Introduction to Vertex Operator Algebras

Prof. Christoph A. Keller
ETH Zürich

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Contents

1 Formal Power Series

1.1 One Variable

Let $a_n \in U$, U some vector space. Define formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^n \ . \tag{1.1}$$

Clearly these form a vector space, which we denote by $U[[z, z^{-1}]]$. We denote formal Laurent polynomials as $U[z, z^{-1}]$. We will also encounter $U[[z]][z^{-1}]$. Sometimes these are called *formal Laurent series*, and are denoted by $U((z))$. (The logic of the notation is that $[[z]]$ denotes infinitely many non-negative powers in z, and $[z]$ only finitely many non-negative powers.) For $\lambda \in \mathbb{C}$ we define $u(\lambda z) := \sum_{n\in\mathbb{Z}} \lambda^n u_n z^n$.

Example: 'multiplicative δ-distribution'

$$\delta(z) := \sum_{n\in\mathbb{Z}} z^n \tag{1.2}$$

Let us now take U to be an associative algebra with unit 1, so that multiplication is defined. In practice, we will work with two types of examples:

- $U = \mathrm{End}(V)$ for some vector space V.
- $U = \mathbb{C}$

We now want to try to define a multiplication of formal power series:

$$a(z)b(z) = \sum_{n\in\mathbb{Z}} c_n z^n \qquad c_n = \sum_{\text{"}k=-\infty\text{"}}^{\text{"}\infty\text{"}} a_k b_{n-k} \tag{1.3}$$

This product of two formal series is in general not well defined, since the infinite sum is only defined if it is secretly a finite sum! (Remember that we are doing algebra, and have no concept of convergence.) Multiplication therefore only works in some cases. An example that fails is for instance $\delta(z)\delta(z)$. We can for example multiply by Laurent polynomials: $f(z)u(z) \in U[[z, z^{-1}]]$ if $f(z) \in U[z, z^{-1}]$.

Proposition 1.1. *For $f(z) \in U[z, z^{-1}]$,*

$$f(z)\delta(z) = f(1)\delta(z)$$

Exercise 1.1. *Define*

$$(1-z)^{-1} := \sum_{n\geq 0} z^n \tag{1.4}$$

Show: $(1-z)^{-1}(1-z)$ is well-defined and $= 1$.

General criterion: $a^1(z)a^2(z)\cdots a^k(z)$ exists if for all n the set

$$I_n = \{(n_1, \ldots, n_k) : n_1 + n_2 + \ldots + n_k = n, a^1_{n_1} a^2_{n_2} \cdots a^k_{n_k} \neq 0\} \subset \mathbb{Z}^n \tag{1.5}$$

only has finitely many elements. We then set

$$a^1(z)a^2(z)\cdots a^k(z) = b(z) \qquad \text{with} \quad b_n = \sum_{(n_1,\ldots,n_r)\in I_n} a^1_{n_1} a^2_{n_2} \cdots a^k_{n_k} \tag{1.6}$$

We then say that the product exists, or is well-defined, or that it is summable. Typical example that works: multiplying formal Laurent series $a(z), b(z) \in U[[z]][z^{-1}]$: $a(z)b(z) \in U[[z]][z^{-1}]$.

Usually it is pretty obvious when we are illegally trying to multiply formal power series, since we get manifestly infinite sums. Sometimes things can go wrong in a more subtle manner, as in the following apparent paradox:

$$\delta(z) = 1\delta(z) = (1-z)^{-1}(1-z)\delta(z) = (1-z)^{-1}\cdot 0 = 0 \tag{1.7}$$

What went wrong here?

Exercise 1.2. *If $a(z)b(z), b(z)c(z)$ and $a(z)b(z)c(z)$ all exist, then $(a(z)b(z))c(z) = a(z)(b(z)c(z)) = a(z)b(z)c(z)$.*

We can define derivatives,

$$\partial_z a(z) := \sum_{n\in\mathbb{Z}} (n+1)a_{n+1} z^n \ , \tag{1.8}$$

and residues,

$$\mathrm{Res}_z a(z) := a_{-1} \ . \tag{1.9}$$

We can solve differential equations:

Lemma 1.2. *$R(z) \in U[[z]]$, initial value f_0. Then*

$$\partial_z f(z) = R(z) f(z)$$

has a unique solution with $f(z) \in U[[z]]$ and with intial data f_0, i.e. $f(z) = f_0 + O(z)$.

