MATH 632, LECTURE 10 JANUARY 30, 2004

Theorem 0.1. *Let X be an arbitrary scheme and Y affine. Then there is a bijection*

$$\mathrm{Hom}(X,Y) \longleftrightarrow \mathrm{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)).$$

Proof. Set $Y = \operatorname{Spec} A$, let $\{U_i\}$ be an affine open cover of X and for each i,j let $\{U_{ijk}\}_{k\in K_{ij}}$ be an affine open cover of $X_i \cap X_j$. Then we the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(X, \operatorname{Spec} A) & \longrightarrow & \prod \mathrm{Hom}(U_i, \operatorname{Spec} A) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \prod_{(i,j)} \prod_{k\in K_{ij}} \mathrm{Hom}(U_{ijk}, \operatorname{Spec} A) \\
\downarrow & & \updownarrow & & \updownarrow \\
\mathrm{Hom}(A, \mathcal{O}_X(X)) & \longrightarrow & \prod \mathrm{Hom}(A, \mathcal{O}_{U_i}(U_i)) & \rightrightarrows & \prod_{(i,j)} \prod_{k\in K_{ij}} \mathrm{Hom}(A, \mathcal{O}_{U_{ijk}}(U_{ijk}))
\end{array}$$

in which the two rows are left exact and the two right-hand columns are bijections. It follows that the first column is also a bijection. ∎

1. EXAMPLES

(1) Take $\mathbf{A}^n_{\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[T_1, \dots, T_n]$. To give a map $X \to \mathbf{A}^n_{\mathbf{Z}}$ for any scheme X is to give a ring map $\mathbf{Z}[T_1, \dots, T_n] \to \mathcal{O}_X(X)$, which amounts to picking an element of $\mathcal{O}_X(X)$ for the image of each T_i. Thus, we have the identifications

$$\mathrm{Hom}(X, \mathbf{A}^n_{\mathbf{Z}}) \longleftrightarrow \mathrm{Hom}(\mathbf{Z}[T_1, \dots, T_n], \mathcal{O}_X(X)) \simeq \mathcal{O}_X(X)^{\oplus n}.$$

(2) The natural map $A \to A_f$ for any ring A and any $f \in A$ induces an open immersion $\operatorname{Spec} A_f \to \operatorname{Spec} A$ onto $X_f = \{x \in X : f_x \neq 0 \text{ in } k(x)\}$ (as any open in X can be covered by sets of the form X_{fg}).

(3) Let k be a field and consider $\mathbf{A}^1_k = \operatorname{Spec} k[T]$. The point $(0) \in \mathbf{A}^1_k$ is open and dense (since every prime ideal of $k[T]$ contains 0), while all other points are closed and have the form (f) for $f \in k[T]$ a monic irreducible polynomial. The residue field at (0) is $k(T)$ while the residue field at f is $k[T]/(f)$, which is a finite field extension of k.

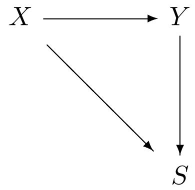
(4) If A has a unique minimal prime (for example, if A is a domain) then $\operatorname{Spec} A$ has a unique point z, open and dense in $\operatorname{Spec} A$.

(5) Consider $\mathbf{A}^1_{\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[X]$. We have a natural mapping $\mathbf{A}^1_{\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$. The points of $\mathbf{A}^1_{\mathbf{Z}}$ are
- The unique open and dense point (0). The residue field is $\mathbf{Z}[X]_{(0)} = \mathbf{Q}(X)$ and the fiber over the point $(0) \in \operatorname{Spec} \mathbf{Z}$ is $\mathbf{A}^1_{\mathbf{Q}}$.
- Prime ideals of the form (f) with $f \in \mathbf{Z}[X]$ irreducible. Such points are open, with closure the set of all prime ideals of the form (p, f) where $p \in \mathbf{Z}$ is a prime such that $f \bmod p$ is reducible. The residue field at f is $\mathrm{Frac}(\mathbf{Z}[X]/(f))$, and such (f) lie in the fiber over (0).
- Maximal ideals of the form (p, f) with f irreducible modulo p. The residue field at (p, f) is $\mathbf{F}_p[X]/(f)$, and these points are in the fiber over (p).

One pictures $\operatorname{Spec} \mathbf{Z}[X]$ as in Mumford's Red Book:

(6) Consider $\mathbf{A}^2_k = \operatorname{Spec} k[x, y]$ with k algebraically closed. The points are
- The unique open and dense point (0). The residue field is $k(x, y)$.
- Prime ideals of the form (f) with $f \in k[x, y]$ irreducible. Such points are open, with closure the set of all maximal ideals of the form $(x-a, y-b)$ with $f(a, b) = 0$. The residue field at (f) is $\mathrm{Frac}(k[x, y]/(f))$, i.e. the function field of the irreducible subvariety $f = 0$.
- Maximal ideals of the form $(x - a, y - b)$. The residue field is just k.

Definition 1.1. Let S be a scheme. Then an S-scheme X is a scheme together with a map $X \to S$. An S-map of S-schemes X, Y is a commutative diagram



As an example, let $S = \operatorname{Spec} A$. By abuse of language, we will often refer to an S-scheme as an A-scheme. An A-scheme X is just a scheme together with a ring map $A \to \Gamma(X, \mathcal{O}_X)$, making $\mathcal{O}_X(U)$ an A-algebra for each open $U \subseteq X$. Thus, $\mathcal{O}_X$ becomes a sheaf of A-algebras, and maps $f : X \to Y$ of A-schemes must have $f^\#$ a map of A-algebras. In classical algebraic geometry, one studies k-schemes for an algebraically closed field k. The base over which a scheme X is considered can make a great difference in the properties and structure of X. For example, consider $X = \operatorname{Spec} \mathbf{C}$. As a scheme over $\mathbf{C}$, X has no nontrivial automorphisms. Over $\mathbf{R}$, $\operatorname{Aut}(X) \simeq \mathbf{Z}/2\mathbf{Z}$, while over $\mathbf{Z}$, $\operatorname{Aut}(X)$ is uncountable. The moral is that algebraic geometry must be developed with respect to an arbitrary base scheme.

Theorem 1.2. *Let $f : X \to Y$ be a map of schemes. Then the following are equivalent:*

(1) *For all open affines $\operatorname{Spec} B \subseteq Y$ and all open affines $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$ the ring A is a finitely generated B-algebra.*
(2) *There exists an open cover $\{\operatorname{Spec} B_i\}$ of Y and an open cover $\{\operatorname{Spec} A_{ij}\}$ of $f^{-1}(\operatorname{Spec} B_i)$ for each i such that each A_{ij} is a finitely generated B_i algebra (for all i).*

Definition 1.3. Such a map f is said to be *locally of finite type*, (with the "locality" referring to the source scheme).

Proof. It suffices to prove that (2) implies (1). Since for $b_i \in B_i$ we have $f^{-1}(\operatorname{Spec}(B_i)_{b_i})$ covered by $\operatorname{Spec}(A_{ij})_{b_i}$, so the hypothesis is inherited by basic opens of $\operatorname{Spec} B_i$. To reduce to the case $Y = \operatorname{Spec} B$ we must cover $\operatorname{Spec} B$ by basic opens $\operatorname{Spec} B_b$ that are also basic opens $\operatorname{Spec}(B_i)_{b_i}$ of $\operatorname{Spec} B_i \subseteq Y$. We will take this up next lecture. ∎