- Ramanujan's first letter to Hardy:

$$\cfrac{e^{-2\pi/5}}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cdots}}} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

$$\cfrac{e^{-\pi/5}}{1+\cfrac{e^{-\pi}}{1+\cfrac{e^{-2\pi}}{1+\cdots}}} = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}$$

- Hardy:

> [These formulas] defeated me completely. I had never seen anything in the least like this before. A single look at them is enough to show they could only be written down by a mathematician of the highest class. They must be true because no one would have the imagination to invent them.

- Where do these formulae come from, and why should they be true?

Let $\tau \in \mathfrak{H}$ and $q = e^{2\pi i\tau}$. We are lead to define

$$F(\tau) \overset{\text{def}}{=} q^{1/5} \left(\frac{1}{1+} \frac{q}{1+} \frac{q^2}{1+\cdots} \right).$$

What is known about $F(\tau)$?

- *Rogers-Ramanujan*:

$$F = q^{1/5} \prod_{n=1}^{\infty} \frac{(1-q^{5n-4})(1-q^{5n-1})}{(1-q^{5n-2})(1-q^{5n-3})}.$$

- Many "singular values." For example, $F(i)$, $F(i/2)$, and

$$F(i/\sqrt{10}) = \left(\sqrt{90(5+2\sqrt{5})} - 18 - 5\sqrt{5} \right)^{1/5}.$$

- Modular equations: If $x = F(\tau)$ and $y = F(3\tau)$ then

$$(x - y^3)(1 + xy^3) = 3x^2y^2.$$

- Arithmetic: If K is an imaginary quadratic field and $\tau \in \mathfrak{H} \cap K$ then $F(\tau)$ is a unit.

How are these things proved?

- Ramanujan's theory of modular functions (mostly identities).
- Clever manipulation of q series.
- Modular forms and multiplier systems.
- Kronecker's Limit Formula

These methods are *unsatisfactory* as they do not provide any structure or framework in which to place the function $F(\tau)$.

One cannot expect to prove more general results about $F(\tau)$ using these methods.

In this talk we will show:

- $j_5 \overset{\text{def}}{=} \frac{1}{F}$ is a modular function of full level 5, and hence an element of the function field of the modular curve $X(5)$.

- The function field $\mathbb{C}(X(5))$ is *rational*, generated over $\mathbb{C}$ by j_5.

This gives us the powerful interpretation of j_5 (equivalently F) as coordinate on the genus 0 modular curve $X(5)$. Using this viewpoint, we prove:

- $x = j_5(\tau)$ and $y = j_5(n\tau)$ satisfy a polynomial $F_n \in \mathbb{Z}[X, Y]$.

- When K is an imaginary quadratic field and $\tau \in K \cap \mathfrak{H}$ then $j_5(\tau)$ is a unit.

- The polynomial $F_n(X, X) \in \mathbb{Z}[X]$ satisfies simple congruences modulo primes p.

$X(5)$ as Riemann Surface

- Any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ by fractional linear transformations.

- The quotient space $X(\Gamma) \stackrel{\text{def}}{=} \Gamma \backslash \mathfrak{H}^*$ admits the structure of a compact Riemann surface.

- We consider *congruence* subgroups, and in particular

$$\Gamma(N) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv 1 \mod N\}$$

  and the associated Riemann surface

$$X(N) \stackrel{\text{def}}{=} X(\Gamma(N)).$$

- For any inclusion $f : H \hookrightarrow G$ of congruence subgroups we get a field extension $K(X(H))/K(X(G))$ of degree $[\bar{G} : \bar{H}]$.

- For $1 \leq N \leq 5$ the genus of $X(N)$ is 0, and hence $\mathbb{C}(X(N)) \simeq \mathbb{C}(x)$.

Klein Forms

- $L \subset \mathbb{C}$ a lattice with fixed $\mathbb{Z}$ basis ω_1, ω_2. Put $W = \binom{\omega_1}{\omega_2}$ and for $a \in \mathbb{Q}^2$ set $z = a \cdot W$.

- Weierstrass σ function:
$$\sigma(z, L) \stackrel{\text{def}}{=} z \prod_{\omega \in L - \{0\}} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}.$$

- Weierstrass η function: defined on L by
$$\frac{\sigma'}{\sigma}(z + \omega, L) = \frac{\sigma'}{\sigma}(z, L) + \eta(\omega, L).$$

- Define the Klein form
$$\kappa_a(W) \stackrel{\text{def}}{=} e^{-\eta(z,L)z/2}\sigma(z, L).$$

- $\kappa_a(W)$ depends on the *choice of basis* of L.

