Cohomology: powerful tool in AG. "Converts" AG world (schemes, morphisms... **HARD!**) into linear algebra world (vector spaces, linear maps... **EASY!**).

Characteristic p-methods: try to make a characteristic 0 problem easier by deforming it to characteristic p. Follows general idea of studying problems in "families."

$$\mathbb{C} \xrightarrow{\text{descend}} \mathbb{Q} \xrightarrow{\text{put in ``family''}} \mathbb{Z} \xrightarrow{\text{localize}} \mathbb{Z}_p$$

Any Scheme over $\mathbb{Z}_p$ has *two* fibers; the $\mathbb{Q}_p$ (generic) fiber and the $\mathbf{F}_p$ (special) fiber. For cohomological techniques, need a theory for special fiber that relates to usual theories (dR, top, Zariski...) for generic fiber.

We want our cohomologies to have characteristic 0 coefficients (otherwise, can only get information "mod p").

p-adic cohomology: cohomology theory for schemes over fields of characteristic p, with characteristic 0 coefficients (usually an extension of $\mathbb{Q}_p$). Makes available powerful tools from characteristic p, e.g. Frobenius!

Example: $X_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ proper and finite type. There exists $A \subset \mathbb{C}$, a f.g. $\mathbb{Z}$-algebra and a model $X_A \to \operatorname{Spec} A$ such that:

$$H^i_{\text{ét}}(X_{\overline{s}}, \mathbb{Q}_\ell) \simeq H^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Q}_\ell) \simeq H^i_{\text{top}}(X_{\mathbb{C}}, \mathbb{Q}_\ell).$$

Choosing s with $p^r = q = \#k(s)$, get Frobenius on $H^i_{\text{top}}(X_{\mathbb{C}}, \mathbb{Q}_\ell)$, with eigenvalues α_j,

$$|\alpha_j| = q^{w_j/2} \qquad\qquad w_j \leq i.$$

This gives the **weight** filtration of $H^i_{\text{top}}(X_{\mathbb{C}}, \mathbb{Q}_\ell)$ by subspaces where F acts with eigenvalues of weight at most w_j; it is *independent* of any choices.

Basic Problem: Given two different cohomology theories known to be isomorphic, how to compare different constructions in each theory?

Example: $X/\mathbb{C}$ a smooth projective variety, $\dim X = n$.

$$\begin{array}{ccccc} H^n(X, \Omega^n_{X/\mathbb{C}}) & \xrightarrow{\sim} & H^n(X^{\text{an}}, \Omega^n_{X^{\text{an}}}) & \xrightarrow{\sim} & H^{2n}_{\text{top}}(X^{\text{an}}, \mathbb{C}) \\ \downarrow \text{tr} & & & \swarrow \frac{1}{(2\pi i)^n}\int & \\ \mathbb{C} & & & & \end{array}$$

p-adic Cohomology

Crystalline cohomology:

$$H^i_{\text{cris}} : \begin{Bmatrix} \text{smooth, proper } \mathbf{F}_p\text{-sch} \\ \text{morphisms of } \mathbf{F}_p\text{-sch} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{finite } \mathbb{Z}_p\text{-modules} \\ \mathbb{Z}_p\text{-module homs} \end{Bmatrix}$$

- Motivated by observation: "de Rham cohomology of a lift is independent of the lift"

- Replacement for p-adic étale cohomology, which is *very bad* for varieties over $\mathbf{F}_p$ (e.g. not finitely generated, wrong Betti nos...)

- Derived functor cohomology (like Zariski cohomology)

- Disadvantages: difficult to calculate explicitly; requires properness

Theorem. Let $\mathscr{X}/\mathbb{Z}_p$ be smooth and proper. There is a canonical isomorphism

$$H^i_{\mathsf{cris}}(\mathscr{X}_{\mathbf{F}_p}) \simeq H^i_{\mathsf{dR}}(\mathscr{X}).$$

Get $\mathbb{Z}_p$-lattice with Frobenius action inside

$$H^i_{\mathsf{dR}}(\mathscr{X}_{\mathbb{Q}_p}) \simeq H^i_{\mathsf{dR}}(\mathscr{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Remark: Very important in p-adic Hodgy theory: how does F interact with the Hodge filtration on $H^i_{\mathsf{dR}}(\mathscr{X}_{\mathbb{Q}_p})$?

Question: If $\mathscr{X}/\mathbb{Z}_p$ has smooth $\mathbb{Q}_p$-fiber but only generically smooth $\mathbf{F}_p$-fiber, is there still a functorial $\mathbb{Z}_p$-lattice equipped with Frobenius inside $H^i_{\mathsf{dR}}(\mathscr{X}_{\mathbb{Q}_p})$?

