1. Defects in the classical theory:

(1) What if k is not algebraically closed? If A is a finite type k-algebra then for any $\mathfrak{m} \in \mathrm{Max}(A)$, we have $[A/\mathfrak{m} : k] < \infty$, but perhaps $A/\mathfrak{m} \neq k$ for some or even *all* maximal ideals $\mathfrak{m}$. As an example, consider $A = \mathbf{Q}[X,Y]/(X^2 + Y^2 + 1)$ and $\mathfrak{m} = (X, Y^2 + 1)$. Then $A/\mathfrak{m} \simeq \mathbf{Q}(i)$. Weil studied the $\overline{k}$ points of a variety with defining equations having coefficients in k, but his methods did not adequately address what happens under base change $\overline{k} \to L$. For example, take $k = \mathbf{Q}$ and $L = \mathbf{C}$.

(2) What if there is *no* field at all? In the classical theory, Max Spec is functorial on finite type k-algebras (i.e. a map between f.t. k-algebras induces a map between the Max Spec's). Indeed, if A and B are f.t. k-algebras and $\varphi : A \to B$ a map of k-algebras then for any $\mathfrak{m} \in \mathrm{Max}(B)$ we have $\varphi^{-1}(\mathfrak{m}) \in \mathrm{Spec}(A)$, so that $A/\varphi^{-1}(\mathfrak{m})$ is a domain. Moreover, $A/\varphi^{-1}(\mathfrak{m}) \hookrightarrow B/\mathfrak{m}$, and since $B/\mathfrak{m}$ is a *finite* extension of k (equal to k if k is algebraically closed), we see that the domain $A/\varphi^{-1}(\mathfrak{m})$ is finite dimensional over the field k, and hence also a field. It follows that $\varphi^{-1}(\mathfrak{m}) \in \mathrm{Max}(A)$ so that $A \to \mathrm{Max\,Spec}(A)$ is functorial on the category of f.t. k-algebras, as claimed.

For more general rings, however, this is not the case, the canonical counterexample being $\mathbf{Z} \hookrightarrow \mathbf{Q}$ in which the preimage of the maximal ideal (0) is not maximal (in fact minimal!)

(3) Non-reduced rings, even those that are f.t. k-algebras pose a serious problem in the classical theory, even though they occur quite naturally. For example, let k be algebraically closed with $\mathrm{char}(k) \neq 2$ and consider the variety in $\mathbf{A}^2_k$ defined by $y^2 = x$, The coordinate ring is $A = k[x,y]/(y^2 - x)$ and the natural map $k[x] \to A$ gives a map $\mathrm{Max}(A) \to \mathrm{Max}(k[x])$ which corresponds to projection onto the affine line. We can ask what the fiber over any point $x_0 \in k$ is. Since the point x_0 of $\mathbf{A}^1_k$ corresponds to the maximal ideal $(x - x_0)$, we want to determine which maximal ideals of $\mathrm{Max}(A)$ contain the expansion of $(x - x_0)$ to A; but these are precisely the maximal ideals of

$$A/(x - x_0) = k[x,y]/(y^2 - x, x - x_0) = k[y]/(y^2 - x_0),$$

i.e. the fiber is just $\mathrm{Max}(k[y]/(y^2 - x_0))$, which for $x_0 \neq 0$ consists of the two points $(y - \sqrt{x_0})$ and $(y + \sqrt{x_0})$. Observe, however, that when $x_0 = 0$ the fiber ring is $k[y]/(y^2)$, which is a non-reduced ring and the fiber is the single point (y). Classically, this is the notion of a "branch point" for finite maps of Riemann surfaces.

(4) Products: given two affine varieties $X = \mathrm{Max\,Spec}(A)$ and $Y = \mathrm{Max\,Spec}(B)$ over $k = \overline{k}$ we can form their product $X \times Y = \mathrm{Max\,Spec}(A \otimes_k B)$. This is a natural enough thing to want to do, but if $k \neq \overline{k}$, it can happen that $A \otimes_k B$ is not reduced, even if A, B are fields! For example, consider $A = B = \mathbf{F}_p(T^{1/p})$ over $k = \mathbf{F}_p(T)$. Then $u = 1 \otimes T^{1/p} - T^{1/p} \otimes 1 \in A \otimes_k B$ is nilpotent (or order p).

