1. CODIMINSION

Definition 1.1. Let $Y \subset X$ be an irreducible closed subset. We define the *codimension* of Y in X

$$\operatorname{codim}(Y,X) = \sup_n (Y = Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n = X),$$

with each Y_i closed and irreducible.

For arbitrary closed $Y \subset X$ we define

$$\operatorname{codim}(Y,X) = \inf_{Y_i} \operatorname{codim}(Y_i, X),$$

where the infimum is over all irreducible components Y_i of Y.

Definition 1.2. For $X \neq \emptyset$, we define

$$\dim X = \sup_n (Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subseteq X)$$

with each Y_i irreducible and closed.

When $X = \operatorname{Spec} A$ with $A \neq 0$ then $\dim X$ is the Krull dimension of A since there is an inclusion reversion bijection between irreducible closed sets in X and prime ideals of A.

When A is a finitely generated domain over a field k then we have

$$\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim A,$$

so that if $Y = \overline{\{\mathfrak{p}\}}$ and $X = \operatorname{Spec} A$ we get $\dim Y + \operatorname{codim}(Y,X) = \dim X$. This is not true in general, and we only have $\dim Y + \operatorname{codim}(Y,X) \leq \dim X$.

Definition 1.3. We define the dimension of X at the point $x \in X$ to be

$$\dim_x X = \sup_Y \dim Y,$$

where the supemum is over all irreducible components of X passing through x.

It is not difficult to see that we have a bijection between $\operatorname{Spec} \mathcal{O}_x$ and irreducible closed subsets of X passing through x, and moreover that $\dim X = \sup_{x \in X} \dim \mathcal{O}_x$.

2. CLOSED SUBSCHEMES

Given a closed subset Y of a scheme X we would like to give Y the structure of a closed subscheme, that is, we want to find a sheaf of rings $\mathcal{O}$ on Y such that the topological inclusion map $i : Y \hookrightarrow X$ induces $i_*\mathcal{O} \simeq \mathcal{O}_X/\mathscr{I}$ for some ideal sheaf $\mathscr{I}$, and such that $(Y, \mathcal{O})$ is a scheme. In other words, we seek an ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_X$ such that

(1) $\operatorname{Supp}(\mathcal{O}_X/\mathscr{I}) = Y$, and when this holds,
(2) $\mathcal{O}_X/\mathscr{I} \simeq i_* i^{-1}(\mathcal{O}_X/\mathscr{I})$,

and such that $(Y, i^{-1}(\mathcal{O}_X/\mathscr{I}))$ is a scheme.

Observe that condition (1) is $Y = \{x \in X \mid f(x) = 0 \text{ for all } f \in \mathscr{I}_x\}$.

Definition 2.1. We say that $\mathscr{I} \subseteq \mathcal{O}_X$ is *radical* if equivalently $\mathscr{I}_x \subseteq \mathcal{O}_x$ is radical for every $x \in X$ or $\mathscr{I}(U) \subseteq \mathcal{O}_X(U)$ is radical for all open U.

Lemma 2.2. *If X is a scheme and $Y \subseteq X$ is a closed subset then there existst a unique radical ideal sheaf $\mathscr{I} \subset \mathcal{O}_X$ with zero locus Y such that $(Y, \mathcal{O}_X/\mathscr{I})$ is a scheme.*

Proof. Let $X = \operatorname{Spec} A$. Then $Y = \operatorname{Spec} A/I$ for a unique radical ideal $I \subseteq A$. For any $a \in A$ we have $X_a \cap Y = \operatorname{Spec} A_a/I_a$ and $I_a \subseteq A_a$ is again radical. Thus, for every open affine $U \subseteq X$ we get a unique radical ideal $I_U \subseteq \mathcal{O}_X(U)$ such that $Y \cap U$ is the zero locus of I_U on U. When $U_a = V \subseteq U$ is a basic open then $(I_U)_a = I_V$ in $\mathcal{O}_X(U_a) = \mathcal{O}_X(U)_a$.

Now we imitate the construction of $\mathcal{O}_X$ on an affine scheme (i.e. the $\mathscr{B}$-sheaf construction) to enhance $\{(I_U)_a\}_{a \in \mathcal{O}_X(U)}$ to an ideal sheaf $\mathscr{I}_{Y \cap U} \subseteq \mathcal{O}_U$ for each affine open $U \subseteq X$.

If $U, U' \subseteq X$ are two open affines then

$$\mathscr{I}_{U\cap Y}\big|_{U\cap U'} = \mathscr{I}_{U'\cap Y}\big|_{U\cap U'}$$

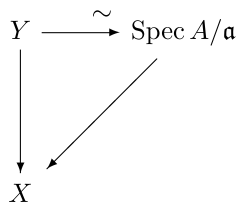
inside $\mathcal{O}_X\big|_{U\cap U'}$ (which can be deduced using Nike's trick locally on $U \cap U'$). Thus that $\mathscr{I}_{U\cap Y}$ glue to give $\mathscr{I}_Y \subseteq \mathcal{O}_X$ such that $\mathscr{I}_Y\big|_U = \mathscr{I}_{Y\cap U}$.

■

We call $(Y, \mathcal{O}_X/\mathscr{I}_Y)$ the *induced reduced scheme structure* on Y. When $Y = X$, the ideal sheaf $\mathscr{I}_Y$ is just the sheaf of nilpotent elements, so we obtain X_{red} in this way.

Given any scheme structure on Y making it a closed subscheme of X, say $\mathscr{I} = \ker(\mathcal{O}_X \to i_*\mathcal{O}_Y)$ we can look at $(Y, \mathcal{O}_X/\mathscr{I}^{n+1})$ for any $n \geq 0$. For example, giving Y the reduced structure, we can contemplate $(Y, \mathcal{O}_X/\mathscr{I}_Y^{n+1})$. On any affine $U = \operatorname{Spec} A \subseteq X$, the sheaf $\mathscr{I}_Y\big|_U$ comes from $I \subseteq A$ so that $(Y, \mathcal{O}_X/\mathscr{I}_Y^{n+1})\big|_U \simeq \operatorname{Spec} A/I^{n+1}$. This is called the n th *infintessimal neighborhood of* Y *in* X.

Theorem 2.3. *If* $X = \operatorname{Spec} A$ *and* $Y \hookrightarrow X$ *is a closed subscheme then there is a unique ideal* $\mathfrak{a} \subseteq A$ *and a unique isomorphism* $Y \simeq \operatorname{Spec} A/\mathfrak{a}$ *such that the diagram*



commutes. Moreover, a map $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A/\mathfrak{a}'$ *exists if and only if* $\mathfrak{a} \supseteq \mathfrak{a}'$.

Proof. This is on the homework. The key point is to show that such a Y as int he statement of the Theorem is affine, for which one uses the criterion for affineness as in Hartshorne Ex. 2.17 (b). ■