1. Sheafification

Theorem 1.1. *Let $\mathcal{F}$ be a presheaf of sets on a topological space X. Then there exists a pair $(\mathcal{F}^+, \theta : \mathcal{F} \to \mathcal{F}^+)$ with $\mathcal{F}^+$ a sheaf, such that for any sheaf $\mathcal{G}$ on X and a map $\mathcal{F} \to \mathcal{G}$, there exists a unique map $\mathcal{F}^+ \to \mathcal{G}$ making the diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\qquad\qquad} & \mathcal{G} \\ & {\scriptstyle\theta}\searrow \qquad \nearrow & \\ & \mathcal{F}^+ & \end{array}$$

commute, i.e. we have a bijection $\operatorname{Hom}_X(\mathcal{F}^+, \mathcal{G}) \overset{\circ\theta}{\longleftrightarrow} \operatorname{Hom}_X(\mathcal{F}, \mathcal{G})$. *Moreover, $\mathcal{F}^+$ is unique up to unique isomorphism and for all $x \in X$ we have an isomorphism $\mathcal{F}_x \simeq \mathcal{F}^+_x$.*

We call $(\mathcal{F}^+, \theta)$ (or by abuse of language, $\mathcal{F}^+$) the *sheafification* of $\mathcal{F}$.

(1) Let $\mathcal{F}$ be the constant presheaf on X associated to the set Σ. Then $\mathcal{F}^+ = \underline{\Sigma}$ is the constant sheaf associated to Σ (i.e. the sheaf of locally constant functions with values in Σ). We claim that $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) = \{\Sigma \to \mathcal{G}(X)\}$. Indeed, since $\mathcal{F}(U) = \Sigma$ for all $U \neq \emptyset$ with restriction maps the identity, to give maps $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subseteq$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(X) = \Sigma & \longrightarrow & \mathcal{G}(X) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \rho_{X,U}} \\ \mathcal{F}(U) = \Sigma & \longrightarrow & \mathcal{G}(U) \end{array}$$

commutes is equivalent to giving a map $\psi : \Sigma \to \mathcal{G}(X)$ since commutativity forces all maps $\mathcal{F}(U) \to \mathcal{G}(U)$ to be induced by ψ.

(2) Let M be a C^∞ manifold and $\mathcal{F}$ the presheaf on M given by $U \mapsto \wedge^k_{\mathcal{O}_M(U)}(\Omega^1_M(U))$. Then we have a canonical map

$$\varphi_U : \wedge^k_{\mathcal{O}_M(U)}(\Omega^1_M(U)) \longrightarrow \Omega^k_M(U)$$

and we claim that the sheaf $U \mapsto \Omega^k(U)$ *is* $\mathcal{F}^+$. Indeed, by the universal property of sheafification, we have a unique map $\mathcal{F}^+ \to \Omega^k_M$ making the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ \theta\ } & \mathcal{F}^+ \\ & \searrow & \downarrow \\ & & \Omega^k_M \end{array}$$

commute. But the map $\theta_x : \mathcal{F}_x \to \mathcal{F}^+_x$ is an isomorphism on stalks, and it is not hard to see that the canonical map $\varphi : \mathcal{F} \to \Omega^k_M$ is also an isomorphism on stalks (because every k-form is locally a k-wedge power of 1-forms). Thus, $\mathcal{F}^+ \to \Omega^k_M$ is an isomorphism on stalks; since $\mathcal{F}^+$ and $\mathcal{G}$ are *sheaves*, it follows that $\mathcal{F}^+ \to \mathcal{G}$ is an isomorphism.

Definition 1.2. A presheaf $\mathcal{F}$ on X is *separated* if the map

$$\mathcal{F}(U) \longrightarrow \prod \mathcal{F}(U_i)$$

is injective for all open $U \subseteq X$ and all open covers $\{U_i\}$ of U.

Proof of Theorem 1.1. Let Σ_U be the set of all indexed open covers $\mathscr{V} = \{V_i\}$ of U. We put a partial ordering on Σ_U by $\{V_i\}_{i\in I} = \mathscr{V} \geq \mathscr{V}' = \{V_j'\}_{j\in J}$ if there exists a map $\tau : I \to J$ such that $V'_{\tau(i)} \supseteq V_i$ for all $i \in I$.

Let $\mathcal{F}$ be a presheaf and define $\mathcal{F}_0$ by

$$\mathcal{F}_0(U) = \varinjlim_{\{V_i\}_{i\in I}\in\Sigma_U} \left\{ (s_i) \in \prod_{i\in I} \mathcal{F}(V_i) \ : \ s_i\big|_{V_i\cap V_j} = s_j\big|_{V_i\cap V_j} \text{ in } \mathcal{F}(V_i \cap V_j) \text{ for all } i,j \in I \right\},$$

where the direct limit is formed as follows: for any $\{V_i'\} \geq \{V_j\}$ and any $\tau : I \to J$ we have the map $\prod \mathcal{F}(V_j) \to \prod \mathcal{F}(V_i')$ given by $(s_j) \mapsto (s_{\tau(i)}\big|_{V_i'})$. It is evident that $s_{\tau(i)}$ and $s_{tau(i')}$ agree on $V_i' \cap V_{i'}'$ because $V_i' \cap V_{i'}' \subseteq V_{\tau(i)} \cap V_{\tau(i')}$ and we know that $s_{\tau(i)}$ and $s_{\tau(i')}$ agree on $V_{\tau(i)} \cap V_{\tau(i')}$ already.