- When $W = \binom{\tau}{1}$ for $\tau \in \mathfrak{H}$, we write $\kappa_a(\tau)$ for $\kappa_a(W)$.

We have

- $\kappa_a(\lambda W) = \lambda \kappa_a(W)$ for any $\lambda \in \mathbb{C}^\times$.

- For any $b = (b_1, b_2) \in \mathbb{Z}^2$ and $a = (a_1, a_2) \in \mathbb{Q}^2$,

$$\kappa_{a+b}(\tau) = \epsilon e^{\pi i (a_1 b_2 - b_1 a_2)} \kappa_a(\tau),$$

where

$$\epsilon = \begin{cases} 1 & \text{if } b \cdot W \in 2L \\ -1 & \text{otherwise} \end{cases}.$$

- For $\alpha \in \mathrm{SL}_2(\mathbb{Z})$,

$$\kappa_a(\alpha W) = \kappa_{a\alpha}(W).$$

- Let $q = e^{2\pi i \tau}$ and $q_z = e^{2\pi i z}$. Then

$$\kappa_a(\tau) = -\frac{q^{1/2(a_1^2 - a_1)}}{2\pi i} e^{\pi i a_2 (a_1 - 1)} (1 - q_z) \times \prod_{n=1}^{\infty} \frac{(1 - q^n q_z)(1 - q^n / q_z)}{(1 - q^n)^2}.$$

Constructing Functions on $X(N)$ for Odd N

- Use Klein forms to construct functions on $\mathfrak{H}$ invariant under $\Gamma(N)$.

- **Theorem 1**[Kubert-Lang]: Let $S \subset \mathbb{Z}^2 - \{0\}$ and $A = \frac{1}{N}S$. To each $a = (a_1, a_2) \in A$, associate an integer $m(a)$ and let

$$f \stackrel{\text{def}}{=} \prod_{a \in A} \kappa_a^{m(a)}.$$

Then f is invariant under the action of $\Gamma(N)$ *if and only if*

1. $\sum_{a \in A} m(a) = 0$

2. $\sum_{a \in A} m(a) N^2 a_i a_j \equiv 0 \mod N$ for each pair $(i, j) \in \{(1,1), (1,2), (2,2)\}$.

A Generator for $\mathbb{C}(X(5))$.

- **Theorem 2**: Let $\zeta = e^{2\pi i/5}$ and define

$$j_5 \stackrel{\text{def}}{=} \zeta^{-1} \prod_{k=0}^{4} \frac{\kappa_{(\frac{2}{5},\frac{k}{5})}}{\kappa_{(\frac{1}{5},\frac{k}{5})}}.$$

  Then j_5 is a modular function of level 5 and $j_5 = 1/F$.

- **Theorem 3**: j_5 as a function on $X(5)$ has a single simple pole at ∞.

- **Corollary 4**: For any number field K, we have $K(X(5)) = K(j_5)$.

- We have

$$j_5(\tau+1) = \zeta^{-1} j_5(\tau)$$
$$j_5(-1/\tau) = \frac{1 + \frac{1+\sqrt{5}}{2} j_5(\tau)}{j_5(\tau) - \frac{1+\sqrt{5}}{2}}.$$

Proofs

- For Theorem 2, use Theorem 1 with $N = 5$,

$$A = \{(\frac{j}{5}, \frac{k}{5}) : j = 1, 2,\ 0 \le k \le 4\},$$

and $m((\frac{j}{5}, *)) = (-1)^j$.

- To prove Theorem 3, compute the order of j_5 at each cusp of $X(5)$. The product expansion makes it clear that j_5 is holomorphic on $\mathfrak{H}$.

Sample: The matrix $\alpha = \left(\begin{smallmatrix} -4 & 1 \\ -9 & 2 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ takes the cusp represented by 2/9 to ∞. Using the properties of Klein forms:

$$j_5 \circ \alpha^{-1} = -\zeta^2 \frac{\kappa_{(0,\frac{2}{5})}\kappa_{(\frac{2}{5},0)}\kappa_{(\frac{1}{5},\frac{1}{5})}\kappa_{(\frac{1}{5},\frac{2}{5})}\kappa_{(\frac{2}{5},\frac{1}{5})}}{\kappa_{(0,\frac{1}{5})}\kappa_{(\frac{1}{5},0)}\kappa_{(\frac{2}{5},\frac{2}{5})}\kappa_{(\frac{1}{5},\frac{3}{5})}\kappa_{(\frac{2}{5},\frac{4}{5})}},$$

and expanding this as a q series:

$$j_5 \circ \alpha^{-1} = -(1+\zeta^4) + (1+3\zeta+\zeta^2)q^{1/5} + \ldots,$$

so the order at the cusp represented by 2/9 is 0.

