## 1. More sheaf constructions

**Definition 1.1.** If $\mathcal{F} \xhookrightarrow{\iota} \mathcal{G}$ is a subsheaf, we define the sheaf $\mathcal{G}/\mathcal{F}$ to be the sheaf $\operatorname{coker} \iota$.

As an example, for any complex manifold $X$ the exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}(1)} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times$$

induces $\mathcal{O}_X/\underline{\mathbf{Z}(1)} \simeq \mathcal{O}_X^\times$.

In general, for any (local) surjection $\varphi : \mathcal{G} \to \mathcal{H}$ of sheaves we obtain $\mathcal{G}/\ker\varphi \simeq \mathcal{H}$, where the existence of a map $\mathcal{G}/\ker\varphi \to \mathcal{H}$ follows from the universal property of a quotient sheaf as a cokernel, and one checks this is an isomorphism on stalks.

**Definition 1.2.** A short exact sequence of sheaves is a sequence

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\iota} \mathcal{F} \xrightarrow{\varphi} \mathcal{F}'' \longrightarrow 0$$

in which

(1) $\varphi \circ \iota = 0$.
(2) $\varphi$ is a (local) surjection.
(3) $\iota$ is injective.
(4) $\varphi$ induces an isomorphism $\mathcal{F}/\mathcal{F}' \simeq \mathcal{F}''$.

Observe from (1), (2), and (3) that (4) is equivalent to $\iota$ inducing $\mathcal{F}' \simeq \ker\varphi$.

Surjectivity is a slightly delicate notion: indeed, a map $X' \to X$ of varieties over a field $k$ can be surjective on $\overline{k}$-points without being surjective on $k$-points. Consider the self-map of $\mathbf{A}^1_k - \{0\}$ given by $t \mapsto t^2$ over a field $k$ in which $k \neq k^2$.

## 2. Pullbacks

Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf on $Y$.

**Definition 2.1.** Let "$f^{-1}\mathcal{F}$" be the presheaf on $X$ given by $U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{F}(V)$, where the direct limit is over all open $V$, One checks that this is a presheaf with the obvious restriction maps. We define the sheaf $f^{-1}\mathcal{F}$ to be the sheafification of "$f^{-1}\mathcal{F}$".

The pullback construction generalizes some already familiar notions: let $y \in Y$ be a point and $X = \{y\}$ with the induced topology and $\iota : X \to Y$ the inclusion map. If $\mathcal{F}$ is any sheaf on $Y$ then $\iota^{-1}\mathcal{F}(X) = \varinjlim_{V \supseteq \{y\}} \mathcal{F}(V) = \mathcal{F}_y$ is the stalk at $y$.

*Remark* 2.2. Pullback and pushforward of sheaves are analogous to restriction and extension of scalars in the category of modules over a ring. To see this, let $A, B$ be a rings, and $A$-module and $f : A \to B$ a map of rings. For any $A$-module $M$ we obtain the $B$-module $M \otimes_A B$, and for any $B$- module $N$, we get an $A$-module ${}_AN$ by restriction of scalars. The $A$-module ${}_AN$ is very much like the pushforward construction, while the $B$-module $M \otimes_A B$ is analogous to the pullback. We will see this more clearly when we investigate sheaves of modules; but for now, observe that we have the adjoint property

$$\operatorname{Hom}_B(B \otimes_A M, N) = \operatorname{Hom}_A(M, {}_AN).$$

**Theorem 2.3.** *Let $\mathcal{F}$ be a presheaf of sets on the topological space $Y$ and let*

$$X' \xrightarrow{g} X \longrightarrow Y$$

*be continuous maps of topological spaces. There exists a natural isomorphism*

$$(f^{-1}\mathcal{F})_x \xrightarrow{\sim} \mathcal{F}_{f(x)} \tag{1}$$

*Moreover, we have*

$$f^{-1}\theta : f^{-1}\mathcal{F} \xrightarrow{\sim} f^{-1}(\mathcal{F}^+) \tag{2}$$

$$\alpha_{g,f} : g^{-1}f^{-1}\mathcal{F} \xrightarrow{\sim} (f \circ g)^{-1}\mathcal{F} \tag{3}$$

*and (3) induces, via (1)*

$$\mathrm{id} : \mathcal{F}_{f(g(x'))} \xrightarrow{\sim} (f^{-1}\mathcal{F})_{g(x')} \xrightarrow{\sim} \mathcal{F}_{f\circ g(x')}.$$

*Finally, (1) is functorial in $\mathcal{F}$, so that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of presheaves, we have*

$$\begin{array}{ccc} (f^{-1}\mathcal{F})_x & \xrightarrow{\sim} & \mathcal{F}_{f(x)} \\ {\scriptstyle (f^{-1}\varphi)_x}\downarrow & & \downarrow{\scriptstyle f(x)} \\ (f^{-1}\mathcal{G})_x & \xrightarrow{\sim} & \mathcal{G}_{f(x)}. \end{array}$$

*Proof.* We will construct (1). First observe that $(f^{-1}\mathcal{F})_x = \text{“}f^{-1}\mathcal{F}\text{”}_x$, so that

$$(f^{-1}\mathcal{F})_x = \varinjlim_{U\ni x} \varinjlim_{V\supseteq f(U)} \mathcal{F}(V).$$

Now any $V$ containing $f(U)$ with $U$ containing $x$ must contain $f(x)$. This gives a map

$$(f^{-1}\mathcal{F})_x = \varinjlim_{U\ni x} \varinjlim_{V\supseteq f(U)} \mathcal{F}(V) \longrightarrow \varinjlim_{W\ni f(x)} \mathcal{F}(W).$$

