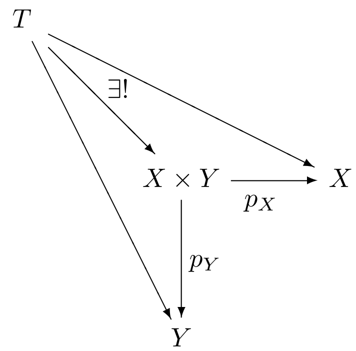
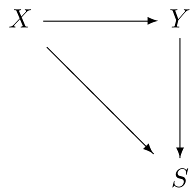
Math 632, Lecture 16 February 13, 2004

## 1. Products

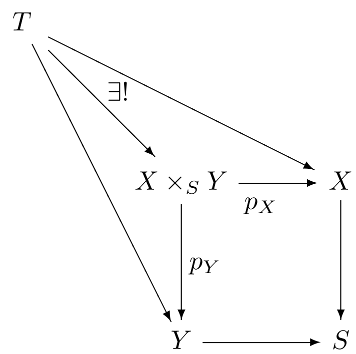
In any category $\mathscr{C}$, one can contemplate products. If $X, Y$ are objects then their product $(X \times Y, p_X, p_Y)$ is an object, together with projection maps, such that for any object $T$ and morphisms $T \to X$ and $T \to Y$ there exists a unique morphism $T \to X \times Y$ making the following diagram commute:




It is easy to see from this definition that the product $X \times Y$ is unique up to unique isomorphism, if it exists. As an example, let $S$ be an object of $\mathscr{C}$. The slice category $\mathscr{C}/S$ has as objects pairs $(X, \pi)$ of an object $X$ of $\mathscr{C}$ and a morphism $\pi : X \to S$. The morphisms $X \to Y$ of $\mathscr{C}/S$ are morphisms of $\mathscr{C}$ making the diagram




commute. In the category $\mathscr{C}/S$ a product of $X, Y$ denoted $X \times_S Y$ is an object together with morphisms $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ such that for any $T/S$ having morphisms to $X$ and $Y$ there exists a unique morphism $T \to X \times_S Y$ making the diagram




commute. Here all morphisms are morphisms over $S$ in the sense described above.

For ease of notation let us put

$$X(T) = \mathrm{Hom}_S(T, X).$$

The universal property of $X \times_S Y$ means precisely that the map

$$(X \times_S Y)(T) \longrightarrow X(T) \times Y(T)$$

is bijective, where the product on the right is as sets. Or, suppressing the $S$-structure, we have a bijection

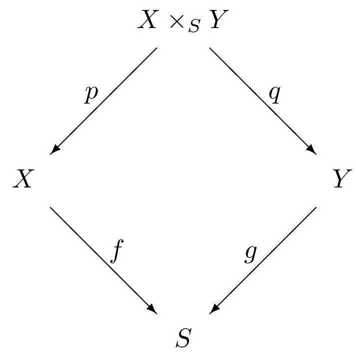
$$\mathrm{Hom}(T, X \times_S Y) \longrightarrow \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, Y).$$

As an example, let $\mathscr{C}$ be the category of $C^\infty$ manifolds. Then $X \times_S Y$ exists if either $X \to S$ or $Y \to S$ is a submersion.