- Corollary 4 follows from the fact that $j_5 \in \mathbb{Z}((q^{1/5}))$.

- To prove the formulae for $j_5(-1/\tau)$, use the strategy in the proof of Theorem 3: we have

  $$j_5(-1/\tau) = -(\zeta^2+\zeta^3)+(3+\zeta+\zeta^4)q^{1/5}+\ldots,$$

  so that

  $$\frac{3+\zeta+\zeta^4}{j_5(-1/\tau)+(\zeta^2+\zeta^3)} - j_5(\tau)$$

  is a function on $X(5)$ with no poles and is therefore constant. Inspection of the q series shows this constant to be $\zeta^2+\zeta^3$.

- In this way, we find value of j_5 at each cusp. In particular, j_5 has a simple zero at 2/5.

- We can now prove Ramanujan's evaluation for $F(i) = 1/j_5(i)$. Indeed, i is fixed by $\tau \mapsto -1/\tau$ so $x = j_5(i)$ satisfies

  $$x^2 - (1+\sqrt{5})x - 1 = 0.$$

- Ramanujan's continued fraction **is** coordinate on $X(5)$.

- Gives analogy between j_5 and j.
  - For $n > 1$ and prime to 5, $j_5(\tau)$ and $j_5(n\tau)$ satisfy $F_n(X,Y) \in \mathbb{Z}[X,Y]$.
  - If n is squarefree, $H_n(X) = F_n(X,X)$ is monic.

- Because level is 5: *more structure*.
  - There exists $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $j_5 \circ \alpha = -1/j_5$
  - For K imaginary quadratic and $\tau \in K \cap \mathfrak{H}$, $j_5(\tau)$ is a *unit*.
  - For $p \equiv \pm 1 \mod 5$

    $$F_p(X,Y) \equiv (X^p - Y)(X - Y^p) \mod p.$$

  - For $p \equiv \pm 2 \mod 5$

    $$F_p(X,Y) \equiv (X^p - Y)(XY^p + 1) \mod p.$$

An Involution on $X(5)$

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- Let $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ satisfy $\sigma_a \equiv \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} \mod 5$. Then
$$j_5 \circ \sigma_a = \begin{cases} j_5 & a \equiv \pm 1 \mod 5 \\ -\frac{1}{j_5} & a \equiv \pm 2 \mod 5 \end{cases}.$$

- Observe that σ_a is in the normalizer of $\Gamma(5)$ in $\mathrm{SL}_2(\mathbb{Z})$.

- We expect σ_a to be an involution since for any a, $\sigma_a^2 \in \pm\Gamma(5)$.

- Proof is simple and follows from transformation properties of Klein forms.
  - Reduce to case $a \equiv 2 \mod 5$ since -1 acts trivially on $\mathfrak{H}$.

Modular Equations for j_5

- Existence follows from algebraic geometry, but we proceed classically.

- (Double coset decomposition): Let

  $A = \left\{ \sigma_a \left(\begin{smallmatrix} a & 5b \\ 0 & d \end{smallmatrix} \right) : (a,b,d) = 1,\ ad = n,\ 0 \le b \le d \right\}.$

  Then

  $$\Gamma(5) \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix} \right) \Gamma(5) = \bigcup_{\alpha \in A} \Gamma(5)\alpha.$$

- Let s_k be the k^{th} symmetric polynomial on $\{j_5 \circ \alpha : \alpha \in A\}$. Then $s_k = P_k(j_5)/j_5^{m_k}$ where $P_k \in \mathbb{C}[X]$.

  – s_k is invariant under $\Gamma(5)$ action, so is a rational function of j_5.

  – Key is that $(n,5) = 1$ so $j_5 \circ \alpha$ has poles *only* at $\infty, 2/5$.

- Since $j_5, 1/j_5 \in \mathbb{Z}((q^{1/5}))$ we find $j_5 \circ \alpha \in \mathbb{Z}[\zeta]((q^{1/5}))$.
  - Galois action on coeffs. of q series permutes the set $\{j_5 \circ \alpha : \alpha \in A\}$.
  - Hence $P_k(X) \in \mathbb{Z}[X]$ by Hasse Principle.

- Put

$$f_n(X) = j_5^{m_0} \prod_{\alpha \in A} (X - j_5 \circ \alpha).$$

  - Have shown $f_n(X) \in \mathbb{Z}[X, j_5]$, so let $F_n(X, Y)$ be such that $F_n(X, j_5) = f_n(X)$. Hasse Principle shows $F_n(X, Y) \in \mathbb{Z}[X, Y]$.

