Recall the following Theorem from last lecture:

Theorem 0.1. *Let $f : X \to Y$ be a map of schemes. Then the following are equivalent:*

(1) *For all open affines* $\operatorname{Spec} B \subseteq Y$ *and all open affines* $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$ *the ring A is a finitely generated B-algebra.*
(2) *There exists an open cover* $\{\operatorname{Spec} B_i\}$ *of Y and an open cover* $\{\operatorname{Spec} A_{ij}\}$ *of* $f^{-1}(\operatorname{Spec} B_i)$ *for each i such that each A_{ij} is a finitely generated B_i algebra (for all i).*

Proof. It suffices to prove that (2) implies (1). Since for $b_i \in B_i$ we have $f^{-1}(\operatorname{Spec}(B_i)_{b_i})$ covered by $\operatorname{Spec}(A_{ij})_{b_i}$, so the hypothesis is inherited by basic opens of $\operatorname{Spec} B_i$. To reduce to the case $Y = \operatorname{Spec} B$ we must cover $\operatorname{Spec} B$ by basic opens $\operatorname{Spec} B_b$ that are also basic opens $\operatorname{Spec}(B_i)_{b_i}$ of $\operatorname{Spec} B_i \subseteq Y$.

Let us prove the more general result that if X is a scheme and $\operatorname{Spec} R, \operatorname{Spec} R'$ are affine opens in X with $x \in \operatorname{Spec} R \cap \operatorname{Spec} R'$ then there exists an open affine $U \ni x$ in $\operatorname{Spec} R \cap \operatorname{Spec} R'$ that is a basic open set of $\operatorname{Spec} R$ and $\operatorname{Spec} R'$.

Since $\operatorname{Spec} R \cap \operatorname{Spec} R'$ is open in $\operatorname{Spec} R'$, we can find a basic open set $U' = \operatorname{Spec} R'_{r'}$ contained in $\operatorname{Spec} R \cap \operatorname{Spec} R'$ and containing x. Since any basic open of U' is of the form $\operatorname{Spec} R'_{r's}$, we see that any basic open of U' is also a basic open in $\operatorname{Spec} R'$. Thus we can replace $\operatorname{Spec} R'$ by $\operatorname{Spec} R'_{r'}$ so that we must now prove:

Given $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ there exists a basic open $\operatorname{Spec} R_r$ of $\operatorname{Spec} R$ containing x and lying inside $\operatorname{Spec} R'$ that is also a basic open in $\operatorname{Spec} R'$. However, the containment $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ gives a map $\varphi : R \to R'$. Pick r so that $\operatorname{Spec} R_r$ is contained in $\operatorname{Spec} R'$. Then we claim that $\operatorname{Spec} R_r = \operatorname{Spec} R'_{\varphi(r)}$. But we have

$$\operatorname{Spec} R_r = \{x \in \operatorname{Spec} R : r(x) \neq 0 \text{ in } k(x)\} = \{x \in \operatorname{Spec} R' : \varphi(r)(x) \neq 0 \text{ in } k(x)\} = \operatorname{Spec} R'_{\varphi(r)}$$

simply by virtue of the containments $\operatorname{Spec} R_r \subseteq \operatorname{Spec} R' \subseteq \operatorname{Spec} R$.

Returning to the proof of the theorem, we have just shown that we can cover $\operatorname{Spec} B$ by basic opens $\operatorname{Spec} B_b = \operatorname{Spec}(B_i)_{b_i}$ that are also basic opens of $\operatorname{Spec} B_i$, so we have reduced to the case $Y = \operatorname{Spec} B$. Now let $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$. We have assumed that we have a covering $\operatorname{Spec} A_i$ of $f^{-1}(\operatorname{Spec} B)$ with each A_i a finitely generated $(B_i)_{b_i} \simeq B_b$ algebra, and hence a finitely generated B-algebra. But by the above trick, we obtain a covering of $\operatorname{Spec} A$ by basic open affines $\operatorname{Spec} A_{a_i} = \operatorname{Spec}(A_i)_{\alpha_i}$ such that each A_{a_i} is a finitely generated B-algebra (since each $(A_i)_{\alpha_i} = A_i[1/\alpha_i]$ is). We are thus reduced to the following:

