Isogeny invariance of the BSD formula

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1 Introduction

In these notes we prove that if $f : A \to B$ is an isogeny of abelian varieties whose degree is relatively prime to the characteristic of the field of definition, then the BSD formula holds for A if and only if it holds for B. We closely follow Milne's notes [3], especially I,§7.

2 Notation

We fix our notation for the remainder of these notes.

$X \mapsto X^*$ is the Pontryagin duality functor (from locally compact Hausdorff abelian groups to abelian groups) $X \mapsto \mathrm{Hom}_{\mathrm{cont}}(X, \mathbf{S}^1)$.

A, B will be used for abelian varieties, and $\widehat{A}, \widehat{B}$ their dual abelian varieties.

K will be a global field.

S is any set (usually finite) of places (=equivalence classes of valuations) of K including all archimedean places.

K_S is the maximal subfield of a fixed separable closure K^{sep} unramified outside S.

G_S is the galois group $\mathrm{Gal}(K_S/K)$

$R_{K,S} = \cap_{v \notin S} \mathcal{O}_v$ is the ring of S-integers.

v will denote a place of K, with K_v the completion of K at v and $k(v)$ the residue field at v.

$G_v = \mathrm{Gal}(K_v^{\mathrm{sep}}/K_v)$ and for $v \notin S$, $D_v, I_v \subseteq G_S$ the decomposition and inertia groups at v respectively. Observe we have the identification $G_v \simeq D_v$.

$g_v = \mathrm{Gal}(k(v)^{\mathrm{sep}}/k(v)) \simeq D_v/I_v$ is generated by Frob_v

M will be a *finite* G_S-module and $M^D = \mathrm{Hom}(M, \mathbf{G}_{\mathrm{m}})$.

A G_S-module M is also a G_v-module via $G_v \hookrightarrow G_K \to G_S$ (identification with the decomposition subgroup induced by the inclusion $K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$). We say M is *unramified* at v if $M^{I_v} = M$, so M becomes a g_v-module.

If v is archimedean and M is unramified at v, the map $G_v \to G_v/I_v \simeq g_v$ defines a map $H^r(g_v, M) \to H^r(G_v, M)$; We denote the image of this map by $H^r_{\mathrm{nr}}(K_v, M)$.

We define

$$\mathcal{H}^r(K_v, M) = \begin{cases} H^r(K_v, M) & v \text{ nonarchimedean} \\ H^r_T(K_v, M) & v \text{ archimedean} \end{cases}. \tag{1}$$

That is, $\mathcal{H}^r(K_v, M)$ is the r th cohomology group $H^r(G_v, M)$ if v is nonarchimedean, and the r th Tate cohomology group $H^r_T(G_v, M)$ if v is archimedean. In particular, $\mathcal{H}^0(\mathbf{R}, M) = M^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})}/N_{\mathbf{C}/\mathbf{R}}M$ and $\mathcal{H}^0(\mathbf{C}, M) = 0$.

$P^r_S(K, M)$ is the restricted topological product $\prod_{v \in S} \mathcal{H}^r(K_v, M)$ relative to the subgroups $H^r_{\mathrm{nr}}(K_v, M)$ for $v \in S$ nonarchimedean.

μ_v is the unique Haar measure on K_v such that $\mathcal{O}_v$ has measure 1 when v is nonarchimedean, and is the usual Lebesgue measure on K_v when v is archimedean.

3 L-series and the BSD formula

Let A be an abelian variety of dimension d over K and S a set of primes containing all archimedean primes and the primes at which A has bad reduction. For any prime v of good reduction, the reduction of A at v, denoted $A(v)$ is again an abelian variety. If $\ell \in \mathbf{Z}$ is any prime distinct from $\operatorname{char} k(v)$ then the criterion of Neron-Ogg-Shafaravich ensures that $V_\ell A := \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} T_\ell A$ is *unramified* at v, i.e. that I_v acts trivially (with the Galois action on the second factor). The characteristic polynomial

$$P_v(A,t) := \det(1 - t \cdot \operatorname{Frob}_v \big|_{V_\ell A})$$

therefore makes sense.

