TATE CONJECTURE OVER NUMBER FIELDS

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1. Introduction

In this lecture, we discuss the proof of the Tate conjecture for abelian varieties over number fields as presented in Falting's seminal paper "Finiteness Theorems for Abelian Varieties over Number Fields" [3]. We will follow his argument closely adding additional details as needed.

This is the beginning of the payoff of all our work throughout this seminar so I will freely reference and quote results from previous talks.

We prove the following theorem:

Theorem 1.1 (Tate conjecture)**.** *Let K be a number field with absolute Galois group Γ_K. Let A and B be abelian varieties over K. For all primes ℓ, the natural map*

$$\mathrm{Hom}_K(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell[\Gamma_K]}(T_\ell(A), T_\ell(B))$$

is an isomorphism, where T_ℓ denotes the ℓ-adic Tate module.

There are two key inputs into the proof. The first is the following finiteness result for Faltings' height which was proven last lecture (L?):

Theorem 1.2. *For an abelian variety A/K, denote it's Faltings height by $h(A)$. Fix integers g, d, and N. Then, there exists finitely many abelian varieties over K up to isomorphism of dimension g with a polarization of degree d and height less than N.*

The second key input which will take up the majority of our talk is a good understanding of how height behaves under isogeny. This takes as input both Tate's results on p-divisible groups and Raynaud's results on finite flat group schemes.

2. Formula for Change of Height under Isogeny

For this section, we assume $\mathcal{A}$ and $\mathcal{B}$ are semi-abelian schemes over Spec $(\mathcal{O}_K)$ where $\mathcal{O}_K$ is a ring of integers of a number field K with proper generic fibers A and B.

Let $\phi : A \to B$ be an isogeny over Spec (K). Then, there is a unique homomorphism $\varphi_{\mathcal{O}_K} : \mathcal{A} \to \mathcal{B}$ inducing ϕ over K (since semi-abelian schemes are the identity component of the Neron model of their generic fibers, see L13 Th'm 4.4 and L12 Lemma 35). The morphism φ is surjective, and its kernel $\mathcal{G}$ is a quasi-finite separated flat group scheme over Spec $\mathcal{O}_K$.

Let ω_A and ω_B be Neron differentials on A and B. By Neron differential, I mean a generator for the top degree invariant forms on the Neron model. If $\mathcal{O}_K$ has non-trivial class group, the module of top degree invariant forms my not be free, however, we can pass to the Hilbert class field where it becomes free, and as defined the height is invariant under field extension. Throughout the discussion, we will always assume we have Neron differentials.

Recall that the Faltings height is defined by

$$h(A) := \frac{1}{2[K:\mathbb{Q}]} \sum_{i:K\to\mathbb{C}} \log(|\int_{i^*(A)(\mathbb{C})} \omega_A \wedge \bar{\omega}_A|^{-1})$$

and similarly for B. The following key lemma computes $h(A) - h(B)$ in terms of the kernel G.

Formula 2.1. *Let s be the identity section of $G = \ker\varphi$. Then,*

$$h(A) - h(B) = \frac{1}{[K:\mathbb{Q}]} \log(|s^*\Omega^1_{G/\mathcal{O}_K}|) - \frac{1}{2}\log(\deg(\phi))$$

where $|\cdot|$ means the size an abelian group.

Proof. The space of top-dimensional invariant forms of $\mathcal{A}$ over Spec $\mathcal{O}_K$ is free of rank 1 over $\mathcal{O}_K$. Thus, we can write $\varphi^*(\omega_B) = a * \omega_A$ for some $a \in \mathcal{O}_K$.

Fix an embedding $i : K \to \mathbb{C}$. Using i, we think of A and B as complex tori. We can identify $A_i(\mathbb{C}) = \mathbb{C}^g/\Lambda_A$ and $B_i(\mathbb{C}) = \mathbb{C}^g/\Lambda_B$. The isogeny ϕ identifies $\Lambda_A \subset \Lambda_B$ as a sublattice of index $\deg(\phi)$.

Computing the i-part of the height A corresponds to computing the volume of $\mathbb{C}^g/\Lambda_A$ with respect to the form ω_A.

Set

$$H_i(A) = |\int_{\mathbb{C}^g/\Lambda_A} \omega_A \wedge \overline{\omega}_A|^{-1},$$

and similarly for B. Now,

$$\begin{aligned} H_i(B) &= |\int_{\mathbb{C}^g/\Lambda_B} \omega_B \wedge \overline{\omega}_B|^{-1} \\ &= \deg(\phi)|\int_{\mathbb{C}^g/\Lambda_A} \omega_B \wedge \overline{\omega}_B|^{-1} \\ &= \deg(\phi)|\int_{\mathbb{C}^g/\Lambda_A} i(a)\overline{i(a)}\omega_A \wedge \overline{\omega}_A|^{-1} \\ &= \frac{deg(\phi)}{i(a)\overline{i(a)}} H_i(A). \end{aligned}$$

Note that $2[K:\mathbb{Q}](h(A)-h(B)) = \sum_i [\log(H_i(A)) - \log(H_i(B))]$. This gives the formula

$$2[K:\mathbb{Q}](h(A)-h(B)) = 2\log(\mathrm{Norm}_{K/\mathbb{Q}}(a)) - [K:\mathbb{Q}]\log(deg(\phi)).$$

We are reduced to showing that $\mathrm{Norm}_{K/\mathbb{Q}}(a) = |s^*\Omega^1_{G/\mathcal{O}_K}|$. □

Lemma 2.2. *Let ω_A and ω_B and a be as in the previous lemma, then* $\mathrm{Norm}_{K/\mathbb{Q}}(a) = |s^*\Omega^1_{G/\mathcal{O}_K}|$.

Proof. We begin by considering the standard exact sequence

$$\phi^*(\Omega^1_B) \to \Omega^1_A \to \Omega^1_{A/B} \to 0$$

as sheaves on A. A Neron differential is a global invariant differential, so it is determined by its value at the identity. Pulling back along the identity section, we have

$$s^*\phi^*(\Omega^1_B) = s^*(\Omega^1_B) \xrightarrow{\phi_s} s^*(\Omega^1_A) \to s^*(\Omega^1_{A/B}) \to 0.$$

The pullbacks $s^*(\omega_A)$ and $s^*(\omega_B)$ of the Neron differentials generate the determinant of $s^*(\Omega^1_A)$ and $s^*(\Omega^1_B)$ respectively which are both free over $\mathcal{O}_K$ of rank 1. We see then that $(a) = (\det(\phi_s))$. It is an easy exercise to check that $|\operatorname{coker}(\phi_s)| = |\operatorname{coker}(\det(\phi_s))| = |\mathcal{O}_K/a\mathcal{O}_K| = |\mathrm{Norm}_{K/\mathbb{Q}}(a)|$.