Let $B \to A$ is a map of rings with $a_1, \ldots, a_n$ in A generating the unit ideal. If each A_{a_i} is a finitely generated B-algebra, so is A. To see this, let $x_1, \ldots, x_n \in A$ be such that $\sum x_i a_i = 1$ and let $A_{a_i} = B[z_{i,1}/a_i^N, \ldots, z_{i,n_i}/a_i^N]$. Then $A' = B[a_i, x_j, z_{k,l}]$ is a finitely generated B sub-algebra of A with the evident property that $A_{a_i} = A'_{a_i}$. For any prime ideal $\mathfrak{p}$ of A, there exists some a_i with $a_i \notin \mathfrak{p}$ so that $A_{\mathfrak{p}}$ is a further localization of A_{a_i}, from which it follows that $A_{\mathfrak{p}} = A'_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Hence $A = A'$, as required. ∎

Theorem 0.2. *Let $f : X \to Y$ be a map of schemes. Then the following are equivalent:*

(1) *For all quasi-compact open $U \subseteq Y$ we have $f^{-1}(U)$ quasi-compact.*
(2) *There exists an open covering* $\operatorname{Spec} B_i$ *of Y such that* $f^{-1}(\operatorname{Spec} B_i)$ *is covered by finitely many open affines* $\operatorname{Spec} A_{ij}$.

Definition 0.3. Any morphism satisfying these equivalent conditions will be called *quasi-compact.*

Proof. Assuming (2), let $U \subset Y$ be quasi-compact and cover U by finitely many open affines $\operatorname{Spec} C_\alpha$. Then since $f^{-1}(U)$ is covered by the $f^{-1}(\operatorname{Spec} C_\alpha)$, it suffices to show that each $f^{-1}(\operatorname{Spec} C_\alpha)$ is quasi-compact, that is, we reduce to the case of affine $U = \operatorname{Spec} C$. Now we can cover $\operatorname{Spec} C$ by finitely many sets of the form $\operatorname{Spec}(B_i)_{b_i}$ since $\operatorname{Spec} B_i$ cover Y. We must show that $f^{-1}(\operatorname{Spec}(B_i)_{b_i})$ is quasi-compact given that $f^{-1}(\operatorname{Spec} B_i)$ is. But we have

$$f^{-1}(\operatorname{Spec}(B_i)_{b_i}) = \bigcup \operatorname{Spec}(A_{ij})_{b_i}$$

(where the union is finite), where $f^{-1}(\operatorname{Spec} B_i) = \cup \operatorname{Spec} A_{ij}$. Thus $f^{-1}(\operatorname{Spec}(B_i)_{b_i})$ is quasi-compact. ∎

Definition 0.4. A morphism of schemes $f : X \to Y$ is *finite type* if it is quasi-compact and locally of finite type, or equivalently, if every open affine $\operatorname{Spec} B \subseteq Y$ has $f^{-1}(\operatorname{Spec} B)$ covered by finitely many open affines $\operatorname{Spec} A_{ij}$ with each A_{ij} a finitely generated B_i-algebra.

Corollary 0.5. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g locally of finite type (resp. quasi-compact) then $g \circ f : X \to Z$ is locally of finite type (resp. quasi-compact).*

Proof. We prove the assertion for locally of finite type as the proof for quasi-compact is similar. Let $\operatorname{Spec} C \subseteq Z$ be open affine. Then $g^{-1}(\operatorname{Spec} C)$ is covered by $\operatorname{Spec} B_i$ with each B_i a finitely generated C algebra. Similarly, since f is locally of finite type, each $f^{-1}(\operatorname{Spec} B_i)$ is covered by $\operatorname{Spec} A_{ij}$, with A_{ij} a finitely generated B_i-algebra. Thus $f^{-1}g^{-1}(\operatorname{Spec} C)$ is covered by $\operatorname{Spec} A_{ij}$ with A_{ij} a finitely generated C-algebra for all i, j. ■

Lemma 0.6. *Let X be a scheme. Then the following are equivalent:*

(1) *$\mathcal{O}_X(U)$ is reduced for all $U \subseteq X$.*
(2) *There exists an open affine cover $\operatorname{Spec} A_i$ of X with each A_i reduced.*
(3) *For all $x \in X$, the ring $\mathcal{O}_{X,x}$ is reduced.*

Proof. Clearly (1) implies (2). Since $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_X(U)$ for any $U \ni x$, and since the localization of the nilradical of a ring is the nilradical of the localization, we see that $\mathcal{O}_{X,x}$ is reduced if $\mathcal{O}_X(U)$ is. Finally, because of the injection

$$\mathcal{O}_X(U) \hookrightarrow \prod_{x \in U} \mathcal{O}_{X,x},$$

we see that if each $\mathcal{O}_{X,x}$ is reduced, so is their product, and hence $\mathcal{O}_X(U)$ is reduced as well. ■