We claim that our definition of $\mathcal{F}_0$ is independent of the choices of maps $\tau : I \to J$ that are used in forming the direct limit as described above. To see this, we must show that for any $\sigma, \tau : I \to J$ the sections $s_{\sigma(i)}$ and $s_{\tau(i)}$ agree on V_i', where $V_i' \subseteq V_{\sigma(i)} \cap V_{\tau(i)}$. But this is clear, as $s_{\sigma(i)}$ and $s_{\tau(i)}$ already agree on $V_{\sigma(i)} \cap V_{\tau(i)}$.

We define transition maps $\rho_{U,W} : \mathcal{F}_0(U) \to \mathcal{F}_0(W)$ as follows: given $(s_i) \in \prod \mathcal{F}(V_i)$ with $\{V_i\}$ a cover of U, we obtain a cover of W as $\{W_i = V_i \cap W\}$ and an element $(s_i\big|_{V_i\cap W}) \in \prod \mathcal{F}(V_i \cap W)$ with the s_i compatible on overlaps; hence we get an element of $\mathcal{F}_0(W)$.

Now we assert that:

(1) $\mathcal{F}_0$ is a separated presheaf.
(2) For any separated presheaf $\mathcal{G}$ and any map $\mathcal{F} \to \mathcal{G}$ there exists a unique map $\mathcal{F}_0 \to \mathcal{G}$ making the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta_0} & \mathcal{F}_0 \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

commute.

We first prove (1). We need to show that given an open cover $\{U_\alpha\}$ of U and sections $s, t \in \mathcal{F}_0(U)$ with $s\big|_{U_\alpha} = t\big|_{U_\alpha}$ in $\mathcal{F}_0(U_\alpha)$ then $s = t$ in $\mathcal{F}_0(U)$. Therefore, suppose we have such s, t and pick an open cover $\{V_i\}$ of U such that there exist $(s_i) \in \prod \mathcal{F}(V_i)$ and $(t_i) \in \prod \mathcal{F}(V_i)$ representing $s, t \in \mathcal{F}_0(U)$. Now for each α, we see that $\{V_i \cap U_\alpha\}_{i\in I}$ is a cover of U_α. Since $s\big|_{U_\alpha} = t\big|_{U_\alpha}$ in $\mathcal{F}_0(U_\alpha)$, for each α there exists a refinement of $V_i \cap U_\alpha$ (covering U_α) such that the s_i and t_i agree under restriction. Putting these refinements together across all α we obtain a cover of $\{W_j\}$ of U together with "refinements" $(s_j) \in \prod \mathcal{F}(W_j)$ and $(t_j) \in \prod \mathcal{F}(W_j)$ such that $s_j = t_j$ in $\mathcal{F}(W_j)$. Therefore, $s = t$ as elements of $\mathcal{F}_0(U)$ and $\mathcal{F}_0$ is separated.

We now dispense with (2). Since $\mathcal{G}$ is a sheaf, we evidently have an isomorphism $\mathcal{G} \tilde{\to} \mathcal{G}_0$ and any map $\varphi : \mathcal{F} \to \mathcal{G}$ induces a natural map $\varphi_0 : \mathcal{F}_0 \to \mathcal{G}_0$ such that the diagram

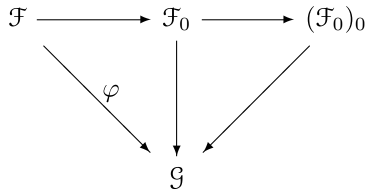
$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ {\scriptstyle\theta_0}\downarrow & & \downarrow \\ \mathcal{F}_0 & \xrightarrow[\varphi_0]{} & \mathcal{G}_0 \end{array}$$

commutes. We need only show that φ_0 is unique. But since $\mathcal{G}$ is a sheaf, it suffices to show that the $(\varphi_0)_x : (\mathcal{F}_0)_x \to (\mathcal{G}_0)_x$ are unique for all x. But from the definition of $\mathcal{F}_0$, it is clear that $(\theta_0)_x : \mathcal{F}_x \to (\mathcal{F}_0)_x$ is an isomorphism for all x. Since the two vertical maps in the diagram

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow{\varphi} & \mathcal{G}_x \\ {\scriptstyle(\theta_0)_x}\downarrow & & \downarrow \\ (\mathcal{F}_0)_x & \xrightarrow[(\varphi_0)_x]{} & (\mathcal{G}_0)_x \end{array}$$

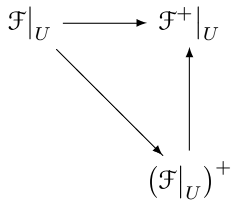
are isomorphisms, we see that $(\varphi_0)_x$ is uniquely determined by φ_x; hence φ_0 is unique.

Now given a map $\varphi : \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf, consider the following diagram:



We have seen that φ induces a unique map $\mathcal{F}_0 \to \mathcal{G}$, and applying this fact twice, we get a unique map $(\mathcal{F}_0)_0 \to \mathcal{G}$. We claim that if $\mathcal{F}$ is any separated presheaf, then $\mathcal{F}_0$ is a sheaf. This essentially follows from the definition of $\mathcal{F}_0$ as the space of "solutions to glueing problems" and the fact that when $\mathcal{F}$ is separated, such solutions are *unique*. ■

We end by recording one obvious property of sheafification: If $U \subseteq X$ is any open set and $\mathcal{F}$ is a presheaf on X, then there is a unique map $\left(\mathcal{F}\big|_U\right)^+ \to \mathcal{F}^+\big|_U$ making the diagram



commute.