Proof. Recursion (Lemma 4.1 in [1]). □

1.2 Multiple variables

Similar definition, *e.g.* for two variables:

$$a(z,w) = \sum_{n,m\in\mathbb{Z}} a_{n,m} z^n w^m \qquad \in U[[z,z^{-1},w,w^{-1}]] \tag{1.10}$$

Multiplication of two different variables is fine: $a(z)b(w) \in U[[z,z^{-1},w,w^{-1}]]$. We define the binomial expansion convention:

$$(z+w)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^{n-k} w^k \in U[[z,z^{-1},w]] \tag{1.11}$$

Note that for $n<0$, $(z+w)^n \neq (w+z)^n$! Sometimes people use the notation $\iota_{z,w}(z+w)^n := (z+w)^n$, $\iota_{w,z}(z+w)^n := (w+z)^n$

Exercise 1.3. *Show*

- $(z+w)^n (z+w)^{-n} = 1$
- $(1-z)^{-1} - (-z+1)^{-1} = \delta(z)$

We can use this to 'shift' the argument of power series to get:

$$a(z+w) \in U[[z,z^{-1},w]] \tag{1.12}$$

(Proof: $z^n w^m$ has only one contribution, only for $m \geq 0$.) We can 'set $w=0$', that is extract the term w^0, to get back $a(z+0) = a(z)$.

We can now work with the δ function in two variables, $\delta(z/w)$. (Note that there is a clash of notation with [1]! We are using multiplicative notation, and also differ by a factor of w.) We can now multiply by any power series in z, not just Laurent polynomials:

$$f(z)\delta(z/w) = f(w)\delta(z/w) \qquad \forall f(z) \in U[[z,z^{-1}]] \ . \tag{1.13}$$

Some more useful properties of the delta distribution:

Proposition 1.3. *δ satisfies the following properties:*

1. $\forall f(z) \in U[[z, z^{-1}]]: \ \mathrm{Res}_z f(z) w^{-1} \delta(w/z) = f(w)$
2. $\delta(z/w) = \delta(w/z)$
3. $(z-w)\partial_w^{j+1}\delta(w/z) = (j+1)\partial_w^j \delta(w/z)$
4. $(z-w)^{j+1}\partial_w^j \delta(w/z) = 0$

Proof. Exercise. □

2 Fields and Locality

2.1 Fields

From now on we will fix $U = \mathrm{End}(V)$. In that case, note that for multiplications to exist, a weaker condition than (1.5) can be imposed: It is enough that for every $v \in V$,

$$I_n^v = \{(n_1, \ldots, n_k) : n_1 + n_2 + \ldots + n_k = n, a^1_{n_1} a^2_{n_2} \cdots a^k_{n_k} v \neq 0\}$$

only has finitely many elements. $a^1(z)a^2(z)\cdots a^k(z)$ is then a well defined element of $\mathrm{End}(V)[[z, z^{-1}]]$, since $\sum a^1_{n_1} a^2_{n_2} \cdots a^k_{n_k} v$ is a finite sum for all v.

Just to annoy everybody, we now switch conventions.

Definition 2.1. *A formal power series*

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \qquad \in \mathrm{End}(V)[[z, z^{-1}]] \tag{2.1}$$

is called a field *if for all $v \in V$ there is a K (sometimes called the* order of truncation of a on v*) such that*

$$a_n v = 0 \qquad \forall n \geq K \ . \tag{2.2}$$

An equivalent (and sometimes more useful) definition is that $\forall v \in V$, $a(z)v \in V[[z]][z^{-1}]$. That is, $a(z) \in \mathrm{Hom}(V, V[[z]][z^{-1}])$. We will call the space of fields $\mathcal{E}(V)$. Note: $\partial a(z)$ is again a field, so $\mathcal{E}(V)$ is closed under taking derivatives.

We now want to define a multiplication on $\mathcal{E}(V)$. To do this, we first need to define the notion of normal ordering. For $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ define the 'annihilation' and 'creation' part as

$$a(z)_- = \sum_{n\geq 0} a_n z^{-n-1} \qquad a(z)_+ = \sum_{n<0} a_n z^{-n-1} \ .$$

Note: We choose this particular definition to ensure that $(\partial a(z))_\pm = \partial(a(z)_\pm)$. We can then define the normal ordered product:

$$:a(z)b(w): := a(z)_+ b(w) + b(w)a(z)_- \tag{2.3}$$

On the level of modes this means

$$:a_m b_n: = \begin{cases} a_m b_n & : m < 0 \\ b_n a_m & : m \geq 0 \end{cases} \tag{2.4}$$

Note: The normal ordered product is neither associative nor commutative! We have $\partial_z :a(z)b(w):=:\partial_z a(z)b(w):$.

Proposition 2.1. *Let $a(z), b(z)$ be fields. Then $:a(z)b(z):$ is well-defined and is a field.*

Proof. $a(z)_+ b(z)v \in \mathrm{End}(V)[[z]]V[[z]][z^{-1}] \subset V[[z]][z^{-1}]$. $a(z)_- v \in V[z^{-1}]$, that is a finite linear combination of terms. So by linearity we have $b(z)a(z)_- v \in V[[z]][z^{-1}]$. □

$\Rightarrow \mathcal{E}(V)$ is a (non-commutative, non-associative) algebra closed under derivatives

Lemma 2.2.

$$:a(w)b(w) := \mathrm{Res}_z \left(a(z)b(w)(z-w)^{-1} - b(w)a(z)(-w+z)^{-1}\right)$$