Some reassurance is in order, so that we don' think all of our work over fields was for nought. Indeed, even when working with general rings, the theory over fields plays a central role. For example, consider a map of rings $A \xrightarrow{\varphi} B$. This gives a map $\mathrm{Spec}(B) \xrightarrow{\varphi^*} \mathrm{Spec}(A)$ via $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$, and we can ask what the fiber over $\mathfrak{p}$ is. As we saw above, the fiber consists of all prime ideals of B which contract to $\mathfrak{p}$. These are precisely the prime ideals of B which contain $\varphi(\mathfrak{p})$ and which do not meet $S = \varphi(A - \mathfrak{p})$ (for if a prime $\mathfrak{q}$ of B meets S then $\varphi^*(\mathfrak{q})$ contains an element of $A - \mathfrak{p}$ and hence is not equal to $\mathfrak{p}$). But these are exactly the prime ideals of $S^{-1}B/\mathfrak{p}S^{-1}B = B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. In fact, ${\varphi^*}^{-1}(\mathfrak{p})$ is homeomorphic to $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ (see Atiyah-MacDonald, pg. 47).

contain the expansion of $\mathfrak{p}$; this corresponds to the maximal ideals of the ring

2. Sheaves and ringed spaces.

Let X be a topological space and $\mathscr{C}$ any category with final object. For example, we might take $\mathscr{C}$ to be the category of abelian groups, rings, or sets in which the final objects are, respectively, the trivial group, the zero-ring, and the one-element set.

Definition 2.1. A *presheaf* $\mathcal{F}$ on X with values in $\mathscr{C}$ is an assignment

$$\mathcal{F} : U \to \mathcal{F}(U)$$

which associates to every open set $U \subseteq X$ an object $\mathcal{F}(U)$ of $\mathscr{C}$, together with the data of restriction maps

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

whenever $V \subseteq U$, such that for any $W \subseteq V \subseteq U$ we have $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$. We require that $\mathcal{F}(\emptyset)$ be the final object of $\mathscr{C}$.

A more sophisticated way to phrase this is to say that a presheaf $\mathcal{F}$ is a functor $\mathcal{F} : \mathfrak{Top}(X) \to \mathscr{C}$ from the open sets of X to $\mathscr{C}$.

Definition 2.2. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves is a collection of maps $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subseteq X$ such that for any $V \subseteq U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho_{U,V;\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \rho_{U,V;\mathcal{G}}} \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V) \end{array}$$

commutes.

As an example, let X be a $\mathbf{C}^\infty$ manifold, take $\mathscr{C}$ to be the category of $\mathbf{R}$-vector spaces, and consider the presheaf $\mathcal{O}_X$ on X to be the assignment $U \to \{\mathbf{C}^\infty \text{ functions } U \to \mathbf{R}\}$ (with usual restriction of functions). Let Ω^1_X be the presheaf given by $U \to \{\omega \in \Omega^1(U)\}$ (that is, the presheaf of $\mathbf{C}^\infty$ 1-forms). Then we have a morphism of sheaves

$$d : \mathcal{O}_X \to \Omega^1_X$$

given on $\mathcal{O}_X(U) \to \Omega^1_X(U)$ as $f \mapsto df$.

Definition 2.3. If $\mathcal{F}$ is a presheaf on X, a subpresheaf is a pair $(\mathcal{G}, i)$ where $\mathcal{G}$ is a presheaf on X and $i : \mathcal{G} \to \mathcal{F}$ is a morphism of presheaves with $i_U : \mathcal{G}(U) \to \mathcal{F}(U)$ injective (monic) for all open $U \subseteq X$.

(1) Let X be a topological space and $\mathcal{F}$ the presheaf of continuous $\mathbf{R}$-valued functions.
(2) Let $X' \xrightarrow{f} X$ be a continuous map of topological spaces. The presheaf of sections of f (on X) is given by
$$\mathcal{F}(U) = \{s : U \to f^{-1}(U) : f \circ s = \mathrm{id}_U\}.$$
This is a presheaf of sets and it explains the practice of calling the elements of $\mathcal{F}(U)$ "sections over U" for arbitrary presheaves $\mathcal{F}$.
(3) Let $X' \subseteq X$ be $\mathbf{C}$-manifolds and $\mathcal{O}_{X'}, \mathcal{O}_X$ the presheaves of holomorphic functions. Let $\mathcal{I}_{X'}$ be the presheaf on X defined by
$$\mathcal{I}_X(U) = \ker\left(\mathcal{O}_X(U) \overset{\rho_{U,X'\cap U}}{\longrightarrow} \mathcal{O}_{X'}(U \cap X')\right).$$
This is called the ideal sheaf associated to X'.