1. Last time

Recall that in classical geometrical examples of ringed spaces we have

(1) If $(M, \mathcal{O}_M)$ is a ringed space then $\mathcal{O}_{M,m}$ is a local ring with maximal ideal consisting of those functions that vanish at m.
(2) If $f : M' \to M$ is a map and $f^\#$ is the "compose with f" map $\mathcal{O}_{M'} \to f_*\mathcal{O}_M$ then the induced map

$$(f^\#)_{f(m)} : \mathcal{O}_{M',f(m)} \longrightarrow (f_*\mathcal{O}_M)_{f(m)} \longrightarrow \mathcal{O}_{M,m}$$

are local maps (that is, $(f^\#)_{f(m)}(\mathfrak{m}_{f(m)}) \subseteq \mathfrak{m}_m$).

Observe that there are many examples maps of local rings which are not local maps: $\mathbf{C}[[t]] \hookrightarrow \mathbf{C}((t))$ and $\mathbf{Z}_{(p)} \hookrightarrow \mathbf{Q}$ are two examples.

2. Locally ringed spaces

Definition 2.1. A *locally $\mathscr{C}$-ringed space* is a $\mathscr{C}$-ringed space $(X, \mathcal{O}_X)$ such that for all $x \in X$, the ring $\mathcal{O}_{X,x}$ is local (in particular nonzero).

The category of locally $\mathscr{C}$-ringed spaces has objects consisting of locally $\mathscr{C}$-ringed spaces and has morphisms $\varphi = (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that for all $x \in X$,

$$\varphi_{f(x)} : \mathcal{O}_{Y,f(x)} \longrightarrow (f_*\mathcal{O}_X)_{f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local map.

Remark 2.2. **Evaluation**: Let $(X, \mathcal{O}_X)$ be a locally ringed space and choose $x \in X$ and open $U \subseteq X$ containing x. For $f \in \mathcal{O}_X(U)$ we have

$$f \in \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x).$$

The image of f in $k(x)$ is denoted $f(x)$. For example, let $(X, \mathcal{O}_X)$ be a real analytic manifold and let $f \in \mathcal{O}_X(U)$. Suppose that $f(x) = c$ (in the usual sense of evaluating a function at a point). Then $f(x) - c \in \mathfrak{m}_x$ so maps to zero in $k(x)$.

Now suppose that K is a field and $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces of K-algebras such that $K \to k(x)$ and $K \to k(y)$ are isomorphisms for all $x \in X$ and $y \in Y$. Then any morphism of ringed spaces

$$(f, f^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is *automatically* local. Indeed,

$$\mathcal{O}_{Y,f(x)} \xrightarrow{f^\#_{f(x)}} (f_*\mathcal{O}_X)_{f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a K-algebra map of local K-algebras having residue field K, so it will suffice to demonstrate the following lemma:

Lemma 2.3. *Let K be a field and $\varphi : A \to B$ a K-algebra map of local K-algebras, each having residue field K. Then φ is local.*

Proof. Choose $a \in \mathfrak{m}_A$ and suppose that $\varphi(a) \notin \mathfrak{m}_B$. Then there exists $c \in K^\times$ such that under the isomorphism $K \tilde{\to} B/\mathfrak{m}_B$ we have $c \mapsto \varphi(a)$. Thus, $\varphi(a) - c \in \mathfrak{m}_B$ and since φ is a K-algebra map, we have $\varphi(a - c) \in \mathfrak{m}_B$. But $c \in K^\times$ and $a \in \mathfrak{m}_A$ so that $a - c$ is a unit in A, whence $\varphi(a - c)$ is a unit in B. This is a contradiction. ■

Let $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of $\mathscr{C}$-locally ringed spaces. Then the map $\varphi_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ induces a unique map $k(\varphi(x)) \to k(x)$: this follows from the fact that $\varphi^\#(\mathfrak{m}_{\varphi(x)}) \subseteq \mathfrak{m}_x$. We therefore have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{\varphi^{\#}} & \mathcal{O}_X(\varphi^{-1}(U)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,\varphi(x)} & \xrightarrow[\varphi_x]{} & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ k(\varphi(x)) & \longrightarrow & k(x) \end{array}$$

By chasing around an element $f \in \mathcal{O}_Y(U)$ in the above diagram, we see that the map $k(\varphi(x)) \to k(x)$ carries $f(\varphi(x))$ to $(\varphi^{\#}(f))(x)$.