It remains to show that $|s^*(\Omega^1_{A/B})| = |s^*(\Omega^1_{G/\mathcal{O}_K})|$. In fact, these two modules over $\mathcal{O}_K$ are isomorphic. Let $f : G \hookrightarrow A$ denote the inclusion. Note also that $G \to \operatorname{Spec} \mathcal{O}_K$ is the base change of $A \to B$ with respect to identity on B so $f^*(\Omega^1_{A/B}) \cong \Omega^1_{G/\mathcal{O}_K}$. The morphism s factors through f, so clearly

$$s^*f^*(\Omega^1_{A/B}) \cong s^*(\Omega^1_{A/B}) \cong s^*(\Omega^1_{G/\mathcal{O}_K}).$$

□

Corollary 2.3. *Let $\phi : A \to B$ be an isogeny over K as above. Then,*

$$\exp(2[K:\mathbb{Q}](h(A)-h(B)))$$

is a rational number r. The primes dividing the numerator and denominator appear in the prime factors of $\deg(\phi)$*, and their exponents are bounded in terms of their exponents in* $\deg(\phi)$.

Proof. By Formula 2.1,

$$\exp(2[K:\mathbb{Q}](h(A)-h(B))) = |s^*(\Omega^1_{G/R})|^2 \deg(\phi)^{[K:\mathbb{Q}]}.$$

The only thing to check is that $|s^*(\Omega^1_{G/R})|$ is divisible by primes only dividing $\deg(\phi)$ and with controlled exponent. This follows because G is a commutative group scheme killed by $\deg(\phi)$. It follows that the abelian group $s^*(\Omega^1_{G/R})$ is also killed by $\deg(\phi)$ and so its order is only divisible by primes dividing $\deg(\phi)$. Bounding the exponents follows from a more concrete description of $|s^*(\Omega^1_{G/R})|$, see Cor. 2.6 and Prop. 2.9 in the next section. □

Remark 2.4. The above Corollary says that the change in height is bounded in terms of $\deg(\phi)$ so to prove certain finiteness result, we can allow isogenies of bounded degree without a problem. For example, later using the Tate conjecture we will show that for a fixed A, there exists a number N such that any B isogenous to A admits an isogeny to A of degree $\leq N$. This shows that heights in a given isogeny class are bounded.

2.1. **Quasi-finite Flat Group Schemes.** Before we continue, we give a more concrete description of the value $|s^*(\Omega^1_{G/\mathcal{O}_K})|$ which appears in the change in height formula. To do this, we must go over some results about quasi-finite separated flat group schemes.

The $\mathcal{O}_K$-module $s^*(\Omega^1_{G/\mathcal{O}_K})$ has finite support, which we would like to identify.

Proposition 2.5. *Let G be a quasi-finite separated flat group scheme over a base S. Assume the fibers of G have orders which are divisible on the base S. Then G is etale over S.*

Proof. G is already flat so it remains to show that G is unramified. This can be checked on fibers which are all etale because their orders are invertible on the base. □

Corollary 2.6. *Let G be a quasi-finite separated group scheme killed by a power of ℓ over* $\operatorname{Spec} \mathcal{O}_K$*. For each place v above ℓ, let $\mathcal{O}_v$ be the completion at v and let $G_v := G \otimes_{\mathcal{O}_K} \mathcal{O}_v$. Then,*

$$|s^*(\Omega^1_{G/\mathcal{O}_K})| = \prod_v |s^*(\Omega^1_{G_v/\mathcal{O}_v})|.$$

Proof. The previous proposition tells us that the support of $\Omega^1_{G/\mathcal{O}_K}$ lies over the complement of $\mathcal{O}_K[1/\ell]$.

Let $M = s^*(\Omega^1_{G/\mathcal{O}_K})$. It is a finite $\mathcal{O}_K$-module which commutes with base change on $\operatorname{Spec} \mathcal{O}_K$. The support of M lies above ℓ so the Chinese remainder theorem says that $|M| = \prod_{v|\ell} |M_v|$, where M_v is the localization at the prime corresponding to v. Since M_v is finite length, $M_v = \widehat{M}_v = M \otimes_{\mathcal{O}_K} \mathcal{O}_v$. This completes the proof. □

We now set about computing $s^*(\Omega^1_{G_v/\mathcal{O}_v})$. We work over a complete dvr R, for example, one of the $\mathcal{O}_v$.

We first recall the structure theorem for quasi-finite, separated morphisms discussed in L13 Th'm 4.10:

Theorem 2.7. *Let X be a quasi-finite separated over a Henselian local ring R. There is a unique decomposition $X = X_f \coprod X_\eta$ where X_f is R-finite and X_η has empty special fiber. Furthermore, X_f satisfies the univeral property that any R-morphism $Y \to X$, where Y is finite over R, factors through X_f.*

Because of the functoriality of the X_f, if G is a quasi-finite separated flat group scheme over a Henselian local fing R, then G_f is a finite flat group scheme which is open and closed in G with the same special fiber(i.e $\overline{G}_f \cong \overline{G}$. We call this the finite part.

We will need the following lemma later:

Lemma 2.8. *Let H be quasi-finite separated flat over a Henselian local ring R with generic characteristic 0. Let G be a closed subgroup scheme of H. Then,*

$$G_f = G \cap H_f.$$

Proof. By the universal property of finite part, $G_f \subset H_f$ so one inclusion is clear. We want to show that

$$G \cap H_f \subset G_f.$$

It is straightforward to check that G_f and $G \cap H_f$ have the same special fiber (only the finite part persists). Now, let's look at the generic fiber. Over K, everything is finite etale (and hence disjoint union of closed points over finite extensions). H_K is finite etale, so we need only consider points over finite extensions of K'/K.