Proof.

$$\mathrm{Res}_z a(z)(z-w)^{-1} = a(w)_+ \qquad \mathrm{Res}_z a(z)(-w+z)^{-1} = -a(w)_- \tag{2.5}$$

□

It turns out to be useful to define a whole family of normal ordered products. We can generalize for $n \in \mathbb{Z}$,

$$a(w)_n b(w) := \mathrm{Res}_z(a(z)b(w)(z-w)^n - b(w)a(z)(-w+z)^n) \tag{2.6}$$

Note for $n \geq 0$, we have $a(w)_{-n-1}b(w) = \frac{1}{n!}:\partial^n a(w)b(w):$ (Take derivatives ∂_z^n in (2.5), use $\mathrm{Res}_z(\partial_z f(z)) = 0$).

Lemma 2.3. *For all $n \in \mathbb{Z}$, $a(w)_n b(w)$ is well defined and a field.*

Proof. For $n < 0$, this follows from 2.1. For $n \geq 0$, apply both terms to $v \in V$. We can neglect the polynomial $(z-w)^n$ which only gives a finite linear combination of shifted terms. First term: $a_0 b(w)v \in V[[w]][w^{-1}]$. Second term: $b(w)a_0 v \in V[[w]][w^{-1}]$. □

We also have

Lemma 2.4.

$$\partial_w(a(w)_n b(w)) = \partial a(w)_n b(w) + a(w)_n \partial b(w)$$

Proof. Exercise. □

2.2 Locality

We say $a(z)$ and $b(z)$ are *mutually local* if $\exists K > 0$ such that

$$(z-w)^K [a(z), b(w)] = 0 \tag{2.7}$$

K is sometimes called the *order of locality*. One might be tempted to multiply by $(z-w)^{-K}$ and conclude that the commutator vanishes. This however is in general illegal. In fact, we know from prop 1.3 that δ distributions are counterexamples. All counterexamples are of this form:

Theorem 2.5. *Let $f(z,w) \in U[[z, z^{-1}, w, w^{-1}]]$ be such that $(z-w)^K f(z,w) = 0$. We can then write*

$$f(z,w) = \sum_{j=0}^{K-1} c^j(w) \frac{1}{j!} z^{-1} \partial_w^j \delta(z/w) ,$$

where the series $c^j(w)$ are given by

$$c^j(w) = \mathrm{Res}_z f(z,w)(z-w)^j .$$

Proof. Define $b(z,w) := f(z,w) - \sum_{j=0}^{K-1} c^j(w) z^{-1} \frac{1}{j!} \partial_w^j \delta(z/w)$. We have $\mathrm{Res}_z (z-w)^n b(z,w) = 0$ for all $n \geq 0$: For $n \geq K$, both terms in the definition of b give zero by 4 in prop 1.3. For $n < K$ we can evaluate Res_z. Both contributions cancel by using 3 in prop 1.3. Next write $b(z,w) =: \sum_{n \in \mathbb{Z}} a_n(w) z^n$. By taking Res_z with $n = 0$ we conclude that $a_{-1}(w) = 0$. From this and taking Res_z with $n = 1$ we conclude $a_{-2}(w) = 0$ etc. $b(z,w)$ thus only contains non-negative powers of z. Since $(z-w)^K b(z,w) = 0$, it then follows that $b(z,w) = 0$, establishing the claim. □

It follows that we can write

$$[a(z), b(w)] = \sum_{j=0}^{K-1} (a(w)_j b(w)) \frac{1}{j!} z^{-1} \partial_w^j \delta(z/w) \tag{2.8}$$

If $a(z)$ and $b(z)$ are mutually local, then so are $\partial a(z)$ and $b(z)$. (To see this, take derivative of (2.7) and multiply by $(z-w)$.) We want to establish the analogue of lemma 2.3 for local fields, *i.e.* establish that local fields form an algebra under normal products. The following Lemma due to Dong does that:

Lemma 2.6 (Dong). *If $a(z)$, $b(z)$ and $c(z)$ are pairwise mutually local, then $a(z)_n b(z)$ and $c(z)$ are mutually local.*

Proof. As in 5.5.15 in [2]. □

(In particular: $:a(z)b(z):$ and $c(z)$ are mutually local.) Local fields and their derivatives thus form an algebra under the normal ordered product!

2.3 Heisenberg algebra

Let us now construct an explicit example of a local field. To do this, we need to take a brief detour to the theory of Lie algebras and their universal enveloping algebras.