- When n is squarefree, lead term in q series of $j_5 - j_5 \circ \alpha$ is a unit. Follows that $H_n(X) = F_n(X, X)$ is monic.

- $H_n(j_5(\tau)) = 0$ iff $j_5(\alpha\tau) = j_5(\tau)$ for some $\alpha \in A$.
  - For some $\beta \in A$ and $\gamma \in \Gamma(5)$, we have $\sigma_2\alpha = \gamma\beta\sigma_2$.
  - Follows that $j_5(\sigma_2\tau) = j_5(\beta\sigma_2\tau)$.
  - Hence $H_n(-1/j_5(\tau)) = 0$.

- Hence for any root z of $H_n(X)$, $-1/z$ is also a root.

- By classical methods, if K is imaginary quadratic and for $\tau \in K \cap \mathfrak{H}$ then $j_5(\tau)$ is a root of H_n for some squarefree n. It follows that $j_5(\tau)$ is a unit.

$$F_p(X,Y) \mod p.$$

- For a prime $p \neq 5$, let $\alpha_b = \left(\begin{smallmatrix} 1 & 5b \\ 0 & p \end{smallmatrix}\right)$ and $\alpha_p = \sigma_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $A = \{\alpha_b : 0 \leq b \leq p\}$.

- Expanding as q series, find for $0 \leq b < p$

$$j_5 \circ \alpha_0 - j_5 \circ \alpha_b \equiv 0 \mod (1-\zeta_p).$$

- Similarly $j_5 \circ \alpha_p \equiv j_5^p \mod p$ if $p \equiv \pm 1 \mod 5$ and $j_5 \circ \alpha_p \equiv -\frac{1}{j_5^p} \mod p$ if $p \equiv \pm 2 \mod 5$.

- Finally, $(j_5 \circ \alpha_0)^p \equiv j_5 \mod p$.

- Since $(1-\zeta_p)$ is the unique prime of $\mathbb{Z}[\zeta_p]$ over p, we can piece these congruences together.

Arithmetic of Singular Values

- Well known result [Shimura] that for K imaginary quadratic and $\tau \in K \cap \mathfrak{H}$ the field $K(\zeta, f(\tau))$ is a certain class field of K, where we adjoin all values $f(\tau)$ for modular functions f of level 5 with coefficients in $\mathbb{Q}$.

- By Corollary 4, we know that $K(\zeta, j_5(\tau))$ is a class field.

- In fact, if $\tau = t_1/t_2$ with $(t_1, t_2) \subset \mathcal{O}_K$ then $K(\zeta, j_5(\tau))$ is the ray class field of K of conductor 5.

Computations

- Can compute $F_p(x,y)$ for small p using linear algebra. We find:

$$F_2(x,y) = yx^3 - y^3x^2 + x + y^2$$

$$F_3(x,y) = yx^4 - y^4x^3 + 3y^2x^2 + x - y^3$$

$$F_5(x,y) = y^5(x^4 - 2x^3 + 4x^2 - 3x + 1) - (x^5 + 3x^4 + 4x^3 + 2x^2 + x)$$

$$F_7(x,y) = yx^8 + (-y^8 + 7y^3)x^7 + 7y^5x^6 + (-7y^7 + 7y^2)x^5 + 35y^4x^4 + (-7y^6 - 7y)x^3 - 7y^3x^2 + (7y^5 + 1)x - y^7$$

$$F_{11}(x,y) = x^{12} + (-y^{11} + 11y^6 - 11y)x^{11} + 66y^2x^{10} - 220y^3x^9 + 495y^4x^8 - 792y^5x^7 + (11y^{11} + 803y^6 - 11y)x^6 - 792y^7x^5 + 495y^8x^4 - 220y^9x^3 + 66y^{10}x^2 + (-11y^{11} - 11y^6 - y)x + y^{12}$$

Questions

- Why are the coefficients of F_n so small?

- Let K be imaginary quadratic, $\tau \in K \cap \mathfrak{H}$ and $L = K(\zeta, j_5(\tau))$. Let $\mathcal{O}_L^\times$ be the full group of units in $\mathcal{O}_L$ and let U be the subgroup of $\mathcal{O}_L^\times$ generated by all values $j_5(\tau) \in L$ with $\tau \in K \cap H$. What is the index $[\mathcal{O}_L^\times : U]$?

Preprint and References

- www.math.lsa.umich.edu/∼bcais/papers.html