Let $x \in G(K') \cap H_f(K')$. Because H_f is finite flat over R, any K'-point comes from an R'-point, where R' is the integral closure of R in K'. Then, we get a map $x : \text{Spec } R' \to G$. Since R' is finite over R, by the universal property x factors through G_f and we are done. □

Formula 2.9. *Let G be a quasi-finite separated flat group scheme killed by a power of ℓ over a complete dvr R with finite residue field. Then, G has a canonical subgroup scheme G^0 which is finite flat and connected over R such that*

$$|s^*(\Omega^1_{G/R})| = |R/\operatorname{disc}(G^0)|^{\frac{1}{\#G^0}}.$$

Proof. By the structure theorem for quasi-finite and separated morphism over a complete local ring, G decomposes as a disjoint union $G_f \coprod G_\eta$ where G_f is finite flat subgroup scheme over R. Since s factors through G_f, we see that

$$s^*(\Omega^1_{G/R}) \cong s^*(\Omega^1_{G_f/R}).$$

Over a complete local ring, we have the connected etale sequence for G_f. Let G^0 be the connected component. By the same argument as above, then $s^*(\Omega^1_{G/R}) \cong s^*(\Omega^1_{G^0/R})$. The formula then follows from the next proposition. □

Proposition 2.10. *Let H be a connected finite flat group scheme over a complete discrete valuation ring R with finite residue field. Then,*

$$|s^*(\Omega^1_{H/R})|^{\#H} = |R/\operatorname{disc}(H)|.$$

Proof. Write $H = \operatorname{Spec} A$. Then, A is a local ring free of rank $\#H$ over R. Let I be the kernel of the identity morphism $A \to R$. Note that $I/I^2 = s^*(\Omega^1_{H/R})$. Furthermore, it is a general fact that for any noetherian ring R and any Hopf algebra A over R, $\Omega^1_{A/R} = A \otimes_R I/I^2$. This is just the fact that $\Omega^1_{H/R}$ is generated by invariant differentials.

Since A is free over R, we get that

$$|\Omega^1_{A/R}| = |s^*(\Omega^1_{H/R})|^{\#H}$$

so it suffices to compare $|\Omega^1_{A/R}|$ with the right-hand side.

The extension $R \to A$ is monogenic, generated by an element α with minimal polynomial f, so $\Omega^1_{H/R}$ is an A-module generated by dx, which is annhilated by $f'(\alpha)$. On the other hand, the different $\delta_{H/R}$ is generated by $f'(\alpha)$, so $\Omega^1_{H/R}$ is a free rank 1 module over $A/\delta_{H/R}$. Thus, we are reduced to comparing the size of $A/\delta_{H/R}$ with the size of $R/\operatorname{disc}_{H/R} = R/N_{H/R}(\delta)$. So suppose that $\mathfrak{P}$ is the prime ideal of A, $\mathfrak{p}$ is the prime ideal of R, and $\delta_{H/R} = \mathfrak{P}^n$. Let e, f be the ramification and inertial degrees of A/R. Then $\operatorname{disc}_{H/R} = \mathfrak{p}^{fn}$, so

$$|R/\operatorname{disc}_{H/R}| = |R/\mathfrak{p}|^{fn} = |A/\mathfrak{P}|^n = |A/\delta_{H/R}|.$$

□

Definition 2.11. Let R be a henselian local ring and let G be a quasi-finite separated flat group over R. We define the *connected part* G^0 of G to be the connected subgroup of G_f. Note the G^0 is functorial, and satisfies a universal property for connected finite flat group schemes over R.

3. Heights in l-power towers of Isogenies

As we will recall in the next section, a key step in the proof of the Tate conjecture involves showing that among a certain family of ℓ^n-power isogenies $B_n \to A$, there are infinitely many B_n, which are isomorphic. By Theorem 1.2, it would suffice to show that all B_n have polarizations of fixed degree d and that their heights are bounded. In this section, we show that the sequence $h(B_n)$ is bounded.

Remark 3.1. In Faltings' original paper, he asserted that $h(B_n) = h(A)$ for all n. This is in fact not necessarily true. We will prove the correct statement that the heights stabilize for large n.

Theorem 3.2. *Let $\pi : A \to \operatorname{Spec} \mathcal{O}_K$ be a semi-abelian variety with proper generic fiber, ℓ a prime number, and $G/K \subset A[\ell^\infty]/K$ a sub-ℓ-divisible group. Furthermore, let G_n be kernel of ℓ^n in G and $B_n := A/G_n$. Then the set $h(B_n)$ is bounded.*

We will prove this theorem in a series of steps. The key inputs will be the Hodge-Tate decomposition for p-divisible groups and Weil conjectures for abelian varieties.

Since A has everywhere semi-stable reduction over K so do all the B_n (by the inertial criterion if you like). By our discussion earlier, each isogeny $A \to B_n$ extends to an isogeny $\mathcal{A} \to \mathcal{B}_n$ over $\mathcal{O}_K$ with kernel $\mathcal{G}_n$, where $\mathcal{G}_n$ is a quasi-finite flat group schemes over $\mathcal{O}_K$ which controls the change in height $h(A) - h(B_n)$ by Formula 2.1.

By Prop. 2.9, it suffices to study $\mathcal{G}_{n,v} := \mathcal{G}_n \otimes_{\mathcal{O}_K} \mathcal{O}_v$ for each place v dividing ℓ. In fact, we are interested in something even smaller, the finite part $\mathcal{G}^f_{n,v}$. The functoriality of all our constructions gives us inclusions $i_{n,v} : \mathcal{G}^f_{n,v} \to \mathcal{G}^f_{n+1,v}$. Unfortunately, these need not form an ℓ-divisible group (this is roughly where Faltings made his error in the original paper).

Lemma 3.3. *For some N sufficiently large, the systems*

$$H_{n,v} := \mathcal{G}^f_{N+n,v}/\mathcal{G}^f_{N,v}$$

form an ℓ-divisible groups over $\mathcal{O}_v$ for all places v dividing ℓ.

Proof. Clearly, it suffices to do it for each v separately so we drop v and work entirely over $\mathcal{O}_v$. This goes back to an argument of Tate in his article [6] on p-divisible groups. If you go back to the proof of Lemma 1.7 in L10, you will see that it suffices to show that the generic fiber of the system $\mathcal{G}^f_n$ of finite flat group schemes is an ℓ-divisible group.

Roughly, the idea is that a system (H_n, i_n) is an ℓ-divisible group if it has the right orders and if the multiplication by ℓ-map $H_{n+1}/H_n \to H_n/H_{n-1}$ is an isomorphism. Given that the generic fiber is an ℓ-divisible group, and we are in generic characteristic 0, H_{n+1}/H_n are orders in the same etale algebra for all n and so the tower eventually stabilizes.