If we work with locally ringed spaces of K-algebras such that all residue fields are K then we have a map $\mathcal{O}_X \to C_{X,K}$, where $C_{X,K}$ is the sheaf of K-valued functions on X and for *any* $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{\varphi^{\#}} & \varphi_* \mathcal{O}_X \\ \downarrow & & \downarrow \\ C_{Y,K} & \xrightarrow[\text{old pullback}]{} & C_{X,K} \end{array}$$

However, observe that the map $\mathcal{O}_X \to \mathbf{C}_{X,K}$ need not be injective. As an example, take $X = \operatorname{Spec} \mathbf{C}[T]/T^2$ and let $P = (T)$ be the unique point of X. Define the structure sheaf $\mathcal{O}_X(X) = \mathbf{C}[T]/T^2$ and consider the function T. Then the value of T at P is the image of T in $O_{X,P}/\mathfrak{m}_P = (\mathbf{C}[T]/T^2)/T = \mathbf{C}[T]/T$, which is 0. Thus, the function T, while nonzero, is identically zero on X.

Exercise 2.4. Show that for the category of $\mathbf{C}$-manifolds, the preceding construction identifies morphisms of ringed spaces of K-algebras with morphisms in the "old-fashioned" sense (i.e in the sense of complex manifolds).

3. Some generalities on sheaves

We can now prove some fundamental results about sheaves. We start with

Theorem 3.1. *Let X be a topological space and $\mathcal{F}, \mathcal{G}$ sheaves on X. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then φ is an isomorphism if and only if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all x.*

Proof. If φ is an isomorphism, then for all U the maps $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ are isomorphisms. It follows that the maps $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ are isomorphisms for all $x \in X$. (Put differently, the association $\varphi \to \varphi_x$ is functorial).

Now suppose that $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for each x. Fix U and let $s, t \in \mathcal{F}(U)$ be two sections over U with $\varphi_U(s) = \varphi_U(t)$. Then because of the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow[\varphi_x]{} & \mathcal{G}_x \end{array}$$

we see that $\varphi_x(s_x) = \varphi_x(t_x)$. Since φ_x is injective, we must have $s_x = t_x$ in $\mathcal{F}_x$ for all $x \in U$. By definition of $\mathcal{F}_x$ as a direct limit, for every point $x \in U$ there exists an open set U_x containing x such that $s\big|_{U_x} = t\big|_{U_x}$ in $\mathcal{F}(U_x)$. Since the U_x cover U and $\mathcal{F}$ is a sheaf, we have $s = t$ (by unique glueing). Hence $\varphi : \mathcal{F} \hookrightarrow \mathcal{G}$, so in particular, for every U we may consider $\mathcal{F}(U)$ as a *subset* of $\mathcal{G}(U)$.

Now suppose $s \in \mathcal{G}(U)$. Then since $\mathcal{F}_x \simeq \mathcal{G}_x$, there exists a neighborhood U_x of x and a section $\sigma_{U_x} \in \mathcal{F}(U_x)$ such that $\varphi_{U_x}(\sigma_{U_x}) = s\big|_{U_x} \in \mathcal{G}(U_x)$ (since we have some $\sigma_x \in \mathcal{F}_x$ which maps to s_x). Since $\varphi : \mathcal{F} \to \mathcal{G}$ is injective, such σ_{U_x} are unique. Now observe that $\sigma_{U_x}\big|_{U_x \cap U_{x'}}$ and $\sigma_{U_{x'}}\big|_{U_x \cap U_{x'}}$ both map to $s\big|_{U_x \cap U_{x'}}$ in $\mathcal{G}(U_x \cap U_{x'})$; since the map $\varphi_{U_x \cap U_{x'}} : \mathcal{F}(U_x \cap U_{x'}) \to \mathcal{G}(U_x \cap U_{x'})$ is injective, we conclude that $\sigma_{U_x}\big|_{U_x \cap U_{x'}} = \sigma_{U_{x'}}\big|_{U_x \cap U_{x'}}$ for all $U_x, U_{x'}$. Since $\mathcal{F}$ is a sheaf, we obtain $\sigma \in \mathcal{F}(U)$ such that $\varphi(\sigma)\big|_{U_x} = s\big|_{U_x}$ (using the same commutative diagram as above). Finally, since $\mathcal{G}$ is a sheaf, unique glueing holds, so $\varphi(\sigma) = s$. ∎