Definition 2.2. *Lie algebra: $\mathfrak{g}$ vector space. A bilinear map $[\cdot,\cdot]_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a* Lie bracket *if*

1. $[a,b]_\mathfrak{g} = -[b,a]_\mathfrak{g}$ *(antisymmetry)*
2. $[a,[b,c]_\mathfrak{g}]_\mathfrak{g} + [c,[a,b]_\mathfrak{g}]_\mathfrak{g} + [b,[a,c]_\mathfrak{g}]_\mathfrak{g} = 0$ *(Jacobi identity)*

The pair $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ is called a Lie algebra*.*

Typical example:

Exercise 2.1. *Let $\mathfrak{g}$ be some associative algebra. Check: $[a,b]_\mathfrak{g} := [a,b] = ab - ba$ defines a Lie bracket.*

Heisenberg algebra: $\mathfrak{h}$ is spanned by the basis vectors $\alpha_n, n \in \mathbb{Z}$ and the central element $\mathbf{k}$. The Lie bracket is defined as

$$[\alpha_m, \alpha_n]_\mathfrak{h} = m\delta_{m+n,0}\mathbf{k} \ , \qquad [\alpha_m, \mathbf{k}]_\mathfrak{h} = [\mathbf{k}, \alpha_m]_\mathfrak{h} = [\mathbf{k},\mathbf{k}]_\mathfrak{h} = 0 \ . \tag{2.9}$$

Exercise 2.2. *Show this is a Lie algebra.*

We now want to turn a Lie algebra into an (associative) algebra.

Definition 2.3. $\mathfrak{g}$ *Lie algebra. Its* tensor algebra *is given by*

$$T(\mathfrak{g}) := \bigoplus_{i\geq 0} \mathfrak{g}^{\otimes i} = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g}\otimes\mathfrak{g}) \oplus \dots$$

Its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ *of $\mathfrak{g}$ is the associative algebra*

$$\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/I$$

where the ideal I is given by

$$I = \langle a\otimes b - b \otimes a - [a,b]_\mathfrak{g} : a, b \in \mathfrak{g}\rangle$$

This simply means that we are modding out by the relation $a\otimes b - b\otimes a - [a,b]_\mathfrak{g} \sim 0$. Note: In the future we will simply write ab for $a\otimes b$. $\mathcal{U}(\mathfrak{h})$ will be the associative algebra for defining our fields.

We now want to turn $\mathcal{U}(\mathfrak{h})$ into some subalgebra of $\mathrm{End}(V)$ for some vector space V. To this end, we first construct the *Fock space* V. Decompose $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_-$ with

$$\mathfrak{h}_+ = \bigoplus_{n>0} \mathbb{C}\alpha_n \qquad \mathfrak{h}_- = \bigoplus_{n<0} \mathbb{C}\alpha_n \qquad \mathfrak{h}_0 = \mathbb{C}\alpha_0 \oplus \mathbb{C}\mathbf{k} \ . \tag{2.10}$$

Note that the Lie bracket restricted to the individual subspaces vanishes. In particular we have $\mathcal{U}(\mathfrak{h}_-) \simeq \mathcal{S}(\mathfrak{h}_-)$, *i.e.* the (commutative!) algebra of polynomials in the variables $\{\alpha_n : n < 0\}$. Define the Fock space V in the following way. Take the 'vacuum vector' $|0\rangle$ and define the action of $\mathfrak{h}_{\geq 0} = \mathfrak{h}_+ \oplus \mathfrak{h}_0$ as

$$\mathbf{k}|0\rangle = |0\rangle, \qquad \alpha_{n\geq 0}|0\rangle = 0 \ . \tag{2.11}$$

Then define

$$V := \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{h}_{\geq 0})} (\mathbb{C}|0\rangle) \simeq \mathcal{S}(\mathfrak{h}_-) \ . \tag{2.12}$$

(Here $\otimes_{\mathcal{U}}$ means that $au \otimes_{\mathcal{U}} b = a \otimes_{\mathcal{U}} ub$ for $u \in \mathcal{U}$.) $\mathcal{U}(\mathfrak{h})$ then acts on V by multiplication from the left, *i.e.* $\mathcal{U}(\mathfrak{h}) \subset \mathrm{End}(V)$.

Operational Summary: We have

- an associative algebra generated by α_m satisfying (2.9)
- the Fock space V spanned by vectors

  $$\alpha_{-n_1} \cdots \alpha_{-n_k}|0\rangle \ , \qquad n_i > 0$$

  generated from the vacuum vector $|0\rangle$ by acting with operators α_{-n}
- the α_n act on V by multiplying from the left, with $\mathbf{k} = 1$ and $\alpha_{n\geq 0}|0\rangle = 0$.

Proposition 2.7. *The Heisenberg field (or free boson)* $\alpha(z) := \sum_n \alpha_n z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]]$ *acting on the Fock space V is a (self-)local field.*

Proof.
- Field: note that V has grading ('weights') given by wt $(\alpha_{-n_1} \cdots \alpha_{-n_k}|0\rangle) =: \sum_i n_i$, and α_n changes the weight by $-n$. WLOG v homogeneous. Since there are no states of negative weight, $\alpha_n u = 0$ for n large enough.
- Local:

  $$[\alpha(z), \alpha(w)] = \sum_m m z^{m-1} w^{-m-1} = w^{-1}\partial_z \delta(z/w)$$

  From prop 1.3: $(z-w)^2 \partial_z \delta(z/w) = 0$.