Conversely, $W \supseteq f(f^{-1}(W))$ and since $f$ is continuous, $f^{-1}(W)$ is open for any open $W$ and contains $x$ when $f(x) \in W$, so we obtain a map in the other direction. It is not difficult to see that these maps are inverse to eachother. ■

2.4. **Adjointness.** Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}, \mathcal{G}$ sheaves on $X, Y$ respectively. Then we have

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

To see this, it suffices to produce natural maps $\mathcal{G} \longrightarrow f_*(f^{-1}\mathcal{G})$ and $f^{-1}(f_*\mathcal{F}) \to \mathcal{F}$, for given such maps the functoriality of $f_*$ and $f^{-1}$ show that any map $f^{-1}\mathcal{G} \longrightarrow \mathcal{F}$ gives rise to a map

$$\mathcal{G} \longrightarrow f_*f^{-1}\mathcal{G} \longrightarrow f_*\mathcal{F}$$

and any map $f\mathcal{G} \longrightarrow f_*\mathcal{F}$ gives rise to a map

$$f^{-1}\mathcal{G} \longrightarrow f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}.$$

The commutative algebra analog is instructive: Let $A, B$ be rings, $f : A \to B$ a ring-map and $N$ a $B$-module and $M$ an $A$-module. Then we have maps

$$\alpha_M : M \longrightarrow {}_A(B \otimes_A M)$$

and

$$\beta_N : B \otimes_A ({}_AN) \longrightarrow N$$

given by $m \mapsto (1 \otimes m)$ and $b \otimes n \mapsto bn$ respectively. It is straightforward to check that these maps give the natural bijection

$$\mathrm{Hom}_B(B \otimes_A M, N) = \mathrm{Hom}_A(M, {}_AN).$$

2.5. **Examples of pushforward.**

**Proposition 2.6.** *Let $f : X \to Y$ be a continuous map of topological spaces and $\underline{\Sigma}_Y, \underline{\Sigma}_Y$ the constant sheaf on $Y, X$ respectively, associated to the set $\Sigma$. Then there exists a natural isomorphism $f^{-1}(\underline{\Sigma}_Y) \xleftarrow{\sim} \underline{\Sigma}_X$.*

Observe that the analog for pushforward is false as the preimage of a connected open set need not be connected.

*Proof.* We have a map

$$\Sigma \longrightarrow \text{“}f^{-1}\underline{\Sigma}_Y\text{”}(X) \longrightarrow f^{-1}\underline{\Sigma}_Y(X)$$

which, by the property that a map of $\Sigma$ to the global sections of any sheaf determines a map of $\underline{\Sigma}$ to that sheaf, gives a unique $\underline{\Sigma}_X \xrightarrow{\sim} f^{-1}\underline{\Sigma}_Y$ such that the induced map of stalks $\Sigma \simeq (\underline{\Sigma}_X)_x \longrightarrow (\underline{\Sigma}_Y)_{f(x)} \simeq \Sigma$ is the identity mapping of $\Sigma$ (which is evident from the construction). ■

Because of this Proposition, we often abuse notation and write $\underline{\Sigma}$ for the constant sheaf associated to $\Sigma$ and omit the space on which it is considered.

As another example, let $f : X \to Y$ be a map of topological spaces, and $\pi : Y' \to Y$ a covering space. Put $X' = X \times_Y Y'$, that is, $X' = \{(x, y') : f(x) = \pi(y')\}$. Let $\mathcal{F} = \Gamma_{Y'/Y}$ be the sheaf of sections of $Y'$ over $Y$. Then we have

$$f^{-1}\mathcal{F} \simeq \Gamma_{X'/X}.$$

**Definition 2.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. If $U$ is an open set we define the induced ring space structure on $U$ to be $(U, \mathcal{O}_U)$ where $\mathcal{O}_U = \mathcal{O}_X\big|_U$. A map $(f, f^{\#}) : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ is an *open immersion* if $f : X' \to X$ is a homeomorphism onto an open $U \subseteq X$ and $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ is adjoint to $f^{-1}\mathcal{O}_X \to \mathcal{O}_{X'}$.

As an example, let $k$ be a field and consider the map $f : \operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[y]/y^2$ given by mapping the unique point of $\operatorname{Spec} k$ to the unique point of $\operatorname{Spec} k[y]/y^2$ and the surjective $k$-algebra map $f^{\#} : k[y]/y^2 \twoheadrightarrow k$ given by $y \mapsto 0$. This is *not* an open immersion.

**Definition 2.8.** A *complex manifold* is a ringed space of **C**-algebras $(X, \mathcal{O}_X)$ such that there exists an open cover $(U_i, \mathcal{O}_{U_i})$ that are isomorphic (as ringed spaces) to opens in finite dimensional complex vector-spaces endowed with the sheaf of holomorphic functions.

## 3. Glueing of ringed spaces

Let $(X_i, \mathcal{O}_{X_i})$ be a collection of ringed spaces and $X_{ij} \subseteq X_i$ open sets, together with isomorphisms of ringed spaces $\varphi_{ij} : X_{ij} \simeq X_{ji}$ such that

(1) $X_{ii} = X_i$ and $\varphi_{ii} = \text{id}$.
(2) $\varphi(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ji}$.
(3) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on triple overlaps.

Then there exists a unique ringed space $(X, \mathcal{O}_X)$ equipped with open immersions

$$\psi_i : (X_i, \mathcal{O}_{X_i}) \hookrightarrow (X, \mathcal{O}_X)$$

covering with overlaps “$X_{ij} = X_{ji}$” heving the correct universal property: