1. Closed immersions

Definition 1.1. A map of ringed spaces $(\iota, \iota^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a *closed immersion* if

(1) $\iota : Y \to X$ is a closed embedding, i.e. a homeomorphism onto a closed subset of X.
(2) $\iota^{\#} : \mathcal{O}_X \to \iota_* \mathcal{O}_Y$ is surjective.

Condition (2) roughly says that functions on Y locally lift to X.

Remark 1.2. If $\iota : Y \hookrightarrow X$ is an embedding and $\mathcal{F}$ is a sheaf on Y then $(\iota_* \mathcal{F})_x = \mathcal{F}_x$ if $x \in \iota(Y)$. If $x \notin \iota(Y)$, the situation is more complicated, but if $\iota(Y)$ is closed then $U = X - \iota(Y)$ is open and $U \cap Y = \emptyset$, so that $(\iota_* \mathcal{F})_x = 0$ when $x \notin \iota(Y)$.

As an example, suppose that X, Y are C^∞ manifolds. Then $\iota : Y \to X$ is a closed immersion if and only if $\iota(Y)$ is a closed submanifold. This follows from the Immersion Theorem. Indeed the difficult direction is to show that surjectivity of $\mathcal{O}_{X,\iota(y)} \to \iota_* \mathcal{O}_{Y,\iota(y)} \to \mathcal{O}_{Y,y}$ implies that Y is locally described as the zero-locus of some local functions. That is, we need to show that $T_y(Y) \xrightarrow{d\iota(y)} T_{\iota(y)}(X)$ is injective, or dually, that $T_y(Y)^* \longleftarrow T_{\iota(y)}(X)^*$ is surjective. But we have the identification $\mathcal{O}_{X,x}/\mathfrak{m}_x^2 \simeq \mathbf{R} \oplus \mathfrak{m}_x/\mathfrak{m}_x^2$ and $T_x(X)^* \simeq \mathfrak{m}_x/\mathfrak{m}_x^2$, so that surjectivity of the map $\mathcal{O}_{X,\iota(y)} \to \mathcal{O}_{Y,y}$ implies the surjectivity of $T_y(Y)^* \longleftarrow T_{\iota(y)}(X)^*$.

Remark 1.3. Given a closed immersion $\iota : Y \to X$ we let $\mathcal{I}_Y = \ker(\mathcal{O}_X \to \iota_* \mathcal{O}_Y)$. Then $\mathcal{I}_Y$ is a sheaf of ideals in $\mathcal{O}_X$ and $\iota_* \mathcal{O}_Y \simeq \mathcal{O}_X / \mathcal{I}$, so pulling back we obtain $\mathcal{O}_Y \simeq \iota^{-1}(\mathcal{O}_X / \mathcal{I}_Y)$. We conclude that closed immersions are "classified" by sheaves of ideals. One should take this interpretation with a grain of salt, however, as many ideal sheaves will correspond to "crazy" spaces Y that one would not want to work with.

Definition 1.4. Let X be a topological space and $\mathscr{B}$ a base for the topology. A *$\mathscr{B}$-presheaf* is an assignement $U \mapsto \mathcal{F}(U)$ for all $U \in \mathscr{B}$ with restriction maps $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subseteq U$ for $U, V \in \mathscr{B}$. A $\mathscr{B}$-presheaf is a $\mathscr{B}$-sheaf if

(1) it is separated: for all $U \in \mathscr{B}$ and $\{U_i\}$ an open cover of U by $U_i \in \mathscr{B}$ the map
$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i)$$
is injective.
(2) glueing works: if $U \in \mathscr{B}$ and $\{U_i\}$ is a cover of U by $U_i \in \mathscr{B}$ then for any covers $\{U_{ijk}\}_{k \in K_{ij}}$ of $U_i \cap U_j$ with $U_{ijk} \in \mathscr{B}$, the sequence
$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{\substack{(i,j) \\ k \in K_{ij}}} \mathcal{F}(U_{ijk})$$
is exact.

Observe that condition (2) says that for a collection of $s_i \in \mathcal{F}(U_i)$ with $s_i\big|_{U_{ijk}} = s_j\big|_{U_{ijk}}$ for all k there exists $s \in \mathcal{F}(U)$ with $s\big|_{U_i} = s_i$. This s is unique by condition (1). We remark that any sheaf on X restricts to a $\mathscr{B}$-sheaf and that if $f : X' \to X$ is continuous, and $\mathscr{B}', \mathscr{B}$ are bases on X', X respectively, such that $f^{-1}(U) \in \mathscr{B}'$ fir all $U \in \mathscr{B}$, then the pushforward under f of any $\mathscr{B}'$-sheaf is a $\mathscr{B}$-sheaf.

