Math 632, Lecture 6 January 21, 2004

1. More properties of sheafification

Let X be a topological space and denote by $\mathscr{A}(X)$ the category of sheaves on X (it is an abelian category). Let $\mathcal{F}, \mathcal{G} \in \mathscr{A}(X)$ and $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. For any opens $V \subseteq U$ we have the commutative diagram of abelian groups (or sets or ...)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \ker(\varphi_U) & \longrightarrow & \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) & \longrightarrow & \operatorname{coker}(\varphi_U) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho_{U,V;\mathcal{F}}} & & \downarrow{\scriptstyle \rho_{U,V;\mathcal{G}}} & & & & \\ 0 & \longrightarrow & \ker(\varphi_V) & \longrightarrow & \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) & \longrightarrow & \operatorname{coker}(\varphi_V) & \longrightarrow & 0 \end{array}$$

and this diagram induces unique compatible maps $\ker(\varphi_U) \to \ker(\varphi_V)$ and $\operatorname{coker}(\varphi_U) \to \operatorname{coker}(\varphi_V)$. This motivates the following definition:

Definition 1.1. Let $\ker \varphi$ be the presheaf $U \mapsto \ker(\varphi_U)$ with the unique compatible restriction maps described above. Similarly, let "$\operatorname{coker} \varphi$" be the presheaf $U \mapsto \operatorname{coker}(\varphi_U)$ with the above restriction maps.

A-priori, $\ker \varphi$ and "$\operatorname{coker} \varphi$" are presheaves. We shall see later that $\ker \varphi$ is in fact a sheaf, while "$\operatorname{coker} \varphi$" is in general not, and must be sheafified.

Now since the exact sequence

$$0 \longrightarrow \ker \varphi(U) \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \longrightarrow \text{“}\operatorname{coker} \varphi\text{”}(U) \longrightarrow 0$$

is compatible with restriction to any $V \subseteq U$ and since direct limit is an *exact* functor, we obtain an exact sequence on stalks

$$0 \longrightarrow \ker \varphi_x \longrightarrow \mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \longrightarrow \text{“}\operatorname{coker} \varphi\text{”}_x \longrightarrow 0$$

from which it follows immediately that $\ker(\varphi_x) = (\ker \varphi)_x$ (since both are contained in $\mathcal{F}_x$) and "$\operatorname{coker} \varphi$"$_x \simeq \operatorname{coker}(\varphi_x)$ as quotients of $\mathcal{G}_x$.

In a similar manner, one defined the presheaf "$\operatorname{im} \varphi$" by $U \mapsto \operatorname{im} \varphi_U \subseteq \mathcal{G}(U)$ with the unique restriction maps that make the corresponding diagram (as above) commute. This is a subpresheaf of $\mathcal{G}$ and we have, just as above, that "$\operatorname{im} \varphi$"$_x = \operatorname{im} \varphi_x$ inside of $\mathcal{G}_x$.

Lemma 1.2. *The presheaf* $\ker \varphi$ *is a sheaf.*

Proof. Since $\ker \varphi \subseteq \mathcal{F}$ and $\mathcal{F}$ is separated, we see that $\ker \varphi$ is also separated: for and open U and any covering U_i of U we have the diagram

$$\begin{array}{ccc} \ker \varphi(U) & \hookrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \prod \ker \varphi(U_i) & \hookrightarrow & \prod \mathcal{F}(U_i) \end{array}$$

from which it follows that $\ker \varphi(U) \longrightarrow \prod \ker \varphi(U_i)$ must be an injection. So in order to show that $\ker \varphi$ is a sheaf, we need only show that for any $s_i \in \ker \varphi(U_i)$ with $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$ in $\ker \varphi(U_i \cap U_j)$ we have a global section $s \in \ker \varphi(U)$ with $s\big|_{U_i} = s_i$. But since $\mathcal{F}$ is a sheaf, there exists a unique $s \in \mathcal{F}(U)$ with $s\big|_{U_i} = s_i$ for all i; we thus need only check that $s \in \ker \varphi(U) = \ker(\mathcal{F}(U) \to \mathcal{G}(U))$. But this follows from the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod \mathcal{F}(U_i) & \longrightarrow & \prod \mathcal{G}(U_i) \end{array}$$

so that since $s \in \mathcal{F}(U)$ maps to zero via $\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{G}(U_i)$, it must map to zero vie $\mathcal{F}(U) \to \mathcal{G}(U) \to \prod \mathcal{G}(U_i)$; injectivity of $\mathcal{G}(U) \to \prod \mathcal{G}(U_i)$ forces s to map to 0 under $\mathcal{F}(U) \to \mathcal{G}(U)$ so that $s \in \ker \varphi(U)$ as claimed. ■

Definition 1.3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. We say that φ is *locally surjective* if for any open $U \subseteq X$ and any $s \in \mathcal{G}(U)$ there existst an open covering U_i of U and sections $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s\big|_{U_i}$.

Observe that this definition is equivalent to surjectivity of $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$. Indeed, one implication is clear. For the other, suppose that $s \in \mathcal{G}(U)$. Then for all $x \in U$ we have a $t_x \in \mathcal{F}_x$ such that $\varphi_x(t_x) = s_x$ so that we have an open $U_x \ni x$ such that t_x is represented in $\mathcal{F}_x$ by $(\tilde{t}_x, U_x)$ with $\tilde{t}_x \in \mathcal{F}(U_x)$. Since $\varphi_{U_x}(\tilde{t}_x)$ has the same stalk at x as s, we can shrink each U_x to some $\tilde{U}_x$ so that s_x is represented in $\mathcal{G}_x$ by $(s\big|_{\tilde{U}_x}, \tilde{U}_x)$ and $\varphi_{\tilde{U}_x}(\tilde{t}_x) = s\big|_{\tilde{U}_x}$.