Now let $\omega \in \Gamma(A, \Omega_A^d)$ be any global differential d-form (the space of such forms is 1-dimensional) and define

$$\mu_v(A,\omega) = \int_{A(K_v)} |\omega|_v \mu_v^d.$$

Let $\mu = \{\mu_v\}_v$ be the unique Haar measure on the adeles $\mathbf{A}_K$ induced by the μ_v. We set

$$|\mu| := \int_{\mathbf{A}_K/K} \mu.$$

Definition 3.1 (The L-Series). Choose $\omega \in \Gamma(A, \Omega_A^d)$ and let S be any set of primes containing all archimedean primes and all those nonarchimedean primes for which A has bad reduction or for which ω does not generate $\Gamma(\mathcal{A}_v, \Omega^1_{\mathcal{A}_v})$ as a 1-dimensional $\mathcal{O}_{K,v}$-module ($\mathcal{A}_v$ is the Neron model of A at v). Define

$$L_S(s,A) = \frac{|\mu|^d}{\prod_{v\in S}\mu_v(A,\omega)} \prod_{v\notin S} P_v(A, Nv^{-s})^{-1}.$$

It is known that this defines a holomorphic function of s in the half-plane $\Re s > 3/2$ and that the definition does not depend on the choice ω. It does depend on the choice of the set S, but it can be shown that the asymptotic behavior of L_S near $s = 1$ is independent of S. For proofs of these facts, see [3, §7].

Let

$$\langle\ ,\ \rangle : \widehat{A}(K) \times A(K) \to \mathbf{R}$$

be the canonical height piring.

Conjecture 1 (BSD). *The function $L_S(A,s)$ admits an analytic continuation to a neighborhood of 1 and $Ш(K,A)$ is finite. Moreover, let A have rank r and choose generators $a_i \in A(K)$ and $a_i' \in \widehat{A}(K)$ for $1 \le i \le r$ of $A(K)/A(K)_{\text{tors}}$ and $\widehat{A}(K)/\widehat{A}(K)_{\text{tors}}$. Then*

$$\lim_{s\to 1} \frac{L_S^*(s,A)}{(s-1)^r} = \frac{\#Ш(K,A) \cdot |\det\langle a_i', a_j\rangle|}{[\widehat{A}(K) : \sum \mathbf{Z} a_i'] \cdot [A(K) : \sum \mathbf{Z} a_i]}.$$

4 Preliminary results

In this section, we accumulate (without proof) some fundamental results that will be indispensable in the sequel. We sketch the main ideas of proofs or give precise references.

Theorem 4.1 (Local Tate duality). *Suppose that* $\operatorname{char} K \nmid \#M$. *For any place v of K, and any* $0 \le i \le 2$, *the cup-product pairing gives a duality*

$$\mathcal{H}^i(K_v, M) \times \mathcal{H}^{2-i}(K_v, M^D) \to \mathbf{Q}/\mathbf{Z}$$

in which, for archimedean v, the subgroups $H^i_{\text{nr}}(K_v, M)$ and $H^{2-i}_{\text{nr}}(K_v, M^D)$ are orthogonal complements.

Proof. [3, I,2.3,2.6,2.13] or [5, II,§5]. ■

Theorem 4.2 (Local Tate duality for abelian varieties). *For a place v of K, let A be an abelian variety over K_v. There is a canonical pairing*

$$\mathcal{H}^r(K_v, \widehat{A}) \times \mathcal{H}^{1-r}(K_v, A) \to \mathbf{Q}/\mathbf{Z}$$

which induces isomorphisms $\widehat{A}(K_v) \xrightarrow{\sim} H^1(K_v, A)^*$ and $H^1(K_v, \widehat{A}) \xrightarrow{\sim} A(K_v)^*$.

Proof. See [3, I,Corollary 3.4] for nonarchimedean v and [3, I,Remark 3.7] for archimedean v. The pairing can be formulated as an augmented cup-product pairing. ■

Theorem 4.3 (Poitou-Tate exact sequence). *Suppose that* $\operatorname{char} K \nmid \# M$ *There is an exact sequence of locally compact groups and continuous homomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & H^0(G_S, M) \arrow[r, "\beta^0"] & P^0_S(K, M) \arrow[r, "\gamma^0"] & H^2(G_S, M^D)^* \arrow[d] & \\
& H^1(G_S, M^D)^* \arrow[d] & P^1_S(K, M) \arrow[l, "\gamma^1"'] & H^1(G_S, M) \arrow[l, "\beta^1"'] & \\
& H^2(G_S, M) \arrow[r, "\beta^2"] & P^2_S(K, M) \arrow[r, "\gamma^2"] & H^0(G_S, M^D)^* \arrow[r] & 0,
\end{tikzcd} \tag{2}$$

where the groups in this sequence have the following topological descriptions:

finite	*compact*	*compact*
compact	*loc. compact*	*discrete*
discrete	*discrete*	*finite*

