is given by $A = \left(\begin{smallmatrix} P & -1 \\ E^h & 0 \end{smallmatrix}\right)$ and $P \in \mathfrak{m}_F[u]$. Define $\mathfrak{M} = \mathfrak{S}_\Lambda^{\oplus 2}$ with the matrix of φ being given by A. Clearly $\mathfrak{M}$ is a Kisin module over $\mathfrak{S}_\Lambda$ of height $\leq h$, and

$$\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} R_2 \cong \mathcal{M}_2 = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} R_2$$

as φ-modules over R_2. Thus, by Proposition 2.1 we deduce $\mathfrak{M} = \mathfrak{M}(T)$ for some lattice $T \subseteq V_{h+1,\mathcal{L}}$. Furthermore, $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathbb{F}[u^{-1}]$ is a φ-module over $\mathbb{F}(\!(u)\!)$ with Frobenius given by $\left(\begin{smallmatrix} 0 & -1 \\ u^h & 0 \end{smallmatrix}\right)$. This shows, in particular, that $\overline{V}_{h+1,\mathcal{L}}$ is the same for any $\mathcal{L}$ satisfying (4.1) (see [2, Corollary 2.3.2]).

Let $V_{h+1,\infty}$ be as in the introduction. By [2, Corollary 5.2.2], for $V_{h+1,\infty}$ there exists a Kisin module $\mathfrak{M}'$ so that $M' := \mathfrak{M}' \otimes_{\mathfrak{S}_\Lambda} \mathbb{F}[u^{-1}]$ has Frobenius also given by $\left(\begin{smallmatrix} 0 & -1 \\ u^h & 0 \end{smallmatrix}\right)$ and M' determines $\overline{V}_{h+1,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^h \chi)$. Therefore, $\overline{V}_{h+1,\mathcal{L}} \cong \overline{V}_{h+1,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^h \chi)$. □

Remark 4.8. We return to Remark 4.2. Suppose we replace (4.1) with

$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} + v_p((h-1)!). \tag{4.6}$$

This has the impact of scaling z by a p-adic unit multiple of p, thus increasing $v_{R_2}(z)$ by 2 throughout our estimates in Section 4.1. The reader may check that Proposition 4.5 holds with these new estimates, and so the proof goes through for all $h \geq 2$ and $p \geq 3$ under the assumption (4.6).

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