Lemma 1.5. *If X and $\mathscr{B}$ are as above and $\mathcal{F}$ is a $\mathscr{B}$-sheaf on X, then there exists a sheaf $\mathcal{F}^+$ on X together with an isomorphism $\mathcal{F}^+\big|_{\mathscr{B}} \simeq \mathcal{F}$ which is universal in the sense that given any sheaf $\mathcal{G}$ on X and any map of $\mathscr{B}$-sheaves $\varphi : \mathcal{F} \to \mathcal{G}\big|_{\mathscr{B}}$, there exists a unique sheaf map $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi^+\big|_{\mathscr{B}} = \varphi$.*

Proof. For any open $V \subset X$ let
$$\mathcal{F}^+(V) = \left\{ (s_U) \in \prod_{\substack{U \in \mathscr{B} \\ U \subseteq V}} \mathcal{F}(U) \ : \ s_U\big|_W = s_{U'}\big|_W \text{ for all } W \subseteq U \cap U' \text{ with } W, U, U' \in \mathscr{B} \right\}.$$

Since $\mathcal{F}$ is a $\mathscr{B}$-sheaf and $\mathscr{B}$ is a base, it is not hard to check that $\mathcal{F}^+$ is a sheaf with $\mathcal{F}^+\big|_{\mathscr{B}} = \mathcal{F}$. Verification of the stated universal property is left as an exercise, or one can consult EGA I, §3.2. ∎

Let A be a commutative ring and put $X = \operatorname{Spec} A$ with the Zariski topology. A base of opens for X is given by

$$X_a = \operatorname{Spec} A_a \simeq \{\mathfrak{p} \in X : a \notin \mathfrak{p}\} = \{\mathfrak{p} \in X : a \neq 0 \text{ in } \kappa(\mathfrak{p}) = \operatorname{Frac}(A/\mathfrak{p})\}.$$

Lemma 1.6. *With this notation, the assignment $X_a \mapsto A_a$ is a $\mathscr{B}$-sheaf.*

Observe that the ring A_a depends only on the set X_a, and not on the particular element a, for we have the description $A_a = S^{-1}A$ where $S = \{\alpha \in A \;:\; \alpha(x) \neq 0 \text{ for all } x \in X_a\}$.

Proof. Notice that $X_{a_1} \cap X_{a_2} = X_{a_1 a_2}$ and that X_a is quasi-compact. It therefore suffices to check the sheaf condition for a finite cover of X_a by $X_{a_1}, \ldots, X_{a_n}$. Moreover, since $X_a \supseteq X_{a_i}$ for all i, we see that $a \in A$ is nonvanishing on X_{a_i}, so that the map $A \to A_{a_i}$ takes a to a unit, that is a is inverted in A_{a_i} so $A_{a_i} = A_{aa_i}$ for all i. Therefore, replacing a_i by aa_i throughout, we reduce to the case $a = 1$, so that $\{X_{a_i}\}$ cover $\operatorname{Spec} A$, and hence $(a_1, \ldots, a_n) = 1$. We must show that

$$A \longrightarrow \prod_{i=1}^{n} A_{a_i} \rightrightarrows \prod_{(i,j)} A_{a_i a_j} \tag{1}$$

is exact.

Suppose $\alpha, \alpha' \in A$ have the same image in A_{a_i} for all i. Since the a_i generate the unit ideal, any $\mathfrak{p} \in \operatorname{Spec} A$ fails to contain some a_i, so that $A_{\mathfrak{p}}$ is a localization of A_{a_i} whence α, α' have the same image in $A_{\mathfrak{p}}$. As this holds for all $\mathfrak{p}$, we see that $\alpha = \alpha'$.

Now given $(s_i) \in \prod_{i=1}^n A_{a_i}$ such that s_i, s_j have the same image in $A_{a_i a_j}$, we must construct some $a \in A$ with $a \mapsto s_i$ in A_{a_i}. For this, we use faithfully flat descent.

Let $A' = \prod_{i=1}^n A_{a_i}$. Since this is a finite product of localizations of A it is faithfully flat over A. Since $A_{a_i a_j} = A_{a_i} \otimes_A A_{a_j}$ we have

$$A' \otimes_A A' = \prod_{(i,j)} A_{a_i} \otimes_A A_{a_j} = \prod_{(i,j)} A_{a_i a_j}.$$

Moreover,

The maps of diagram (1) are *precisely* the maps $A' \to A' \otimes_A A'$ given by $a' \mapsto 1 \otimes a'$ and $a' \mapsto a' \otimes 1$. Since A' is faithfully flat over A, we have an injection $A \hookrightarrow A'$. We must then check that the diagram

$$A \hookrightarrow A' \underset{a' \mapsto 1 \otimes a'}{\overset{a' \mapsto a' \otimes 1}{\rightrightarrows}} A' \otimes_A A'$$

is exact. We begin with this next lecture.

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