(1) Let X be a $\mathbf{C}$-manifold and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Then for any open $U \subseteq X$ we have the exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}(1)}(U) \longrightarrow \mathcal{O}_X(U) \xrightarrow{\exp} \mathcal{O}_X(U)^\times \longrightarrow \text{“}\operatorname{coker}\exp\text{”}(U) \longrightarrow 0$$

Since the logarithm is well defined *locally*, however, we see that the exponential map is locally surjective, and hence that “$\operatorname{coker}\exp$”$_x = \{1\}$ for all x. Beware, however, that the map $\exp : \mathcal{O}_X(U) \to \mathcal{O}_X(U)^\times$ need not be surjective if U is not simply connected.

Lemma 1.4. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a local surjection and $\mathcal{G} \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} \mathcal{H}$ are two maps of sheaves such that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$ then $\psi_1 = \psi_2$.*

Proof. It is enough to show that the two maps $\mathcal{G}_x \underset{(\psi_2)_x}{\overset{(\psi_1)_x}{\rightrightarrows}} \mathcal{H}_x$ agree for all x. But we have

$$(\psi_1)_x \circ \varphi_x = (\psi_1 \circ \varphi)_x = (\psi_2 \circ \varphi)_x = (\psi_2)_x \circ \varphi_x$$

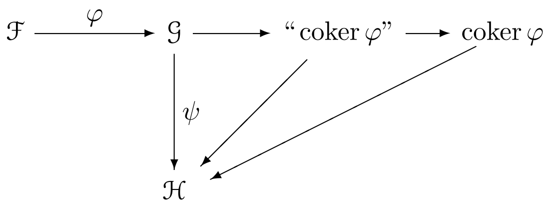
for all x; since φ_x is surjective we conclude that $(\psi_1)_x = (\psi_2)_x$ and we are done. ■

Definition 1.5. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves we define $\operatorname{coker}\varphi$ to be the sheafification of the presheaf “$\operatorname{coker}\varphi$”. Similarly we define $\operatorname{im}\varphi$ to be the sheafification of “$\operatorname{im}\varphi$”.

Observe that with these definitions the sheaves $\operatorname{coker}\varphi$ and $\operatorname{im}\varphi$ have the “right” universal properties. For example, Suppose we have sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ such that $\psi \circ \varphi = 0$. Then over any U, the map $\psi \circ \varphi$ factors through $\operatorname{coker}(\varphi_U) =$ “$\operatorname{coker}\varphi$”$(U)$ by the universal property of cokernals of abelian groups. That is we obtain the diagram of presheaves

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} & \longrightarrow & \text{“}\operatorname{coker}\varphi\text{”} \\ & & {\scriptstyle\psi}\downarrow & \swarrow & \\ & & \mathcal{H} & & \end{array}$$

But this diagram induces (via the universal property of sheafification) a map $\operatorname{coker}\varphi \to \mathcal{H}$ making the diagram

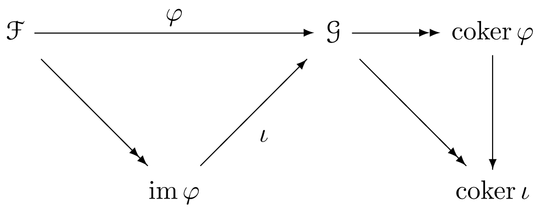


commute. Moreover, this map is unique since it is completely determined on the level of stalks, and in this setting the stalk map is unique by the universal property for abelian groups, so that the sheaf $\operatorname{coker}\varphi$ indeed has the correct universal property.

We can similarly identify $\operatorname{im}\varphi$ with a subsheaf of $\mathcal{G}$. Indeed, we would like to know that the maps $\operatorname{im}\varphi(U) \to \mathcal{G}(U)$ are injective for all U; but for this it is equivalent to show that the stalk maps $(\operatorname{im}\varphi)_x \to \mathcal{G}_x$ are injective. But we know that $(\operatorname{im}\varphi)_x = \operatorname{im}(\varphi_x)$ and evidently $\operatorname{im}(\varphi_x) \hookrightarrow \mathcal{G}_x$ (it is a subgroup).

(1) If $\varphi : \mathcal{F} \to \mathcal{G}$ factors through a subsheaf $\mathcal{H} \subseteq \mathcal{G}$ then $\operatorname{im}\varphi \subseteq \mathcal{H}$. This is because for subsheaves $\mathcal{G}_1, \mathcal{G}_2$ of $\mathcal{G}$, the containment $\mathcal{G}_1 \subseteq \mathcal{G}_2$ is equivalent to the containments $(\mathcal{G}_1)_x \subseteq (\mathcal{G}_2)_x$ inside of $\mathcal{G}_x$ for all x.

(2) Consider the diagram



where ι is the natural injection. By chasing stalks, one can show that $\operatorname{coker}\varphi \to \operatorname{coker}\iota$ is an isomorphism. Moreover, if φ is injective, so that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ then $\mathcal{F}$ *is* the kernel of the map $\mathcal{G} \twoheadrightarrow \operatorname{coker}\iota$. Similarly, if we have a map $\pi : \mathcal{F} \twoheadrightarrow \mathcal{G}$ that is a local surjection, then $\mathcal{G}$ *is* the cokernel of the map $\ker\pi \hookrightarrow \mathcal{F}$.