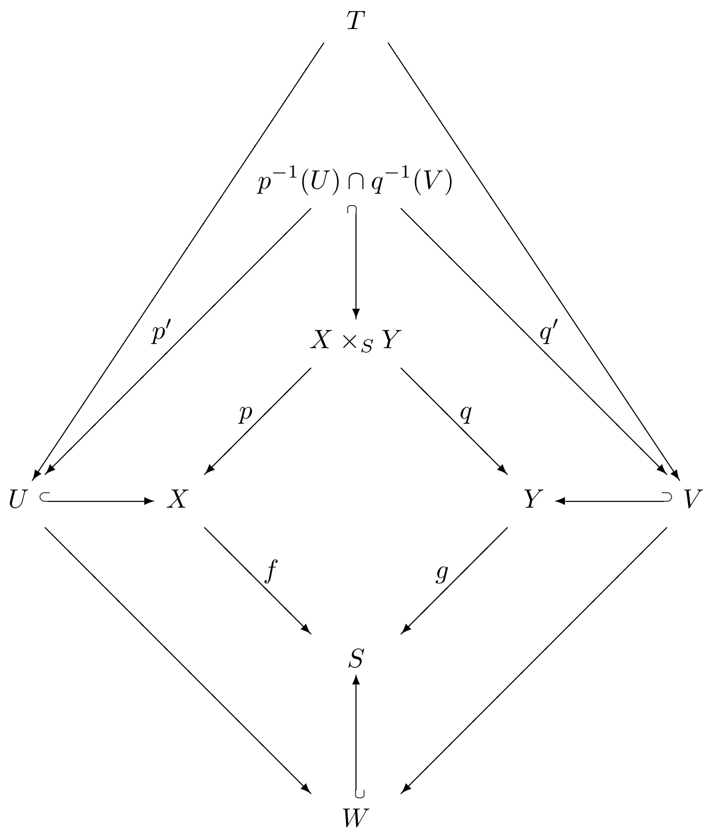
The main goal of this lecture is:

**Theorem 1.1.** *The category of schemes admits fiber products.*

*Proof.* We will proceed in several steps. We begin by showing that if $X \times_S Y$ exists, then the universal property enjoyed by $X \times_S Y$ localizes well. Specifically, suppose $X \times_S Y$ exists so we have the diagram




and let $U \subset X$, $V \subseteq Y$ and $W \subseteq S$ be opens with $U \subseteq f^{-1}(W)$ and $V \subseteq g^{-1}(W)$. Then we claim that the open subsecheme $p^{-1}(U) \cap q^{-1}(V) \subseteq X \times_S Y$ together with the projections $p' = p : p^{-1}(U) \to U$ and $q' = q : q^{-1}(V) \to V$ *is* $U \times_W V = U \times_S V$. Indeed, consider the diagram




By the universal property of $X \times_S Y$, we obtain a unique map $\phi : T \to X \times_S Y$ making the diagram commute. But observe that the compositions $p \circ \phi : T \to X$ and $q \circ \phi : T \to Y$ have images landing in $U, V$ respectively. It follows that the image of $\phi$ is contained in $p^{-1}(U) \cap q^{-1}(V)$ so that in fact $\phi$ is a map $\phi : T \to p^{-1}(U) \cap q^{-1}(V)$ making the diagram commute. This map is unique as any other such map would violate the uniqueness of $\phi$ as a map to $X \times_S Y$. SInce $p^{-1}(U) \cap q^{-1}(V)$ has the same universal property as $U \times_W V = U \times_S V$, we conclude that there exists a unique isomorphism $p^{-1}(U) \cap q^{-1}(V) \simeq U \times_W V$.

Next, we show that it suffices to solve the problem of existence of fiber product locally on $S$. Indeed, let $\{S_i\}$ be an open cover of $S$ such that $X_i = f^{-1}(S_i)$ and $Y_i = g^{-1}(S_i)$ admit a fiber product. This gives the fiber product $X_{ij} \times_{S_{ij}} Y_{ij}$ with $X_{ij} = X_i \cap X_j$, $Y_{ij} = Y_i \cap Y_j$, and $S_{ij} = S_i \cap S_j$ in two different ways: one as a subscheme of $X_i \times_{S_i} Y_i$ and the other as a subscheme of $X_j \times_{S_j} Y_j$. It follows from our discussion above that we have a unique isomorphism $\phi_{ij}$ identifying these two constructions, and we wish to use the $\phi_{ij}$ to glue the schemes $X_i \times_{S_i} Y_i$ along their overlaps. But since the fiber product is unique up to unique isomorphism if it exists, we must have $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ as both sides are isomorphisms of the appropriate fiber products. Thus, by the glueing lemma for spaces and maps, we obtain the following diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & & & Y \\ & \searrow_{f} & & \swarrow_{g} & \\ & & S & & \end{array}$$

and we wish to show that $P$ has the required universal property. But over each $S_i$, the scheme $P$ restricts to $X_i \times_{S_i} Y_i$ so that given a scheme $T \overset{\rightarrow}{\pi} S$ and maps $T \to X$ and $T \to Y$ we obtain unique maps $\phi_i : \pi^{-1}(S_i) \to X_i \times_{S_i} Y_i$, and the uniqueness of the fiber product up to unique isomorphism ensures agreement on overlaps and compatibility of composites of gluing morphisms on triple overlaps, so that we can glue these $\phi_i$ to obtain a unique map $\phi : T \to P$. Hence $P$ has the correct universal property and our problem is local on $S$.