Proof. We will describe the maps β^r, γ^r. For a proof, see [3, I, Theorem 4.10]. There is a natural map $H^r(G_S, M) \to \prod_{v \in S} \mathcal{H}^r(K_v, M)$ induced by the maps $G_v \hookrightarrow G_K \twoheadrightarrow G_S$ with image contained in $P^r_S(K, M)$ [5, II,§6 Prop. 21]; this is the map β^r. Theorem 4.1 shows that $P^r_S(K, M)$ is the algebraic and topological (Pontryagin) dual of $P^{2-r}_S(K, M^D)$, and the maps γ^r are the duals of β^r.

■

Definition 4.4. Let M be a finite G_S-module with $\operatorname{char} K \nmid \# M$. Then the groups $H^r(G_S, M)$ are finite for all r [3, I,Corollary 4.15] and we define

$$\chi(G_S, M) = \frac{\# H^0(G_S, M) \cdot \# H^2(G_S, M)}{\# H^1(G_S, M)}.$$

Theorem 4.5 (Euler characteristic formula). *With the notation as above, suppose that $\# M$ is a unit in $R_{K,S}$. Then the formula*

$$\chi(G_S, M^D) = \prod_{v \text{ arch}} \frac{\# H^0_T(G_v, M)}{\# H^0(G_v, M)}$$

holds, where the product is over all archimedean places v of K.

Proof. See [3, I,Remark 5.2]. ■

5 Isogeny invariance of BSD

In this section we will prove the following theorem:

Theorem 5.1. *Let $f : A \to B$ be an isogeny of abelian varieties over K. If the degree of f is relatively prime to* $\operatorname{char} K$ *then Conjecture 1 holds for A if and only if it holds for B.*

We begin with a definition and a trivial but useful lemma.

Definition 5.2. Let $f : X \to Y$ be a homomorphism of abelian groups with finite kernel and cokernel. We set

$$z(f) = \frac{\#\ker f}{\#\operatorname{coker} f}.$$

Lemma 5.3 (A Trivial Lemma). *1. If $X^\bullet = 0 \to X^0 \to \cdots \to X^n \to 0$ is a complex of finite groups then*

$$\prod (\# X^r)^{(-1)^r} = \prod (\# H^r(X^\bullet))^{(-1)^r}.$$

2. If $f^\bullet : X^\bullet \to Y^\bullet$ is a map of exact sequences of finite length with $z(f^r)$ defined for all r, then

$$\prod z(f^r)^{(-1)^r} = 1.$$

Proof. We prove (1) in the case that $X^\bullet$ is the complex $X^0 \xrightarrow{f^0} X^1$; the general case follows from this. We obviously have

$$\frac{\#X^0}{\#\ker f^0} = \#\operatorname{im} f^0$$

$$\#\operatorname{coker} f^0 = \frac{\#X^1}{\#\operatorname{im} f^0},$$

and multiplying these equalities together gives the desired result.

We prove (2) in the case that $X^\bullet$ and $Y^\bullet$ are short exact sequences, as the general case follows from this. In this case, the snake lemma gives a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker f^0 \arrow[r] & \ker f^1 \arrow[r] & \ker f^2 \arrow[r] & \operatorname{coker} f^0 \arrow[r] & \operatorname{coker} f^1 \arrow[r] & \operatorname{coker} f^2 \arrow[r] & 0 \end{tikzcd}$$

from which the proposed formlua follows. ■

For the remainder of this section, we fix any nonzero $\omega_B \in \Gamma(B, \Omega^d_B)$ and put $\omega_A := f^*\omega_B$. SInce $\operatorname{char} K \nmid \deg(f)$, we have $\omega_A \neq 0$. We take S to be any finite set of places of K including all archimedean places and all nonarchimedean places where A and B have bad reduction (The criterion of Neron-Ogg-Shafarevich shows that the places of bad reduction for A are the same as those for B), or ω_A or ω_B fails to reduce to a nonzero global differential d-form. We fix $a_i \in A(K)$ and $b'_i \in \widehat{B}(K)$ for $1 \leq i \leq r$ generating $A(K)/A(K)_{\text{tors}}$ and $\widehat{B}(K)/\widehat{B}(K)_{\text{tors}}$ respectively, and put $a'_i = \widehat{f}(b'_i)$ and $b_i = f(a_i)$. Since f is an isogeny, it is clear that the b_i and a'_i are $\mathbf{Z}$-linearly independent families.