□

3 Vertex Algebras

3.1 Definition of a vertex algebra

Definition 3.1. *A vertex algebra is a vector space V with a distinguished vector $|0\rangle \neq 0$ (vacuum vector), an endomorphism $T \in \mathrm{End}(V)$ (translation operator) and a linear map Y from V into the space of fields $\mathcal{E}(V)$ (state-field map)*

$$a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

such that the following axioms hold:

VA1 $Y(a,z)|0\rangle = a + O(z)$ (creativity)

VA2 $[T, Y(a,z)] = \partial Y(a,z)$ *and* $T|0\rangle = 0$ (translation covariance)

VA3 *For all* a *and* b, $Y(a,z)$ *and* $Y(b,z)$ *are mutually local* (locality)

Remark: From **VA1** and **VA2** it follows that the translation operator is given by $Ta = a_{-2}|0\rangle$. We will also see below that $Y(|0\rangle, z) = I_V$, the identity on V.

We want to turn our Heisenberg algebra into a vertex algebra. Clearly we want to define $Y(\alpha, z) = Y(\alpha_{-1}|0\rangle) := \alpha(z)$. What about more general states in the Fock space V such as $Y(\alpha_{-1}\alpha, z)$? It turns out that there is no choice: their fields are uniquely determined. To show this and find the correct expression, we need to show a few propositions first.

Proposition 3.1. *1.* $Y(a,z)|0\rangle = e^{zT}a$

2. $e^{zT}Y(a,w)e^{-zT} = Y(a, w+z)$

3. $e^{zT}Y(a,w)_{\pm}e^{-zT} = Y(a, w+z)_{\pm}$

Proof. Show that both sides satisfy the same differential equation, then apply lemma 1.2 □

Proposition 3.2 (Skewsymmetry)**.** $Y(a,z)b = e^{zT}Y(b,-z)a$

Proof. Using locality with proposition 3.1 gives

$$(z-w)^K Y(a,z)e^{wT}b = (z-w)^K Y(b,w)e^{zT}a \ .$$

Now apply proposition 3.1 to get

$$(z-w)^K Y(a,z)e^{wT}b = (z-w)^K e^{zT}Y(b,w-z)a$$

We want to obtain the result by 'setting $w = 0$' and then dividing by z^K. This we can do if we choose K large enough, since then everything is in $(\mathrm{End}V)[[z, z-w]][z^{-1}]$. □

Corollary 3.3.

$$Y(|0\rangle, z) = I_V$$

Proof. For all b we have $Y(|0\rangle, z)b = e^{zT}Y(b,-z)|0\rangle = e^{zT}e^{-zT}b = b$. □

Theorem 3.4 (Uniqueness)**.** *Let* $B(z)$ *be field that is local with respect to all fields in* $\{Y(a,z) : a \in V\}$*. Suppose for some* $b \in V$

$$B(z)|0\rangle = e^{zT}b \ .$$

Then $B(z) = Y(b,z)$.

Proof. Locality gives

$$(z-w)^K B(z)Y(a,w)|0\rangle = (z-w)^K Y(a,w)B(z)|0\rangle$$

which we can write as

$$(z-w)^K B(z)e^{wT}a = (z-w)^K Y(a,w)e^{zT}b = (z-w)^K Y(a,w)Y(b,z)|0\rangle$$

Choosing K large enough, we can apply locality to the RHS to get

$$(z-w)^K B(z)e^{wT}a = (z-w)^K Y(b,z)e^{wT}a$$

Since both sides only contain positive powers of w, we can set $w=0$ and divide by z^K to get $B(z)a = Y(b,z)a$ for all a, establishing the claim. □

Using all this, we are now in a position to prove a central result, which allows to evaluate fields recursively:

Proposition 3.5.

$$Y(a_n b, z) = (Y(a,z)_n Y(b,z)) \tag{3.1}$$

Proof. Define $B(z) := Y(a,z)_n Y(b,z)$. The idea is of course to use theorem 3.4 to show that $B(z) = Y(a_n b, z)$. To do this, first note that by Dong's Lemma $B(z)$ is indeed local. Next we want to show $B(z)|0\rangle = e^{zT}a_n b$ by using differential equations. First note that from (2.6)

$$B(z)|0\rangle = a_n b_{-1}|0\rangle + O(z)\ . \tag{3.2}$$

Both sides of (3.1) thus satisfy the same initial condition. They also satisfy the same differential equation

$$\partial_z f(z) = Tf(z)\ .$$

For $Y(a_n b, w)$ this is immediate. For $Y(a,z)_n Y(b,z)$ this follows that from Lemma 2.4 and the fact that commutators satisfy the Leibniz rule. By Lemma 1.2 they thus agree. It then follows by theorem 3.4 that indeed $Y(a_n b, z) = B(z)$. □