To show, that the generic fibers of $\mathcal{G}^f_n$ form an ℓ-divisible group, we use what we know about the ℓ-divisible group of A. Namely, Brian showed in L13 that for a semi-abelian variety $\mathcal{A}$ with proper generic fiber, $\mathcal{A}[\ell^n]_f$ forms an ℓ-divisible group (See L13 Lemma 5.4).

By Lemma 2.8, $\mathcal{G}^f_n = \mathcal{G} \cap \mathcal{A}[\ell^n]_f$. Thus, the generic fiber of $\{\mathcal{G}^f_n\}$ is just $G \cap A[\ell^\infty]_f$ which is an ℓ-divisible group because the intersection of two etale ℓ-divisible groups over a field is always an ℓ-divisible. One can see this in terms of Tate modules or pass to the separable closure and work with constant group schemes.

Note that it was crucial that we had a good description of the "finite" part of the generic fiber. □

Replacing A with B_N, we can assume that the finite part of the kernels $\mathcal{G}^f_{n,v}$ are ℓ-divisible groups over R_v for all v. By Formula 2.9, we know that the change in height only depends on the connected parts $\mathcal{G}^0_{n,v}$ which also form a ℓ-divisible group.

Tate's formula (see L9 pg. 15) says that

$$\mathrm{disc}(\mathcal{G}^0_{n,v}) = \ell^{d_v n \ell^{h_v n}} = \ell^{d_v n |\mathcal{G}^0_{n,v}|}.$$

where d_v is the dimension of formal group corresponding to $\{\mathcal{G}^0_{n,v}\}$.

Applying Formula 2.9,

$$|s^*(\Omega^1_{G_v/R_v})|^{\ell^{h_v n}} = |R_v/(\ell^{d_v n(\ell^{h_v n}})| = \ell^{m_v d_v n(\ell^{h_v n})}$$

where $m_v = [K_v : \mathbb{Q}_\ell]$.

A straightforward calculation shows that $h(B_n) = h(A)$ exactly when

$$\frac{1}{2}hm = \sum_v m_v d_v$$

where h is the height of G/K and $m = [K : \mathbb{Q}]$. Note the degree of ϕ_n is ℓ^{nh} which is where the h comes from.

This is the formula we will prove in the next section. Before do this, we discuss the basic theory of Hodge-Tate representations. This is necessary because the numbers d_v above are very mysterious. The way we get our hands on them is by analyzing the Galois representations on the generic fiber. The d_v then appear because of the Hodge-Tate decomposition for p-divisible groups which was discussed in L10.

3.1. **Hodge-Tate Representations.** Fix $\mathbb{C}_\ell := \widehat{\overline{\mathbb{Q}_\ell}}$. Let K be a ℓ-adic field and fix an embedding $K \hookrightarrow \overline{\mathbb{Q}_\ell}$, and let $\Gamma_K := \mathrm{Gal}(\overline{\mathbb{Q}_\ell}/K)$.

Let V be a ℓ-adic representation of Γ_K. The theory of Hodge-Tate represents the very first step in studying such V and the most basic operation in p-adic Hodge theory. We recall some key results from the theory to simplify exposition

Definition 3.4. Let ζ_{ℓ^n} be a primitive ℓ^n root of unity. For any $\sigma \in \Gamma_K$, let $\sigma(\zeta_{\ell_n}) = \zeta_{\ell_n}^{n_\sigma}$. The *$\ell$-adic cyclotomic character* $\chi_\ell : \Gamma_K \to \mathbb{Z}_\ell^\times$ is given by

$$\chi_\ell(\sigma) := \lim_{\leftarrow} n_\sigma.$$

It is a continuous ℓ-adic character.

Definition 3.5. Define $\mathbb{C}_\ell(i)$ to be $\mathbb{C}_\ell$ equipped with the twisted Galois action given by

$$\sigma.x = \chi_\ell^i(\sigma) * \sigma(x)$$

Note that the above action is not $\mathbb{C}_\ell$-linear. One should think of $\mathbb{C}_\ell(i)$ is a one-dimensional $\mathbb{C}_\ell$-vector space equipped with the given semi-linear action of Γ_K.

We are interested in studying $V \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell$, for any ℓ-adic representation V, which is a $\mathbb{C}_\ell$-vector space of dimension $= \dim_{\mathbb{Q}_\ell} V$ with a semi-linear action of Γ_K-action (via the diagonal) for any particular V.

Definition 3.6. An ℓ-adic representation V is called *Hodge-Tate* if $V \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \cong \oplus_i \mathbb{C}_\ell(i)^{h_i}$ as a $\mathbb{C}_\ell$-vector space with semi-linear Galois action. If V is Hodge-Tate, then the set of non-zero i are called Hodge-Tate weights and each Hodge-Tate weight occurs with multiplicity h_i.

The above weight decomposition is not canonical but the h_i are well-defined numbers. See Example 2.3.5 in [2]. This is all seen more naturally using a period ring formalism which we don't need here.

Theorem 3.7. *The property of being Hodge-Tate is insensitive to replacing K by a finite extension K' (i.e. restricting to $\Gamma_{K'}$), or to replacing K by $\widehat{K}^{\mathrm{un}}$ (i.e. only depends on the action of I_K). The Hodge-Tate weights are similarly insensitive to such finite extensions.*

Proof. See [2] Th'm 2.4.6. ☐

The number we are actually interested in is $t_H(V) := \sum_i i h_i$.

Proposition 3.8. *Let V be a Hodge-Tate representation, then* $\det(V)$ *is Hodge-Tate and $t_H(V)$ is the unique Hodge-Tate weight of* $\det(V)$*. The Hodge-Tate weight of a one-dimensional representations is a well-defined.*

Proof. If $V \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \cong \oplus_i \mathbb{C}_\ell(i)^{h_i}$, then clearly $\det(V) \otimes \mathbb{C}_\ell \cong \det(V \otimes \mathbb{C}_\ell) \cong \mathbb{C}_\ell(\sum_i i h_i)$.

It remains to show that there are no Γ_K-equivariant isomorphisms $\mathbb{C}_\ell(i) \cong \mathbb{C}_\ell(j)$ for $i \neq j$. For this, note that

$$\mathrm{Hom}_{\mathbb{C}_\ell}(\mathbb{C}_\ell(i), \mathbb{C}_\ell(j))^{\Gamma_K} = \mathbb{C}_\ell(j-i)^{\Gamma_K}.$$

Since $j - i \neq 0$, the character χ_ℓ^{j-i} satisfies the hypotheses of L10 Th'm 3.1 (the main cohomological result of Tate's paper [6]), which says that $\mathbb{C}_\ell(j-i)^{\Gamma_K} = 0$. ☐

We will want to pass between local and global perspectives so for the remainder of the section, choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$, which gives us an inclusion $\Gamma_{\mathbb{Q}_\ell} \hookrightarrow \Gamma_{\mathbb{Q}}$.