The following lemma allows us to make sense of the discussion and calculations that follow.

Lemma 5.4. *If one of $L_S(A, s)$, $L_S(B, s)$ admits an analytic continuation to a neighborhood of $s = 1$, so does the other. Moreover, is one of $\text{Ш}(A, K)$, $\text{Ш}(B, K)$ is finite, so is the other.*

Proof. Because A, B are isogenous, $V_\ell A = V_\ell B$. Since the polynomials

$$P_v(A, t) := \det(1 - t \cdot \operatorname{Frob}_v |_{V_\ell A})$$

depend only the galois action on $V_\ell A = V_\ell B$, the first statement is clear since $L_S(A,s)$ and $L_S(B,s)$ differ from eachother by constant (independent of s) multiples.

Let M be the kernel of $f : A \to B$. Then we have an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker \text{Ш}(f) \arrow[r] \arrow[d] & \text{Ш}(A,K) \arrow[r, "\text{Ш}(f)"] \arrow[d] & \text{Ш}(B,K) \arrow[d] \\
 & H^1(G_S,M) \arrow[r] & H^1(G_S,A) \arrow[r, "f"] \arrow[d] & H^1(G_S,B) \arrow[d] \\
 & & \oplus_{v\in S} H^1(K_v,A) \arrow[r, "f"'] & \oplus_{v\in S} H^1(K_v,B)
\end{tikzcd}$$

Since $H^1(G_S,M)$ is *finite*, the diagram shows that $\ker \text{Ш}(f)$ is finite. Thus, if $\text{Ш}(B,K)$ is finite, so is $\text{Ш}(A,K)$. For the reverse implication, we use the fact that there is an isogeny $g : B \to A$ with $g \circ f = \deg f$. ■

We now proceed with some calculations that will enable us to compare the terms involved in the formula of Conjecture 1 for A, B.

Lemma 5.5. *We have*

$$\det\langle a'_j, a_i\rangle = \det\langle b'_j, b_i\rangle. \tag{3}$$

Proof. The height-pairing is functorial in the sense that we have a commutative diagram

$$\begin{tikzcd}
\widehat{A}(K) & \times & A(K) \arrow[r] \arrow[d, "f"] & \mathbf{R} \\
\widehat{B}(K) \arrow[u, "\widehat{f}"] & \times & B(K) \arrow[r] & \mathbf{R} \arrow[u, equal]
\end{tikzcd} \tag{4}$$

[1, Chap. 5], so we have

$$\langle a'_j, a_i\rangle = \langle \widehat{f}(b'_j), a_i\rangle = \langle b'_j, f(a_i)\rangle = \langle b'_j, b_i\rangle$$

as claimed. ■

Lemma 5.6. *The equalities*

$$z(f(K)) = \frac{[A(K) : \sum \mathbf{Z}a_i]}{[B(K) : \sum \mathbf{Z}b_i]} \qquad\qquad z(\widehat{f}(K)) = \frac{[\widehat{B}(K) : \sum \mathbf{Z}b'_i]}{[\widehat{A}(K) : \sum \mathbf{Z}a'_i]}$$

hold.

Proof. Apply Lemma 5.3 (2) and (1) to the morphisms of short exact sequences given by the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \sum \mathbf{Z}a_i \arrow[r] \arrow[d, "\simeq"] & A(K) \arrow[r] \arrow[d, "f(K)"] & A(K)/\sum \mathbf{Z}a_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sum \mathbf{Z}b_i \arrow[r] & B(K) \arrow[r] & B(K)/\sum \mathbf{Z}b_i \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \sum \mathbf{Z} b_i' \arrow[r] \arrow[d, "\simeq"] & \widehat{B}(K) \arrow[r] \arrow[d, "\widehat{f}(K)"] & \widehat{B}(K)/\sum \mathbf{Z} b_i' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sum \mathbf{Z} a_i' \arrow[r] & \widehat{A}(K) \arrow[r] & \widehat{A}(K)/\sum \mathbf{Z} a_i' \arrow[r] & 0
\end{tikzcd}$$

■

Lemma 5.7. *The equality*

$$\frac{\#Ш(A,K)}{\#Ш(B,K)} = \frac{\#\ker Ш(f)}{\#\ker Ш(\widehat{f})}$$

holds.

