Geometric Iwasawa theory and modular forms (mod p)

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CENTRE DE RECHERCHES MATHÉMATIQUES

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- There are natural quotient maps

$$\pi_r : \mathrm{Ig}(p^{r+1}) \to \mathrm{Ig}(p^r), \qquad \pi_r(E, Q) = (E, V(Q))$$

  making $\mathrm{Ig}(p^r)$ a Galois cover of $\mathrm{Ig}(p)$ with Galois group

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2. *There is a canonical isomorphism*

$$\bigoplus_{k=2}^{p} S_k \simeq H^0(\mathrm{Ig}(p), \Omega^1).$$

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- The $\mathbf{F}_p$-vector space of global differentials with simple poles at the s.s. points

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- Question: What is the structure of $\mathbf{M}$ as a $\mathbf{F}_p[\![\Gamma]\!]$-module?

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$$H^0(\mathrm{Ig}(p^r), \Omega^1((p^r m)\underline{\mathrm{ss}}))^{\Gamma/\Gamma_r} \simeq H^0(\mathrm{Ig}(p), \Omega^1((p^r + (m-1)p)\underline{\mathrm{ss}})).$$

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$$\dim_{\mathbf{F}_p} H^0(\mathrm{Ig}(p^r), \Omega^1)^{\Gamma/\Gamma_r} = (p^r - p)\#\underline{\mathrm{ss}} + \mathrm{genus}(\mathrm{Ig}(p)) - 1.$$

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$\pi : \mathrm{Ig}(p^r) \to \mathrm{Ig}(p)$ is unramified outside $\underline{\mathrm{ss}}$, so

$$\pi^*(\Omega^1_{\mathrm{Ig}(p)}) = \Omega^1_{\mathrm{Ig}(p^r)} \otimes \mathscr{I}_{\underline{\mathrm{ss}}}^N.$$

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Now use formal groups on supersingular elliptic curves to calculate N (see Katz-Mazur, §12). □

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- Since $\omega = \frac{df}{f}$ can only have simple poles, for all $m \geq 1$:

$$M_r(m)^{\mathrm{ord}} = M_r(1)^{\mathrm{ord}}$$

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- *The* $\mathbf{F}_p[\![\Gamma]\!]$*-module* $\mathbf{M}^{\mathrm{ord}}$ *is finite free of rank*

$$\dim_{\mathbf{F}_p} H^0(\mathrm{Ig}(p), \Omega^1(\underline{\mathrm{ss}}))^{\mathrm{ord}}$$

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Proof.

Put $M = H^0(X, \Omega^1(\pi^{-1}(S)))^{\mathrm{ord}}$, so $M^G = H^0(Y, \Omega^1(S))^{\mathrm{ord}}$.

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Let $\pi : X \to Y/\mathbf{F}_p$ be a Galois cover of curves with group G. Let $S \subseteq Y$ be the ramification points of π and assume $\#G = p^m$. Then $H^0(X, \Omega^1(\pi^{-1}(S)))^{\mathrm{ord}}$ is a free $\mathbf{F}_p[G]$-module of rank $d := \dim_{\mathbf{F}_p} H^0(Y, \Omega^1(S))^{\mathrm{ord}}$

Proof.

Put $M = H^0(X, \Omega^1(\pi^{-1}(S)))^{\mathrm{ord}}$, so $M^G = H^0(Y, \Omega^1(S))^{\mathrm{ord}}$. Consider the map of $\mathbf{F}_p[G]$-mods $\alpha : M^G \hookrightarrow \mathbf{F}_p[G]^d$.

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Now $\widehat{\alpha}$ is injective since G is a p-group and α is injective. By the Deuring-Shafaravich formula, $\dim_{\mathbf{F}_p} M = \#G \cdot \dim_{\mathbf{F}_p} M^G$ so $\widehat{\alpha}$ must be surjective too. □

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Proof.

By geometric class field theory, $J_r[p](\mathbf{F}_p) \simeq H^0(X_r, \Omega^1)^{\mathrm{ord}}$. □

Further Questions

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This gives a $\mathbf{F}_p[\![\Gamma]\!]$-filtration

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Problem: For each $s \geq 1$, describe the quotient

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Thank You!