In the case of one-dimensional representations of $\Gamma_{\mathbb{Q}}$, we can read off the Hodge-Tate weight (when restricted to $\Gamma_{\mathbb{Q}_\ell}$) in a straightforward way.

Lemma 3.9. *(Class Field Theory) Let $\Gamma_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$. Let $\chi : \Gamma_{\mathbb{Q}} \to \mathbb{Z}_\ell^\times$ be a continuous character unramified outside finitely many places which is Hodge-Tate at ℓ. Then,*

$$\chi = \chi_\ell^d \cdot \chi_0$$

where χ_0 is a finite order character and d is the Hodge-Tate weight.

Proof. Let d be the Hodge-Tate weight of $\mathbb{Q}_\ell(\chi)$ at ℓ. Set $\chi_0 = \chi \cdot \chi_\ell^{-d}$. We want to show that χ_0 if finite order.

Let Σ be the set of ramified primes of χ_0. Class field theory tells us that χ_0 corresponds to a continuous character

$$\hat{\chi_0} : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / \prod_{p \notin \Sigma} \mathbb{Z}_p^\times \to \mathbb{Z}_\ell^\times.$$

To show χ_0 is finite order, it suffices to show that at each prime p if we restrict $\hat{\chi_0}$ to $\mathbb{Z}_p^\times$, $\hat{\chi_0}$ is trivial on an open subgroup because $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$ modulo any open compact subgroup is finite.

Consider the restriction of $\hat{\chi_0}$ to $\mathbb{Z}_\ell^\times$ which corresponds the inertial group at ℓ via local class field theory. A theorem of Tate (see [2] Th'm 2.2.7) says a character η of a local Galois group is Hodge-Tate of weight 0 if and only if the image of inertia is finite. Since χ_0 is Hodge-Tate of weight 0, $\hat{\chi_0}$ is trivial on some finite index subgroup.

For all other $p \in \Sigma$, we note that $\mathbb{Z}_p^\times$ has a finite index subgroup that is pro-p. The incompatibility between pro-p and pro-ℓ forces the restriction to $\mathbb{Z}_p^\times$ to have finite image. □

Let $W = V_\ell(G) \subset V_\ell(A)$ be the ℓ-adic Galois representations of Γ_K coming from G in Theorem 3.2. We will prove the formula

$$\frac{1}{2} hm = \sum_v m_v d_v$$

by computing $t_H(\mathrm{Ind}_{\mathbb{Q}}^K(W)|_{\Gamma_{\mathbb{Q}_\ell}})$ in two different ways.

Define $V := \mathrm{Ind}_{\mathbb{Q}}^K(W)$ which is an mh-dimensional representation of $\Gamma_{\mathbb{Q}}$ where h is the height of G.

Recall the Hodge-Tate decomposition for ℓ-divisible groups.

Theorem 3.10. [6] *(Tate) Let G be an ℓ-divisible group over the rings of integers $\mathcal{O}_{K_v}$ of a ℓ-adic field K_v. Then, the Tate module $T_\ell(G)$ as a representation of Γ_{K_v} has Hodge-Tate weights 0 and 1 with the multiplicities given by*

$$h_0 = d', h_1 = d,$$

where d is the dimension of G and d' is the dimension of it's Cartier dual.

Proof. See L10 Corollary 2.15, where this result is stated in it's dual form. □

In particular, if a representation comes from the generic fiber of ℓ-divisible group, then it is Hodge-Tate. We want to show V is Hodge-Tate and compute its Hodge-Tate weights.

We will employ the following result from representation theory several times:

Proposition 3.11. *Let H be finite index subgroup of a group M, and H' any subgroup of M. Let $S = H'\backslash M/H$ be the finite double coset space. For each $s \in S$, define $H_s = sHs^{-1} \cap H'$. Let $\rho : H \to \mathrm{GL}(W)$ be a representation. Then,*

$$\mathrm{Res}_{H'}\mathrm{Ind}_H^M(W) \cong \oplus_{s\in S}\mathrm{Ind}_{H_s}^{H'}(W_s)$$

where W_s is the representations given by $\rho^s(x) = \rho(s^{-1}xs)$ acting on W.

Proof. See Serre [5] Prop. 22 §7.3. □

Let v be a place of K dividing ℓ, and let $m_v = [K_v : \mathbb{Q}_\ell]$. As before, let d_v denote the dimension of the connected part of $\mathcal{G}_v^f$.

Lemma 3.12. *Let V and W be as above. Choose embedding $i_v : K_v \hookrightarrow \overline{\mathbb{Q}_\ell}$ for all places v dividing ℓ. Then,*

$$\mathrm{Res}_{\Gamma_{\mathbb{Q}_\ell}}V = \oplus_{v|\ell}\mathrm{Ind}_{\mathbb{Q}_\ell}^{K_v}\mathrm{Res}_{\Gamma_{K_v}}W.$$

Proof. In order to restrict to $\Gamma_{\mathbb{Q}_\ell}$, we have to choose j an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_\ell}$. To apply, Prop. 3.11, we study the coset

$$\Gamma_{\mathbb{Q}_\ell}\backslash\Gamma_{\mathbb{Q}}/\Gamma_K.$$

The right coset space $\Gamma_{\mathbb{Q}}/\Gamma_K$ can be identified with the set of embedding $K \hookrightarrow \overline{Q}$ whcih we denote by Σ. The map j identifies Σ with the set of embeddings $K \hookrightarrow \overline{\mathbb{Q}_\ell}$. This breaks up Σ as a disjoint union

$$\Sigma = \coprod_{v|\ell}\Sigma_v$$

where Σ_v is the set of embeddings of $K_v \hookrightarrow \overline{\mathbb{Q}_\ell}$. The left action of $\Gamma_{\mathbb{Q}_\ell}$ on Σ is via its action on $\overline{\mathbb{Q}_\ell}$. The orbits then are exactly given by Σ_v i.e. $\Gamma_{\mathbb{Q}_\ell}\backslash\Gamma_{\mathbb{Q}}/\Gamma_K$ can be identified with the set of places v_n dividing ℓ.