Proof. We first claim that $Ш(\widehat{A},K)$ and $Ш(\widehat{B},K)$ are both finite. By Lemma 5.4, it suffices to show that $Ш(A,K)$ is finite. However, there are maps $\phi : A \to \widehat{A}$ and $\psi : \widehat{A} \to A$ such that $\phi \circ \psi = \psi \circ \phi = m$ for some integer m, so the map $Ш(m) : Ш(A,K) \to Ш(A,K)$ factors through $Ш(A,K) \xrightarrow{Ш(\phi)} Ш(\widehat{A},K)$ and

$$\ker Ш(\phi) \subseteq Ш(A,K)_m = \ker Ш(m).$$

It is well known [3, Remark 6.14 (c)] that $Ш(A,K)_m$ is finite when $\operatorname{char} K \nmid m$, so the finiteness of $Ш(\widehat{A},K)$ implies that of $Ш(A,K)$. The reverse implication follows upon interchanging the roles of A, $\widehat{A}$ and replacing ϕ with ψ. In the somewhat more subtle case of $\operatorname{char} K = p > 0$, one needs a separate argument as in [4].

Applying Lemma 5.3 (1) to the complex $Ш(A,K) \xrightarrow{Ш(f)} Ш(B,K)$ shows that

$$\frac{\#Ш(A,K)}{\#Ш(B,K)} = z(Ш(f)) = \frac{\#\ker Ш(f)}{\#\operatorname{coker} Ш(f)}.$$

The nondegeneracy of the pairings in the commutative diagram [3, Theorem 6.13 (a)]

$$\begin{tikzcd}
Ш(A,K) \arrow[d, "Ш(f)"] & \times & Ш(\widehat{A},K) \arrow[r] & \mathbf{Q}/\mathbf{Z} \arrow[d, equal] \\
Ш(B,K) & \times & Ш(\widehat{B},K) \arrow[u, "Ш(\widehat{f})"] \arrow[r] & \mathbf{Q}/\mathbf{Z}
\end{tikzcd}$$

implies that $\#\operatorname{coker} Ш(f) = \#\ker Ш(\widehat{f})$ and this completes the proof. ■

Proposition 5.8. *We have*

$$\frac{L_S(A,s)}{L_S(B,s)} = \prod_{v \in S} z(f(K_v))^{-1}.$$

Proof. From Definition 3.1 and our remarks about the polynomials P_v in the proof of Lemma 5.4, we at once see that

$$\frac{L_S(A,s)}{L_S(B,s)} = \prod_{v \in S} \frac{\mu_v(B(K_v),\omega_B)}{\mu_v(A(K_v),\omega_A)},$$

so it will suffice to prove that $z(f(K_v)) = \mu_v(A(K_v),\omega_A)/\mu_v(B(K_v),\omega_B)$. Let U be any subset of $A(K_v)$ mapping isomorphically onto $f(A(K_v))$. Then $f(K_v)\big|_U$ is injective, so since $\omega_A = f^*\omega_B$, we have $\mu_v(U,\omega_A) = \mu_v(f(A(K_v)),\omega_B)$. Since the translates of U by the elements of $\ker f$ give a cover of $A(K_v)$ (with trivial pairwise intersections) and since μ_v is translation invariant, we conclude that

$$\mu_v(A(K_v),\omega_A) = \#\ker f(K_v) \cdot \mu_v(f(A(K_v)),\omega_B). \tag{5}$$

On the other hand, the translates of $f(A(K_v))$ by elements of $\operatorname{coker} f(K_v)$ give a cover (again with pairwise trivial intersections) of $B(K_v)$, so as above,

$$\#\operatorname{coker} f(K_v) \cdot \mu_v(f(A(K_v)), \omega_B) = \mu_v(B(K_v), \omega_B) \tag{6}$$

Combining (5) and (6) gives the desired equality. ■

Theorem 5.9 (The big commutative diagram). *Let M be the kernel of $f : A \to B$; it is a finite Galois module. Suppose that* $\operatorname{char} K \nmid \#M$*, and enlarge S if necessary so $\#M$ is a unit in $R_{K,S}$. There is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & H^0(G_S, M) \arrow[r] & \oplus_{v\in S}\mathcal{H}^0(K_v, M) \arrow[d] & & \\
& & H^2(G_S, M^D)^* \arrow[d] & & \\
0 \arrow[r] & \operatorname{coker} f(K) \arrow[r] \arrow[d, "\varphi'"'] & H^1(G_S, M) \arrow[r] \arrow[d, "\varphi"'] & H^1(G_S, A)_f \arrow[r] \arrow[d, "\varphi''"'] & 0 \\
0 \arrow[r] & \oplus_{v\in S}\operatorname{coker} f(K_v) \arrow[r] \arrow[d, "\psi'"'] & \oplus_{v\in S}H^1(K_v, M) \arrow[r] \arrow[d, "\psi"'] & \oplus_{v\in S}H^1(K_v, A)_f \arrow[r] \arrow[d, "\psi''"'] & 0 \\
0 \arrow[r] & H^1(G_S, \widehat{B})^*_{\widehat{f}} \arrow[r] & H^1(G_S, M^D)^* \arrow[r] & (\operatorname{coker} \widehat{f}(K))^* \arrow[r] & 0
\end{tikzcd} \tag{7}$$