Answer: If $\mathscr{X}_{\mathbf{F}_p}$ is ncd, then yes, but uses log-crystalline cohomology: Frobenius is *very* hard to describe explicitly, and this structure is hard to relate to other theories.

Monsky-Washnitzer Cohomology:

$$H^i_{\mathsf{MW}} : \left\{ \begin{matrix} \text{smooth affine } \mathbf{F}_p\text{-sch} \\ \text{morphisms of } \mathbf{F}_p\text{-sch} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{finite } \mathbb{Q}_p\text{-v.s.} \\ \mathbb{Q}_p\text{-linear maps} \end{matrix} \right\}$$

Motivated by work of Dwork, in the spirit of de Rham cohomology.

Sketch: Let $X = \operatorname{Spec} \mathbf{F}_p[X_i]/(f_j)$.

First idea: Pick lifts $\widetilde{f}_j$ of the f_j to $\mathbb{Z}_p[X_i]$ and set $A := \mathbb{Z}_p[X_i]/(\widetilde{f}_j)$. Use the cohomology of the de Rham complex of $A \otimes \mathbb{Q}_p$. **Bad**: Not functorial in X, moreover cohomology depends on choice of lift.

Second idea: To lift maps from characteristic p, pass to p-adic completions. Use the cohomology of the de Rham complex of

$$\mathbb{Q}_p \otimes \varprojlim A/p^n A = \left\{ \begin{matrix} \text{power series in } X_i \text{ with radius of conver-} \\ \text{gence } |X_i|_p \leq 1 \text{, modulo relations} \end{matrix} \right\}$$

Better: Functorial in X, cohomology independent of lift, BUT almost never finite dimensional, roughly because:

Cannot integrate things like

$$1 + px^{p-1} + p^2 x^{p^2-1} + p^3 x^{p^3-1} + \cdots$$

Third idea: The p-adic completion is "too big," so consider the smaller "weak completion"

$$A^\dagger := \left\{\begin{matrix}\text{power series in } X_i \text{ with radius of conver-} \\ \text{gence } |X_i|_p \le 1+\epsilon, \text{ some } \epsilon > 0, \\ \text{modulo relations}\end{matrix}\right\}$$

of overconvergent power series; use the cohomology of the de Rham complex of $\mathbb{Q}_p \otimes A^\dagger$. **Best**: can still lift maps, so functorial in X, and independent of choice of lift. Even finite dimensional! (hard theorem).

We define $H^i_{\mathsf{MW}}(X)$ to be the ith cohomology group of the de Rham complex of $\mathbb{Q}_p \otimes A^\dagger$.

- Many nice properties: Künneth formula, cup product, Lefschetz fixed-point formula...

- Used to prove first two-thirds of Weil conjectures

- *Very* explicit, easy to calculate: Kedlaya's algorithm

- Requires X to be affine and smooth

Question: How do Monsky-Washnitzer and crystalline cohomology relate to eachother?

Rigid Cohomology:

$$H^i_{\text{rig}} : \left\{ \begin{matrix} \text{q-projective } \mathbf{F}_p\text{-sch} \\ \text{morphisms of } \mathbf{F}_p\text{-sch} \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} \text{finite } \mathbb{Q}_p\text{-v.s.} \\ \mathbb{Q}_p\text{-linear maps} \end{matrix} \right\}$$

- "de Rham type" cohomology, based on the idea of lifting to characteristic 0. Handles singular varieties roughly as in Hartshorne's paper "On the de Rham Cohomology..." by locally embedding in a smooth variety and "completing" in some sense

- "Interpolation" of MW and crystalline cohomology

- Can often be made very explicit

Theorem 1. *If $X/\mathbf{F}_p$ is proper and smooth, there is a natural isomorphism*

$$H^i_{\mathsf{cris}}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^i_{\mathsf{rig}}(X).$$

Theorem 2. *If $X/\mathbf{F}_p$ is affine and smooth, there is a natural isomorphism*

$$H^i_{\mathsf{MW}}(X) \simeq H^i_{\mathsf{rig}}(X).$$

Question: Suppose $X/\mathbb{Q}_p$ and $Y/\mathbb{Q}_p$ are smooth and proper of finite type, and that Y has a smooth and proper model $\mathscr{Y}/\mathbb{Z}_p$ but X has only a non-smooth model $\mathscr{X}/\mathbb{Z}_p$. If U is a subscheme of $\mathscr{X}^{\mathsf{sm}}_{\mathbf{F}_p}$ and $X \to Y$ is a map, there are maps:

$$H^i_{\mathsf{cris}}(\mathscr{Y}_{\mathbf{F}_p})\otimes\mathbb{Q}_p \simeq H^i_{\mathsf{dR}}(Y) \to H^i_{\mathsf{dR}}(X) \to H^i_{\mathsf{rig}}(U).$$

Is the composite map F-equivariant?