Next we show that the problem of constructing a fiber product of $X$ and $Y$ over $S$ is local on both $X$ and $Y$. Let $\{U_i\}$ and $\{V_j\}$ be open covers of $X$ and $Y$ respectively such that $U_i \times_S V_j = P_{ij}$ exists for each $i, j$, and let $p_i : U_i \times_S V_j \to U_i$ and $q_j : U_i \times_S V_j \to V_j$ be the projection maps. Let $U_{ii'} = U_i \cap U_{i'}$ and $V_{jj'} = V_j \cap V_{j'}$. Observe that $p_i^{-1}(U_{ii'}) \cap q_j^{-1}(V_{jj'}) \subseteq P_{ij}$ and $p_{i'}^{-1}(U_{ii'}) \cap q_{j'}^{-1}(V_{jj'}) \subseteq P_{i'j'}$ are both fiber products of $U_{ii'}$ and $V_{jj'}$ over $S$ by our initial remarks, so that we obtain a unique isomorphism

$$\phi_{ii',jj'} : p_i^{-1}(U_{ii'}) \cap q_j^{-1}(V_{jj'}) \longrightarrow p_{i'}^{-1}(U_{ii'}) \cap q_{j'}^{-1}(V_{jj'})$$

respecting the projections to $U_{ii'}$ and $V_{jj'}$. We thus glue the $P_{ij}$ using the $\phi_{ii',jj'}$ as triple overlap compatibility of the isomorphisms $\phi_{ii',jj'}$ is forced by the uniqueness of any isomorphism identifying two objects with the same universal property.

We therefore obtain a global space $P$ that may be shown to have the right universal property by the same argument above. We have now reduced the existence of a fiber product of $X$ and $Y$ over $S$ to the existence of fiber products $X_i \times_{S_k} Y_j$ for any open covers $\{X_i\}$, $\{Y_j\}$, and $\{S_k\}$ of $X, Y, S$. It is therefore enough to show that a fiber product of $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ over $S = \operatorname{Spec} C$ exists. But for any scheme $T$, we have

$$\begin{aligned} \operatorname{Hom}(T,X) \times_{\operatorname{Hom}(T,S)} \operatorname{Hom}(T,Y) &= \operatorname{Hom}(A, \Gamma(T, \mathcal{O}_T)) \times_{\operatorname{Hom}(C,\Gamma(T,\mathcal{O}_T))} \operatorname{Hom}(B, \Gamma(T, \mathcal{O}_T)) \\ &= \operatorname{Hom}(A \otimes_C B, \Gamma(T, \mathcal{O}_T)) = \operatorname{Hom}(T, \operatorname{Spec}(A \otimes_C B)), \end{aligned}$$

by the universal mapping property of the tensor product. This solves the local construction of fiber products and therefore completes the proof. ■

As an example, given a morphism $f : X \to Y$ and a point $y \in Y$ we obtain a morphism $\operatorname{Spec} \kappa(y) \to Y$ and hence have the diagram

$$\begin{array}{ccc} X_y = X \times_Y \operatorname{Spec} \kappa(y) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \kappa(y) & \longrightarrow & Y \end{array}$$

We call $X_y$ the scheme-theoretic fiber. Observe that a point of $X \times_S Y$ is *not* determined by its images in $X$ and $Y$: consider for example

$$\operatorname{Spec} \mathbf{C} \times_{\operatorname{Spec} \mathbf{Q}} \operatorname{Spec} \mathbf{C} = \operatorname{Spec} \mathbf{C} \otimes_{\mathbf{Q}} \mathbf{C}.$$