Remembering definition (2.6) gives a very useful and explicit formula to evaluate fields coming from general states:

$$Y(a_n b, w) = \mathrm{Res}_z(Y(a,z)Y(b,w)(z-w)^n - Y(b,w)Y(a,z)(-w+z)^n) \tag{3.3}$$

As immediate corollaries we can write:

Corollary 3.6.

$$[Y(a,z), Y(b,w)] = \sum_{j=0}^{K-1} Y(a_j b, w)\frac{1}{j!}z^{-1}\partial_w^j \delta(z/w) \tag{3.4}$$

Corollary 3.7.

$$Y(a^1_{-j_1}\cdots a^n_{-j_n}|0\rangle, z) = :\frac{1}{(j_1-1)!}\partial^{j_1-1}Y(a^1,z)\cdots\frac{1}{(j_n-1)!}\partial^{j_n-1}Y(a^n,z): \tag{3.5}$$

$$Y(Ta,z) = \partial Y(a,z) \tag{3.6}$$

Corollary 3.8 (Locality-Truncation Relation)**.** *Let $a, b \in V$, $K > 0$. The two following statements are equivalent:*

1. $(z-w)^K[a(z), b(w)] = 0$
2. $a_n b = 0\ \forall n \geq K$.

Proof. $\Rightarrow$: Use (3.3) and injectivity of Y. $\Leftarrow$: Use (2.8) and prop 3.5. □

3.2 An Example: Heisenberg VA

We are now ready to show that the Heisenberg fields form a vertex algebra.

To construct the Heisenberg VA, we take V to be the Fock space (2.12), with $|0\rangle$ the vacuum. We define the translation operator as

$$T := \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_n \alpha_{-1-n}\ . \tag{3.7}$$

Note that even though this is formally an infinite sum, T is well-defined on V, since for any $a \in V$ only finitely many terms are non-vanishing. Finally we define the state-field map Y as (the linear extension of)

$$Y\left(\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_n}|0\rangle, z\right) := \alpha(z)_{j_1}(\alpha(z)_{j_2}(\cdots(\alpha(z)_{j_n}I_V)) \tag{3.8}$$

From Lemma 2.3 these are indeed fields.

Let us check that this indeed satisfies the three axioms of a vertex algebra:

VA1: Creativity is ensured by applying $a(z)_n b(z)|0\rangle = a_n b + O(z)$ (see (3.2)) recursively.

VA2: To check translation covariance, note first that with our definition of T, $T|0\rangle = 0$. Next, a straightforward computation shows that

$$\begin{aligned}[T,\alpha(z)] = \frac{1}{2}\sum_{n,m\in\mathbb{Z}}[\alpha_n\alpha_{-1-n}, \alpha_m z^{-m-1}] &= \frac{1}{2}\sum_{n,m\in\mathbb{Z}}\left(\alpha_{-1-n}n\delta_{n+m,0} + (-1-n)\alpha_n\delta_{-1-n+m,0}\right)z^{-m-1} \\ &= \frac{1}{2}\sum_{m\in\mathbb{Z}} -2m\alpha_{m-1}z^{-m-1} = \partial\alpha(z) \end{aligned} \tag{3.9}$$

To show that this also holds for general fields Y, we apply Lemma 2.4 and the fact that commutators satisfy the Leibniz rule.

VA3: Locality, is actually the easiest to check: Dong's Lemma ensures that all $Y(a,z)$ are mutually local, since $\alpha(z)$ is already local.

This establishes that V is indeed a vertex algebra. It is clear that we can generalize this construction:

Theorem 3.9 (Existence)**.** *Let V be a vector space, $|0\rangle$ a vector, T an endomorphism. Let $\{a^\alpha(z)\}_{\alpha\in A}$ (A some index set) a local set of fields satisfying*

1. $[T, a^\alpha(z)] = \partial a^\alpha(z)$
2. $T|0\rangle = 0$
3. $a^\alpha(z)|0\rangle \in V[[z]]$ *and with* $a^\alpha := a^\alpha(z)|0\rangle_{z=0}$ *the linear map* $\sum_\alpha \mathbb{C}a^\alpha(z) \to \sum_\alpha \mathbb{C}a^\alpha$ *defined by* $a^\alpha(z) \mapsto a^\alpha$ *is injective*
4. *the vectors* $a^{\alpha_1}_{j_1} a^{\alpha_2}_{j_2} \cdots a^{\alpha_n}_{j_n}|0\rangle$ *with* $j_s \in \mathbb{Z}_-, \alpha_s \in A$ *span* V.

Then the map

$$Y\left(a^{\alpha_1}_{j_1} a^{\alpha_2}_{j_2} \cdots a^{\alpha_n}_{j_n}|0\rangle, z\right) := a^{\alpha_1}(z)_{j_1}(a^{\alpha_2}(z)_{j_2}(\cdots(a^{\alpha_n}(z)_{j_n} I_V)) \tag{3.10}$$

defines a vertex algebra $(V, T, |0\rangle, Y)$.