Choosing coset representatives s_n, we see that

$$s_n\Gamma_K s_n^{-1} \cap \Gamma_{\mathbb{Q}_\ell} = \Gamma_{K_{v_n}}$$

under some choice of embedding $K_{v_n} \hookrightarrow \overline{\mathbb{Q}_\ell}$. The lemma follows then from Prop. 3.11. □

Lemma 3.13. *Let W be an ℓ-representation of Γ_{K_v} whose determinant is Hodge-Tate. Then,* $\det(\mathrm{Ind}_{\mathbb{Q}_\ell}^{K_v}(W))$ *is Hodge-Tate, and*

$$t_H(\mathrm{Ind}_{\mathbb{Q}_\ell}^{K_v}(W)) = [K_v : \mathbb{Q}_\ell] t_H(W).$$

Proof. Let L be a Galois closure of K_v over $\mathbb{Q}_\ell$. By Corollary 3.7, we can restrict to the finite index subgroup Γ_L. Over Γ_L, Proposition 3.11, tells us that $\mathrm{Ind}_{\mathbb{Q}_\ell}^{K_v}(W)$ breaks a part as

$$\oplus_{i=1}^{[K_v:\mathbb{Q}_\ell]} W_i$$

where the W_i are all isomorphic to W as Γ_L representations. Clearly, then

$$\det(\mathrm{Ind}_{\mathbb{Q}_\ell}^{K_v}(W)) \cong \det(W)^{[K_v:\mathbb{Q}_\ell]}$$

over Γ_L. Again by Corollary 3.7, $\det(W)|_{\Gamma_L}$ has Hodge-Tate weight $t_H(W)$ and so we are done. □

Proposition 3.14. *The represention* $\det(W|_{\Gamma_{K_v}})$ *for any v dividing ℓ is Hodge-Tate of weight d_v.*

Proof. This is a very non-trivial result. We will invoke both the Hodge-Tate decomposition for ℓ-divisible groups and the orthogonality theorem for semi-stable abelian varieties.

Everything is over a fixed K_v so we go ahead use G to denote the base change of G to K_v. Recall that $W = V_\ell(G)$ and that G has an ℓ-divisible subgroup G_f which extends to an ℓ-divisible group over $\mathcal{O}_v$. Let $W_f := V_\ell(G)$ which is a Γ_{K_v}-stable subspace.

Theorem 3.10 says that W_f is Hodge-Tate with Hodge-Tate weights 0 and 1 with $h_1 = d_v$. Now, $\det(W) = \det(W_f) \otimes \det(W/W_f)$ so it suffices to show that W/W_f is Hodge-Tate of weight 0. We will show something a bit stronger, namely, that W/W_f is unramified. (Note that Prop 3.7 says unramified implies HT-weight 0.)

W_f is constructed from the the integral model so how do we get a handle on W/W_f. Recall from Lemma 2.8 that $G_f = G \cap A[\ell^\infty]_f$ so that $W_f = W \cap V_\ell(A)_f$, where $V_\ell(A)$ is the lift of the torsion of special fiber of the semi-abelian model.

The Orthogonality theorem (Th'm 5.5 in L13) says that $V_\ell(A)/V_\ell(A)_f$ has trivial I_{K_v} action. Since W/W_f is a sub-representation of $V_\ell(A)/V_\ell(A)_f$ it is also unramified. □

Lemma 3.12 says that

$$\mathrm{Res}_{\Gamma_{\mathbb{Q}_\ell}} \det(V) = \otimes_{v|\ell} \det(\mathrm{Ind}_{\mathbb{Q}_\ell}^{K_v} \mathrm{Res}_{\Gamma_{K_v}} W).$$

By Lemma(3.13) combined with Theorem 3.10, the right-hand side has Hodge-Tate weight $\sum_{v|\ell} d_v m_v$. Thus, $\sum_{v|\ell} d_v m_v = d$ as desired.

We now can compute $t_H(V)$ in another way.

Proposition 3.15. *The unique Hodge-Tate weight of* $\det(V)$ *is equal to*

$$\frac{1}{2}hm.$$

Proof. Let $L := \det(V)$. By Lemma 3.9, L is given by a character $\chi = \chi_\ell^d * \chi_0$. By Prop. **??**, $L|_{\Gamma_{\mathbb{Q}_\ell}}$ is Hodge-Tate with weight d. We want to show that $d = \frac{1}{2}hm$.

The idea is to use global information (Weil conjectures) to determine d. The representation $V \subset \mathrm{Ind}_{\mathbb{Q}}^K V_\ell(A)$. I claim that for almost all primes p, the characteristic polynomial of the Frobenius F_p acting on V, has roots which are algebraic integers which have absolute value $p^{1/2}$ under any complex embedding. These are called Weil numbers of weight 1.

By almost all p, we mean primes of good reduction not equal to ℓ. To show this for V, it suffices to show the same property holds for F_p acting on $\mathrm{Ind}_{\mathbb{Q}}^K V_\ell(A)$. One could deduce this from the Weil conjectures on A and some representation theory. However, one can also note that

$$\mathrm{Ind}_{\mathbb{Q}}^K V_\ell(A) = V_\ell(\mathrm{Res}_{\mathbb{Q}}^K A)$$

where $\mathrm{Res}_{\mathbb{Q}}^K A$ is the Weil restriction of scalars. This is a consequence of the adjunction formula for Weil restriction (see Neron Models [1] §7.5 Lemma 1). The restiction of scalars $A' = \mathrm{Res}_{\mathbb{Q}}^K A$ is an abelian variety, and so we can apply the Weil conjectures to A'. See Mumford [4] Chap. 21 Theorem 4 for the statement and proof of the Weil conjectures for abelian varieties.

Given that the characterstic polynomial for F_p has the desired form, we see that $\det(F_p) = \chi(F_p)$ is an algebraic integer with complex absolute value $p^{mh/2}$ under any embedding, since V has dimension mh. Since χ_0 is finite order, $\chi_0(F_p)$ is a root of unity which does not effect the absolute value. Thus, under any embedding,

$$|\chi(F_p)| = |\chi_\ell^d(F_p)| = |p^d|$$

because $\chi_\ell(F_p) = p$. □

This will complete the proof of Theorem 3.2.

4. Proof of Main Theorem

In this section, we prove the Tate conjecture for abelian varieties over number fields. The proof follows the same line as Tate's proof [7] over finite fields which was discussed in L3. At a critical point, there is the additional input of Zarhin's trick [8].

