1. Last time

Recall that last time we wanted to show that for $X = \operatorname{Spec} A$ with A a ring, the association $X_f \to A_f$ is a $\mathscr{B}$-sheaf. We reduced to the case of a finite open cover $\{X_{a_i}\}$ of X with $(a_i)A = A$ and we want to show that

$$A \xrightarrow{a \mapsto (a)} \prod_i A_{a_i} \xrightarrow{(s_i) \mapsto (s_j - s_k)} \prod_{(j,k)} A_{a_j a_k} \tag{1}$$

is left exact. Set $A' = \prod_{i=1}^n A_{a_i}$. Then since localization and finite products preserve faithful flatness, A' is faithfully flat over A. (N.B. the geometric meaning of A' faithfully flat over A is that $\operatorname{Spec} A' \to \operatorname{Spec} A$ is surjective). One checks that the diagram (1) is equivalent to

$$A \longrightarrow A' \xrightarrow{a' \mapsto a' \otimes 1 - 1 \otimes a'} A' \otimes_A A' \tag{2}$$

Therefore, it suffices to prove the more general result:

Theorem 1.1. *Let A be a ring and A' a faithfully flat A-algebra. Then $a' \in A'$ comes from A if and only if $a' \otimes 1 = 1 \otimes a'$ in $A' \otimes_A A'$.*

Proof. First suppose that we have a section $\sigma : A' \to A$, *i.e.* that we have $A' = A \oplus I$ with I some ideal of A'. Then the map $A' \to A' \otimes_A A'$ is given by

$$A \oplus I \xrightarrow{a + i \mapsto (a,i,0,0) - (a,0,i,0) = (0,i,-i,0)} A \oplus (I \otimes_A A) \oplus (A \otimes_A I) \oplus (I \otimes_A I),$$

so that the image of $a + i$ in $A' \otimes_A A'$ is zero if and only if $i = 0$.

Now we reduce to this case. We wish to show that the sequence of A-modules

$$A \xrightarrow{a \mapsto a} A' \xrightarrow{a' \mapsto a' \otimes 1 - 1 \otimes a'} A' \otimes_A A'$$

is left exact. For any B faithfully flat over A, such exactness occurs if and only if

$$B \otimes_A A \xrightarrow{b \otimes 1 \mapsto b \otimes 1} B \otimes_A A' \xrightarrow{b' \mapsto b' \otimes 1 - 1 \otimes b'} B \otimes_A (A' \otimes_A A') = (B \otimes_A A') \otimes_B (B \otimes_A A')$$

is exact. Thus, with $B' = B \otimes_A A'$, we need only check exactness of

$$B \xrightarrow{b \mapsto b \otimes 1} B' \xrightarrow{b' \mapsto b' \otimes 1 - 1 \otimes b'} B' \otimes_B B'.$$

Taking $B = A'$, we obtain a section $\sigma A' \otimes_A A' \to A' = A' \otimes_A A$ given by $\sigma(a'_1 \otimes a'_2) = a'_1 a'_2$ (which evidently composes with $a' \mapsto a' \otimes 1$ to give the identity on A'). Thus, we are in the situation above and we are done. ∎

2. Spec of a ring

We can now define $\operatorname{Spec} A$:

Definition 2.1. $\operatorname{Spec} A$ is the ringed space whose underlying topological space is the prime spectrum of the ring A and whose structure sheaf is given on a base of opens $D(f)$ by $D(f) \mapsto A_f$.

Theorem 2.2. *Let $(X, \mathcal{O}_X) = \operatorname{Spec} A$.*

(1) *For $a \in A$ there exists a canonical isomorphism $A_a \simeq \Gamma(X_a, \mathcal{O}_X)$, where $X_a = \{x \in X : a(x) \neq 0 \text{ in } k(x)\}$.*
(2) *For $x = \mathfrak{p} \in X$ we have $\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}}$, so that X is a locally ringed space.*

Proof. (1) This was how $\mathcal{O}_X$ was constructed as a $\mathscr{B}$-sheaf.
(2) We have

$$\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \Gamma(U, \mathcal{O}_X) = \varinjlim_{X_a \ni x} A_a = \varinjlim_{a \notin \mathfrak{p}} A_a = A_{\mathfrak{p}}$$

by the universal mapping property of the localization $A_{\mathfrak{p}}$, since the transition maps defining the direct limit are the natural localization maps.