Proof. See Theorem 4.5 in [1]. Idea: Choose basis of a^α. 3 makes (3.10) well defined. Use Dong's lemma for locality. Both ∂ and $[T,\cdot]$ are derivatives wrt normal ordered product. □

3.3 Jacobi identity

Let us discuss some more structural aspects of vertex algebras. In particular, there are other equivalent definitions of VAs.

First let us address the question: In what sense is a vertex algebra actually an algebra?

Note that we can consider the formal power series $Y(a,z)b$ as a generating function of an infinite list of product operations $*_n : V \times V \to V$, $a *_n b := a_n b$. Are these products commutative, *i.e.*

$$a_n b = b_n a \ ? \tag{3.11}$$

A quick look at *e.g.* Proposition 3.2 implies that they are not. Are they associative, *i.e.* do we have something like

$$a_n(b_m c) = (a_n b)_m c \ ? \tag{3.12}$$

Again, they are not, as *e.g.* (3.3) shows.

The situation is however not quite as bad as it seems. Locality implies that two fields *almost* commute, *i.e.* they commute once we multiply with a factor. Locality is therefore sometimes also called *weak commutativity*, Similarly one can show that fields of a VA satisfy *weak associativity*:

Proposition 3.10 (Weak associativity). *For all $a, c \in V$ $\exists k$ (depending only on a and c, not on b!) such that for any b*

$$(z+w)^k Y(a, z+w) Y(b, w) c = (z+w)^k Y(Y(a, z)b, w) c$$

A slightly different point of view is given by identities of the form (3.3): They imply that non-commutativity and non-associativity are related somehow. That is, there is an infinite number of identities that the products $*_n$ have to satisfy. In fact, using the language of formal power series as generating functions, we can write the totality of these identities in the form of the so-called *Jacobi identity*:

Proposition 3.11 (Jacobi Identity, **VA4**). *For any three a, b, c in a vertex algebra V we have*

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1) Y(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(b, z_2) Y(a, z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2) \quad (3.13)$$

All terms of this expression are well-defined. One way to see this is to explicitly expand out (3.13) and read off the coefficient of say $z_0^{-l-1} z_1^{-m-1} z_2^{-n-1}$, giving

$$\sum_{i \geq 0} (-1)^i \binom{l}{i} (a_{l+m-i} b_{n+i} - (-1)^l b_{l+n-i} a_{m+i}) = \sum_{i \geq 0} \binom{m}{i} (a_{l+i} b)_{m+n-i} \quad (3.14)$$

Since a, b, c are fields, the sums over i are actually finite. This is how we obatin the promised infinite list of identities. (3.14) is called the *Borcherds identity*, and from what we have said it is clearly equivalent to the Jacobi identity (3.13).

Why do we call it the Jacobi identity? It is indeed a generalization of the Jacobi identity of Lie algebras. To see this, write the adjoint action of u on v as $(\text{ad}u)v := [u, v]_{\mathfrak{g}}$. The Jacobi identity is then

$$(\text{ad}u)(\text{ad}v) - (\text{ad}v)(\text{ad}u) = \text{ad}((\text{ad}u)v) , \quad (3.15)$$

which is exactly of the form (3.13). (In fact, (3.15) is a special case of (3.13).)

Proof. (Sketch) Indeed one can show (see theorem 4.8 of [1]) that translation and locality imply (3.14). The components of the proof are (3.4) written as

$$[a_m, Y(b, w)] = \sum_{j \geq 0} \binom{m}{j} Y(a_j b, w) w^{m-j} , \quad (3.16)$$

and corollary 3.7. These are two special cases of (3.14), and it turns out that combining them is enough to establish the general case of (3.14), which then in turn is equivalent to (3.13). □

In the context of a vertex algebra, we have thus shown that **VA1**, **VA2**, **VA3** $\Rightarrow$ **VA4**. In fact, one can show that the converse holds: **VA1**, **VA4** $\Rightarrow$ **VA2**, **VA3**: Starting from (3.13), locality holds: Multiply (3.13) by $z_0^k, k > 0$ and take Res_{z_0} to get

$$(z_1 - z_2)^k Y(a, z_1)Y(b, z_2) - (z_1 - z_2)^k Y(b, z_2)Y(a, z_1) = \mathrm{Res}_{z_0} z_0^k z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2) \quad (3.17)$$

Note that on the right hand side, δ only produces positive powers of z_0, so that the only negative powers of z_0 come from $Y(a, z_0)b \in V[[z]][z^{-1}]$. This means that if we choose k large enough, the residue vanishes, establishing locality **VA3**.

Moreover one then defines $Ta := a_{-2}|0\rangle$, and one can show that this then satisfies the commutator equation of the translation axiom **VA2**.

In conclusion, in the definition of a vertex algebra we can replace axioms **VA2** and **VA3** by **VA4**. In practice we will use locality, since that is usually easier to show using constructions theorems like theorem 3.9. In some cases (such as the definition of modules) using the Jacobi axiom (3.13) is more natural though.