In L3, Sam discussed the proof of the Tate conjecture if one knows the strong hypothesis or the weak hypothesis plus some additional assumptions. By the end of the paper, Faltings does establish the strong hypothesis over number fields; however, he uses the Tate conjecture in a critical way so we must prove the Tate conjecture in some other way. His proof really threads the needle in the sense that he doesn't even establish the full strength of the weak hypothesis and yet emerges at the end with everything falling into place.

In L3 §3, Sam showed that to establish Th'm [7] for all abelian varieties over K, it suffices to show the following weaker statement:

Theorem 4.1. *Let A be an abelian variety over a number field K, then the natural map*

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell, \Gamma_K}(V_\ell(A))$$

is an isomorphism.

At the same time that we prove Th'm 4.1, we will prove semi-simplicity of the Galois action.

Theorem 4.2. *Let A be an abelian variety over a number field K. Then, Γ_K acts semi-simple on $V_\ell(A)$ for all ℓ.*

Note that over finite fields we knew Th'm 4.2 already, and we knew that Th'm 4.1 for one ℓ implies it for all ℓ. Both of these facts Tate exploited and we don't have.

Both theorems follow from the following lemma:

Lemma 4.3. *(Projection Lemma) Let $W \subset V_\ell(A)$ be a Γ_K-stable subspace. Then, there exists a operator $u \in \mathrm{End}(A) \otimes \mathbb{Q}_\ell$ such that $u^2 = u$ and $uV_\ell(A) = W$.*

The lemma immediately implies the semi-simplicity of the Galois action (Th'm 4.2). For Th'm 4.1, if you look back of the Tate conjecture in §6 of L3 under the strong finiteness hypothesis, you will see that the only place it gets used is in the proof of Prop 6.1.1, which is exactly the Projection Lemma.

As we are looking for projectors, it is often easier to find multiple elements that do the job. The following fact about the semi-simple algebras over a field says that is enough.

Proposition 4.4. *Let E be a finite semi-simple algebras over a field. For example, $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$. Then any right ideal of E is principal generated by an idempotent element.*

Proof. Unable to find reference, Brian generously provided the following proof:

We work with left ideals, but it's all the same (using the opposite algebra, for example). A finite-dimensional semisimple algebra over a field k is a direct product of finitely many simple algebras, and the module theory decomposes accordingly, so we can pass to the case when E is simple, so its center is a field (rather than a product of several such). We can therefore rename that center as k, so E is a central simple E-algebra. Then $E = \mathrm{Mat}_n(D)$ for a central division algebra D over k. I claim that the left ideals of E are exactly the annihilators of right D-subspaces of $V = D^n$, from which the assertion is obvious.

More specifically, if J in E is a left ideal and if W is the set of elements of V killed by J under the left E-action then W is a right D-subspace of V and the set $\mathrm{Ann}_E(W)$ of elements of E killing W under the left action on V is a left ideal that contains J. So we want the containment of J in $\mathrm{Ann}_E(W)$ to be an equality. We may and do assume J is nonzero (as the case $J = 0$ is obvious). Assume W has dimension r.

If we pick any r right D-hyperplanes H_i such that $W = \cap H_i$, then note that $\mathrm{Ann}_E(W) = \oplus_{i=1}^r \mathrm{Ann}_E(H_i)$ as left E-modules. Such hyperplanes exists because any right D-basis for W can be extended to a right D-basis of V. This is an obvious generalization of a basis fact in linear algebra over a field. Important also is that the number of elements in any right D-basis is the same. The same row reduction arguments work in this situation precisely because D is a division algebra.

Now, one can check by hand that $\mathrm{Ann}_E(H_i)$ is a simple left E-module using that all hyperplanes are translates of the standard one. Hence $\mathrm{Ann}_E(H_i) \cong D^n$; in particular it has D-dimension n. Thus, $\mathrm{Ann}_E(W)$ has D-dimension nr. It suffices to show that J does as well.

If J is simple, then we must show that W is hyperplane. Since J is simple, it is generated by a single element u and $W = \ker u$. Let $x \in E$ such that $\ker(x) \supset W$. If we choose a basis for V compatible with W, then u has the matrix form with 0's in the columns corresponding to W and rank equal to $\dim V - \dim W$. Left multiplication by E corresponds to row operations. Since row space x is contained in row space of u, we can generate x by row operations on u. Thus, $J = \mathrm{Ann}_E(W)$ and since J is simple, W must be a hyperplane.

As a left E-module, J decomposes a the direct sum $\oplus_{i=1}^m J_i$, where J_i are simple left ideals. By the base case, $J_i = \mathrm{Ann}_E(H_i)$ for a right D-hyperplane H_i of V. It is clear that $\cap H_i \subset W$, as J clearly annihilates anything in $\cap H_i$. Thus, $\mathrm{codim}(W) \leq m$ and so $\mathrm{Ann}_E(W)$ has D-dimension at most nm. However, $\mathrm{Ann}_E(W) \subset J$ which has D-dimension exactly nm and so we get equality. □

We will the above proposition to the right ideal of elements u such that $uV_\ell(A) \subset W$.

Proposition 4.5. *Let A be an abelian variety over a number field K. If the Projection Lemma holds over for A_L where L is a finite extension of K, then it holds for A. Hence, we can assume that A has semi-stable reduction.*

Proof. Let W be a Γ_K-stable subspace of $V_\ell(A)$. Then, there exists an idempotent $u \in \mathrm{End}_L(A) \otimes \mathbb{Q}_\ell$ such that $u(V_\ell(A)) = W$.

Let σ_i be coset representatives for Γ_K/Γ_L. Since W is Γ_K-stable, we see that $\sigma_i(u)$ has the same property as u. Consider

$$u' = \frac{1}{[L:K]} \sum_i \sigma_i(u).$$

One checks $u'(V_\ell(A)) = W$. Furthermore, Galois descent of morphisms says that u' actually lies in $\mathrm{End}_K(A) \otimes \mathbb{Q}_\ell$. Applying Prop 4.4, we have some idempotent $u \in \mathrm{End}_K(A) \otimes \mathbb{Q}_\ell$ which works. □

Proposition 4.6. *Let A be an abelian variety over K with semi-stable reduction. Let Ψ be a polarization of* $\deg d$. *This induces a non-degenerate skew symmetric form on $V_\ell(A)$. If W is a maximal isotropic subspace of $V_\ell(A)$ which is Galois stable, then the Projection Lemma holds for W (i.e. there exists a u).*

Proof. In Prop 6.4.1 from L3, this is proved under the weak hypothesis. We don't quite have the weak hypothesis, but we have exactly what we need. Consider $W \cap T_\ell(A)$, this is a Galois stable saturated sub-$\mathbb{Z}_\ell$ module of $T_\ell(A)$, and hence corresponds to an ℓ-divisible group $G \subset A[\ell^\infty]$.