∎

Definition 2.3. An *affine scheme* is a locally ringed space isomorphic (as a locally ringed space) to $\operatorname{Spec} A$ for some ring A.

Definition 2.4. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $\{U_i\}$ of X with $(U_i, \mathcal{O}_X\big|_{U_i})$ affine schemes.

We have already seen that schemes and affine schemes are full subcategories of the category of locally ringed spaces.

Lemma 2.5. *Any scheme has a base of affine opens.*

Proof. It is enough to treat the affine case, say $X = \operatorname{Spec} A$. But X has a base of opens $(X_a, \mathcal{O}_X\big|_{X_a}) \simeq \operatorname{Spec} A_a$, as is evident from the definition of the structure sheaf of $\operatorname{Spec} A_a$. ∎

Corollary 2.6. *If X is a scheme and $U \subseteq X$ open, then $(U, \mathcal{O}_X\big|_U)$ is a scheme.*

Now let $\varphi : A \to B$ be a map of rings. Then we get a map of topological spaces $f : X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ defined by $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$. We want to define a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. For this, it is enough to find a base $\mathscr{B}$ of opens in Y and maps $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ for all $U \in \mathscr{B}$ compatible with restriction. We therefore take $\mathscr{B} = \{Y_a\}$ and we have $f^{-1}(Y_a) = X_{\varphi(a)}$, so that we must define a map $A_a \to B_{\varphi(a)}$ compatible with restriction. To do this, we take as our map the map induced by φ via the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_a & \xrightarrow{\exists!} & B_{\varphi_a}. \end{array}$$

It is straightforward to check that this map is compatible with restriction. Moreover, we claim that the induced map

$$A_{\varphi^{-1}(\mathfrak{q})} = \mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_Y)_x \to \mathcal{O}_{X,x} = B_{\mathfrak{q}}$$

is *local*. Indeed, this follows easily from the definition of $B_{\mathfrak{q}}$ as $\varinjlim_{b \notin \mathfrak{q}} B_b$, or from the fact that the *only* map $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ making

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{\varphi^{-1}(\mathfrak{q})} & \longrightarrow & B_{\mathfrak{q}} \end{array}$$

commute *is* the local map induced by φ (via the universal property of $A_{\varphi^{-1}(\mathfrak{q})}$).

Theorem 2.7. *Let A, B be rings. Then there is a bijection*

$$\operatorname{Hom}_{\text{ring}}(A, B) \longleftrightarrow \operatorname{Hom}_{\text{l.r.spaces}}(\operatorname{Spec} B, \operatorname{Spec} A).$$

Proof. We wish to show that the composite functor

$$\operatorname{Hom}(A, B) \xrightarrow{\text{Spec}} \operatorname{Hom}(\operatorname{Spec} B, \operatorname{Spec} A) \xrightarrow{\Gamma(\cdot, \mathcal{O}.)} \operatorname{Hom}(A, B)$$

is the identity. Suppose that under the map $\operatorname{Hom}(\operatorname{Spec} B, \operatorname{Spec} A) \to \operatorname{Hom}(A, B)$ the map f is carried to φ. We must show that $f = \operatorname{Spec} \varphi$. It is enough to show that for two maps

$$Y = \operatorname{Spec} B \underset{(g, g^\#)}{\overset{(f, f^\#)}{\rightrightarrows}} \operatorname{Spec} A = X$$

which agree on global sections we have $f = g$ and $f^\# = g^\#$. First assume that $f = g$. Then we want $f^\# = g^\#$, and it is enough to check this holds on a base of opens. But the uniqueness of the induced map $A_a = \mathcal{O}_X(X_a) \to$

$\mathcal{O}_Y(f^{-1}X_a) = B_{f^\#(a)}$ in the diagram below shows that if $f^\#, g^\#$ agree on global sections then they agree on a base of opens.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_a & \longrightarrow & B_{f^\#(a)}. \end{array}$$

Thus, we need only check that $f = g$ (as topological maps). Again, there is a unique *local* map $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{\varphi^{-1}(\mathfrak{q})} & \longrightarrow & B_{\mathfrak{q}}. \end{array}$$

commute. Thus, since $f^\#$ and φ agree as maps $A \to B$, we must have $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$, so that $f, \operatorname{Spec} \varphi$ agree as topological maps. ■

We will see next time that this bijection holds when $\operatorname{Spec} B$ is replaced by an arbitrary scheme X and B by $\Gamma(X, \mathcal{O}_X)$.