in which the rows and center column are exact.

Proof. The center column is a portion of the Poitou-Tate exact sequence of Theorem 4.3, and the rows come from the long exact cohomology sequences associated to

$$0 \to M(K_S) \to A(K_S) \xrightarrow{f} B(K_S) \to 0$$
$$0 \to M(K_v^{\text{sep}}) \to A(K_v^{\text{sep}}) \xrightarrow{f} B(K_v^{\text{sep}}) \to 0$$
$$0 \to M^D(K_S) \to \widehat{B}(K_S) \xrightarrow{\widehat{f}} \widehat{A}(K_S) \to 0$$

Observe that requiring $\operatorname{char} K \nmid \deg(f)$ ensures that $\widehat{f}$ is separable and that the map $\widehat{B}(K_S) \xrightarrow{\widehat{f}} \widehat{A}(K_S)$ is surjective. The first two exact sequences follow from the definition of M. The third is the "dual exact sequence;" see, for example, [2, §11]. Surjectivity on K_S-points is a consequence of the fact that A, B have good reduction outside S and is well-known *cf.* [3, Lemma 6.1].

It therefore remains to check that the four squares commute. The two top squares obviously commute, and the bottom two are seen to commute as follows: Theorem 4.2 shows that $B(K_v)$ and $H^1(K_v, \widehat{B})$ are dual, so by dualizing the exact sequence

$$A(K_v) \xrightarrow{f} B(K_v) \to \operatorname{coker} f(K_v) \to 0,$$

we obtain

$$0 \to (\operatorname{coker} f(K_v))^* \to H^1(K_v, \widehat{B}) \to H^1(K_v, \widehat{A}),$$

which shows that $\operatorname{coker} f(K_v)$ is dual to $H^1(K_v, \widehat{B})_{\widehat{f}}$. It follows using the duality of $H^1(K_v, M)$ and $H^1(K_v, M^D)$ in Theorem 4.1 that the diagram

$$\begin{tikzcd}
\oplus_{v\in S} \operatorname{coker} f(K_v) \arrow[r] \arrow[d, "\psi'"'] & \oplus_{v\in S} H^1(K_v, M) \arrow[d, "\psi"'] \\
H^1(G_S, \widehat{B})^*_{\widehat{f}} \arrow[r] & H^1(G_S, M^D)^*
\end{tikzcd}$$

is simply the dual of

$$\begin{tikzcd}
\oplus_{v\in S} H^1(K_v, \widehat{B})_{\widehat{f}} & \oplus_{v\in S} H^1(K_v, M^D) \arrow[l] \\
H^1(G_S, \widehat{B})_{\widehat{f}} \arrow[u] & H^1(G_S, M^D) \arrow[l] \arrow[u]
\end{tikzcd}$$

where the horizontal arrows come from the obviously compatible cohomology sequences

$$0 \to M^D(K) \to \widehat{B}(K) \to \widehat{A}(K) \to H^1(G_S, M^D) \to H^1(G_S, \widehat{B}) \to H^1(G_S, \widehat{A}) \to \dots$$
$$0 \to M^D(K_v) \to \widehat{B}(K_v) \to \widehat{A}(K_v) \to H^1(K_v, M^D) \to H^1(K_v, \widehat{B}) \to H^1(K_v, \widehat{A}) \to \dots$$

Similarly, the squares

$$\begin{tikzcd}
\oplus_{v\in S} H^1(K_v, M) \arrow[r] \arrow[d, "\psi"'] & \oplus_{v\in S} H^1(K_v, A)_f \arrow[d, "\psi''"'] & \oplus_{v\in S} H^1(K_v, M^D) & \oplus_{v\in S} \operatorname{coker} \widehat{f}(K_v) \arrow[l] \\
H^1(G_S, M^D)^* \arrow[r] & (\operatorname{coker} \widehat{f}(K))^* & H^1(G_S, M^D) \arrow[u] & \operatorname{coker} \widehat{f}(K) \arrow[l] \arrow[u]
\end{tikzcd} \tag{8}$$

are duals of each other, so the left square commutes because the right obviously does. ■