The key is to show that among the quotients $B_n := A/G_n$ there are infinitely many which are isomorphic. Then, everything follows just as in Tate's argument. Since W is a maximal isotropic, it follows that B_n has a polarization of $\deg d$ (see Prop 6.4.1 in L3 again). By Th'm 3.2, the heights $h(B_n)$ is bounded. Thus, by Faltings finiteness theorem (1.2), the B_n fall into finitely many isomorphism classes. □

To go from maximal isotropics to any G-stable subspaces, we employ Zarhin's trick. The idea being for a given W to consider appropriately an chosen subspace of $V_\ell(A^2)$, $V_\ell(A^4)$ or $V_\ell(A^8)$. The argument is well-explained in his article "Endomorphisms of Abelian varieties over fields of finite characteristic" [8]. W will make the following simplifying assumption: Assume $\sqrt{-1} \in \mathbb{Q}_\ell$. For the general argument which is very similar see Lemma 2.4 in [8].

Proof of Projection Lemma. Let $\alpha \in \mathbb{Q}_\ell$ be a squareroot of -1 in $\mathbb{Q}_\ell$. Fix a polarization Ψ on A. Consider the following subspace of $V_\ell(A^2)$

$$X := \{(x, \alpha x) \mid x \in W\} + \{(y, -\alpha y) \mid y \in W^\perp\} \subset V_\ell(A) \oplus V_\ell(A).$$

A straightforward calculation shows that X is an isotropic subspace for $\Psi \times \Psi$. This is where we use that $\alpha^2 = -1$.

To show it is a maximal isotropic subspace, we count dimensions. $V_\ell(A^2)$ has dimension $4g$. If W has dimension n, then $\{(x, \alpha x) \mid x \in W\}$ has dimension n and $\{(y, -\alpha y) \mid y \in W^\perp\}$ has dimension $2g - n$. The intersection of these two subspaces is trivial because if $\alpha x = -\alpha x$ then $x = 0$. Thus, X has dimension $2g$ as desired.

By Prop 4.6, there exists $U \in \mathrm{End}(A^2) \otimes \mathbb{Q}_\ell \cong \mathrm{Mat}_{2\times 2}(\mathrm{End}(A) \otimes \mathbb{Q}_\ell)$ which projects onto X. Denote by p_1 and p_2 the two projections from $V_\ell(A^2) = V_\ell(A) \oplus V_\ell(A)$ onto $V_\ell(A)$. Consider the map

$$(p_1 - \alpha p_2) \circ U : V_\ell(A) \oplus V_\ell(A) \to V_\ell(A)$$

whose image is $2W = W$. Thus, we get to elements $u_1, u_2 \in \mathrm{End}(A) \otimes \mathbb{Q}_\ell$ such that $u_1 V_\ell(A) + u_2 V_\ell(A) = W$.

The set of elements $u \in \mathrm{End}(A) \otimes \mathbb{Q}_\ell$ such that $uV_\ell(A) \subset W$ forms a right ideal of the semi-simple algebra $E_\ell = \mathrm{End}(A) \otimes \mathbb{Q}_\ell$. This ideal is principal generated by an idempotent element u which satisfies the condition of the Projection Lemma by Prop 4.4. □

5. Consequences

Corollary 5.1. *Let A_1, A_2 be abelian varieties over K. The following are equivalent:*

1. *A_1 and A_2 are isogenous*
2. *$T_\ell(A_1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong T_\ell(A_2) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ as Γ_K-modules for all ℓ*
3. *$T_\ell(A_1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong T_\ell(A_2) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ as Γ_K-modules for some ℓ*
4. *$L_v(s, A_1) = L_v(s, A_2)$ for almost all places v of K*
5. *$L_v(s, A_1) = L_v(s, A_2)$ for all v*

Proof. (i) implies (ii) is clear. (ii) implies (iii) is clear. (iii) implies (i): By the Tate conjecture, there exists $\phi \in \mathrm{Hom}(A_1, A_2) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ inducing the given isomorphism. Since $\mathrm{Hom}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ is dense in $\mathrm{Hom}(A_1, A_2) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, we can pick a sequence of elements f_n in the former group converging to ϕ. It is an easy exercise to show that for n sufficiently large f_n induces an isomorphism of $\mathbb{Q}_\ell$-Tate modules, and hence corresponds to an isogeny. (ii) implies (v): L-factors can be read off from the Tate module for ℓ not dividing v. Since we have an isomorphism for all ℓ, we have same L-factors for all v. (v) implies (iv) is clear (iv) implies (iii): We have to show that an ℓ-adic Galois representations is determined by the characteristic polynomial of F_v for almost all v. By Cebotarev, the set of all F_v for all but finitely many v is dense in the Galois group, so we know the characteristic polynomial of $g \in G_K$. It is a general fact that a semi-simple representation is determined (up to isomorphism) by the characteristic polynomials, and we know that $T_\ell(A_i) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ are semi-simple by Theorem 4.2. □

Corollary 5.2. *Let A/K be an abelian variety, $d > 0$. Then there are only finitely many isomorphism classes of B/K, with polarization of degree d, such that, for all ℓ, $T_\ell(A) \cong T_\ell(B)$.*

Proof. See Mike's talk. □

References

[1] S. Bosch, W. Lutkebohmert, and Michel Raynaud, "Néron Models", Springer-Verlag, 1980.

[2] O. Brinon and B. Conrad, "CMI Summer School Notes on p-adic Hodge Theory." Available online at http://math.stanford.edu/ conrad/.

[3] G. Faltings, "Finiteness Theorems for Abelian Varieties over Number Fields" in *Arithmetic Geometry*, Cornell-Silverman, 1985.

[4] D. Mumford, *Abelian varieties*, Oxford Univ. Press, 1970.

[5] Jean-Pierre Serre, *Linear Representations of finite groups.*

[6] J. Tate, "p-Divisible Groups" in *Proceedings of a Conference on Local Fields*, Springer-Verlag, 1967.

[7] J. Tate, *Endomorphisms of abelian varieties over finite fields*

[8] Ju. G. Zarhin, *Endomorphism of Abelian varieties over fields of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. **39**(1975), pp. 272–277.