Corollary 5.10 (The five formulae). *We have:*

$$\begin{aligned}
\frac{\#\ker\varphi'}{\#\ker\varphi}\,\frac{\#\ker \mathrm{Ш}(f)}{\#(\ker\psi'/\operatorname{im}\varphi')} &= 1 \qquad (9)\\
\frac{\#\operatorname{coker} f(K)}{\prod_{v\in S}\#\operatorname{coker} f(K_v)}\#H^1(G_S,\widehat{B})_{\widehat{f}} &= \frac{\#\ker\varphi'}{\#(\ker\psi'/\operatorname{im}\varphi')}\#\ker \mathrm{Ш}(\widehat{f}) \qquad (10)\\
1 &= \frac{\#H^1(G_S,\widehat{B})_{\widehat{f}}}{\#H^1(G_S,M^D)}\#\operatorname{coker}\widehat{f}(K) \qquad (11)\\
1 &= \frac{\#\ker f(K)}{\prod_{v\in S}\#\ker f(K_v)}\,\frac{\#H^2(G_S,M^D)}{\#\ker\varphi}\prod_{v\ arch}\frac{\#H^0(K_v,M)}{\#H^0_T(K_v,M)} \qquad (12)\\
\#\ker\widehat{f}(K) &= \#H^0(G_S,M^D). \qquad (13)
\end{aligned}$$

Proof. We view the bottom of the diagram (7) as a short exact sequence of complexes

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0,$$

where $A^\bullet$ is the complex consisting of the first column, $B^\bullet$ is the complex formed by the second column etc. The long exact sequence on cohomology is then

$$0 \to \ker\varphi' \to \ker\varphi \to \ker\varphi'' \to H^1(A^\bullet) \to H^1(B^\bullet).$$

Since the complex $B^\bullet$ is exact (recall it is a piece of the Poitou-Tate exact sequence) we have $H^1(B^\bullet) = 0$. Obviously, $H^1(A^\bullet) = \ker\psi'/\operatorname{im}\varphi'$. We conclude that

$$0 \to \ker\varphi' \to \ker\varphi \to \ker\varphi'' \to \ker\psi'/\operatorname{im}\varphi' \to 0$$

is exact. Therefore,

$$\frac{\#\ker\varphi'}{\#\ker\varphi}\frac{\#\ker\varphi''}{\#(\ker\psi'/\ker\varphi')} = 1. \tag{14}$$

The commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker\text{Ш}(f) \arrow[r] \arrow[d] & \text{Ш}(K,A) \arrow[r, "\text{Ш}(f)"] \arrow[d] & \text{Ш}(K,B) \arrow[d] \\
0 \arrow[r] & H^1(G_S,A)_f \arrow[r] \arrow[d, "\varphi''"'] & H^1(G_S,A) \arrow[r, "f"] \arrow[d] & H^1(G_S,B) \arrow[d] \\
0 \arrow[r] & \oplus_{v\in S}H^1(K_v,A)_f \arrow[r] & \oplus_{v\in S}H^1(K_v,A) \arrow[r, "f"'] & \oplus_{v\in S}H^1(K_v,B)
\end{tikzcd}$$

shows that $\ker\text{Ш}(f) = \ker\varphi''$. Employing this in (14) yields (9).

Applying Lemma 5.3 (1) to the first column of (7) gives

$$\frac{\#\operatorname{coker}f(K)}{\prod_{v\in S}\#\operatorname{coker}f(K_v)}\#H^1(G_S,\widehat{B})_{\widehat{f}} = \frac{\#\ker\varphi'}{\#(\ker\psi'/\operatorname{im}\varphi')}\#\operatorname{coker}\psi'. \tag{15}$$

The commutative diagram

$$\begin{tikzcd}
0 & \oplus_{v\in S}\operatorname{coker}f(K_v) \arrow[l] \arrow[d, "\psi'"'] & \oplus_{v\in S}H^1(K_v,\widehat{B})^* \arrow[l] \arrow[d] & \oplus_{v\in S}H^1(K_v,\widehat{A})^* \arrow[l, "\widehat{f}"'] \arrow[d] \\
0 & H^1(G_S,\widehat{B})^*_{\widehat{f}} \arrow[l] \arrow[d] & H^1(G_S,\widehat{B})^* \arrow[l] \arrow[d] & H^1(G_S,\widehat{A})^* \arrow[l, "\widehat{f}"'] \arrow[d] \\
0 & (\ker\text{Ш}(\widehat{f}))^* \arrow[l] & \text{Ш}(K,\widehat{B})^* \arrow[l] \arrow[d] & \text{Ш}(K,\widehat{A})^* \arrow[l, "\text{Ш}(\widehat{f})"'] \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

shows that $\operatorname{coker}\psi' = (\ker\text{Ш}(\widehat{f}))^*$. Using this in (15) gives (10).

The third row of (7) at once gives (11).

Now the long exact sequence *of unmodified cohomology* shows that $M^{G_v} = H^0(K_v,M) = \ker f(K_v)$, so by (1) we have

$$\#\mathcal{H}^0(K_v,M) = \begin{cases} \#\ker f(K_v) & v \text{ nonarchimedean} \\ \#\ker f(K_v)\frac{\#H^0_T(K_v,M)}{\#H^0(K_v,M)} & v \text{ archimedean} \end{cases}. \tag{16}$$

Similarly, it is clear that $\#\ker f(K) = \#H^0(G_S,M)$. Using this and (16) together with the middle column of (7) gives (12).

Finally, the cohomology sequence of the dual exact sequence [2, §11]

$$0 \to M^D(K_S) \to \widehat{B}(K_S) \xrightarrow{\widehat{f}(K_S)} \widehat{A}(K_S) \to 0$$

shows that $H^0(G_S, M^D) = \ker \widehat{f}(K)$, which is (13). ∎

Corollary 5.11 (The Final Formula). *In the notation of Definition 5.2, we have*

$$\prod_{v \in S} z(f(K_v)) = \frac{\#\ker Ш(\widehat{f})}{\#\ker Ш(f)} \frac{z(f(K))}{z(\widehat{f}(K))}. \tag{17}$$

Proof. Multiplying the equalities (9)–(13) of Corollary 5.10 and cancelling like terms from either side, we find

$$\prod_{v \in S} z(f(K_v)) = \frac{\#\ker Ш(\widehat{f})}{\#\ker Ш(f)} \frac{z(f(K))}{z(\widehat{f}(K))} \cdot \chi(G_S, M^D) \cdot \prod_{v \text{ arch}} \frac{\#H^0(K_v, M)}{\#H^0_T(K_v, M)},$$

where $\chi(G_S, M^D)$ is as defined in Definition 4.4. The corollary now follows from Theorem 4.5. ∎

Corollary 5.12. *We have*

$$\frac{L_S(B,s)}{L_S(A,s)} = \frac{\#Ш(B,K) \cdot |\det\langle b'_j, b_i\rangle|}{[B(K) : \sum \mathbf{Z} b_i][\widehat{B}(K) : \sum \mathbf{Z} b'_i]} \cdot \left(\frac{\#Ш(A,K) \cdot |\det\langle a'_j, a_i\rangle|}{[A(K) : \sum \mathbf{Z} a_i][\widehat{A}(K) : \sum \mathbf{Z} a'_i]} \right)^{-1}.$$

Proof. Proposition 5.8 and Corollary 5.11 show that

$$\frac{L_S(B,s)}{L_S(A,s)} = \frac{\#\ker Ш(\widehat{f})}{\#\ker Ш(f)} \frac{z(f(K))}{z(\widehat{f}(K))},$$

which by Lemmas 5.6 and 5.7 is equal to

$$\frac{\#Ш(B,K)}{\#Ш(A,K)} \cdot \frac{[A(K) : \sum \mathbf{Z} a_i]}{[B(K) : \sum \mathbf{Z} b_i]} \cdot \frac{[\widehat{A}(K) : \sum \mathbf{Z} a'_i]}{[\widehat{B}(K) : \sum \mathbf{Z} b'_i]}.$$

Since $\det\langle a'_j, a_i\rangle = \det\langle b'_j, b_i\rangle$ by Lemma 5.5, the proposed formula follows. ∎

Proof of Theorem 5.1. Assume that Conjecture 1 holds for A. By Lemma 5.4, the function $L_S(B,s)$ admits an analytic continuation to a neighborhood of $s = 1$ and the quantity $\#Ш(B,K)$ makes sense (i.e. is finite). Corollary 5.12 then shows that Conjecture 1 holds for B. The reverse implication follows at once upon interchanging the roles of A and B. ∎

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