4 Conformal invariance

5 Vertex Operator Algebras

Definition 5.1 (Vertex Operator Algebra ("Conformal Vertex Algebra"))**.** *Let V be $\mathbb{Z}$ graded vector space*

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)} \qquad \text{with} \quad \dim V_{(n)} < \infty \ , \qquad V_{(n)} = 0 \text{ for } n \text{ small enough} \ ,$$

where for $v \in V_{(n)}$, $n = \mathrm{wt}\ v$ (the weight *of v). A* Vertex Operator Algebra *is V equipped with a vertex algebra structure $(V, Y, |0\rangle, T)$ together with a distinguished vector ω (the* conformal vector*)*

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \left(= \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1}\right) \ , \quad (5.1)$$

satisfying the following axioms:

VOA1 *The* Virasoro modes *L_n satisfy the Virasoro algebra*

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12} m(m^2 - 1)\delta_{m+n,0} c_V \ . \quad (5.2)$$

Here $c_V \in \mathbb{C}$ is the central charge. (conformal symmetry)

VOA2 $L_0 v = \mathrm{wt}\ (v) v$ (conformal weight)

VOA3 $T = L_{-1}$ (translation generator)

Let us list a few immediate consequences of these axioms:

- We can replace the requirement $T = L_{-1}$ by the weaker requirement $Y(L_{-1}a, w) = \partial Y(a, w)$. To see this, use (3.16) to get
$$Y(L_{-1}a, w) = Y(\omega_0 a, w) = [L_{-1}, a(w)] \ . \tag{5.3}$$
- From **VA1** it follows that
$$L_n|0\rangle = 0 \qquad \forall n \geq -1 \ . \tag{5.4}$$
- In particular from $L_0|0\rangle = 0$ it follows that $|0\rangle \in V_{(0)}$.
- We also have
$$L_0\omega = L_0 L_{-2}|0\rangle = [L_0, L_{-2}]|0\rangle = 2\omega \ , \tag{5.5}$$
so that $\omega \in V_{(2)}$.

Lemma 5.1. *Let $a \in V_{(\mathrm{wt}\ a)}$, i.e. a homogeneous of weight* wt a*. Then the mode a_n (defined from $Y(a, z) = \sum_n a_n z^{-n-1}$ maps*
$$a_n : V_m \to V_{m+\mathrm{wt}\ a-n-1} \ , \tag{5.6}$$
i.e. a_n is a homogeneous operator of weight $\mathrm{wt}\ a_n = \mathrm{wt}\ a - n - 1$.

Proof. Use (3.16) to obtain
$$[L_0, a(w)] = Y(L_0 a, w) + Y(L_{-1}a, w)w = (\mathrm{wt}\ a)a(w) + w\partial_w a(w) \ . \tag{5.7}$$
Multiplying by w^n and extracting the residue gives $[L_0, a_n] = (\mathrm{wt}\ a - n - 1)a_n$, which together with **VOA2** gives the result. □

Physicists prefer a different convention for the modes: For a homogeneous state a they write
$$Y(a, z) = \sum_n a_{(n)} z^{-n-\mathrm{wt}\ a} \ , \tag{5.8}$$
such that $a_{(n)} = a_{n+\mathrm{wt}\ a-1}$. The advantage of that convention is that $\mathrm{wt}\ a_{(n)} = -n$. The disadvantage is of course that it only works for VOAs, and only for homogeneous states. We will continue to use the mathematicians' convention.

Remark: Let a and b be homogeneous. Then almost all terms in $Y(a, z)b$ have positive weight. More precisely, if $k > 0$ is such that $a_n b = 0$ for all $n > k$, then
$$Y(a, z)b \in \left(\bigoplus_{n \geq \mathrm{wt}\ b+\mathrm{wt}\ a-k-1} V_{(n)} \right) [[z]][z^{-1}] \tag{5.9}$$

5.1 Heisenberg VOA

We have already established that the Heisenberg algebra is a vertex algebra. Let us now show that it is actually also a vertex operator algebra. The grading of V was introduced in the proof of proposition 2.7, *i.e.*

$$\text{wt }(\alpha_{-n_1}\cdots\alpha_{-n_k}|0\rangle) =: \sum_i n_i \ , \tag{5.10}$$

so that clearly $V_{(n)} = 0$ for $n < 0$. Next we define the conformal vector

$$\omega := \frac{1}{2}\alpha_{-1}\alpha \ . \tag{5.11}$$

From corollary 3.7, we immediately get

$$L_n = \text{Res}_z z^{n+1} Y(\omega, z) = \frac{1}{2}\text{Res}_z z^{n+1} :Y(\alpha, z)Y(\alpha, z):= \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_m a_{n-m} : \tag{5.12}$$

Note in particular that $L_{-1} = T$, which establishes **VOA3**. Next we want to show **VOA2**. For this we use the following lemma:

Lemma 5.2.

$$[L_m, \alpha_n] = -n\alpha_{m+n} \tag{5.13}$$

Proof. Start with the commutator formula in the form (3.16) and read off the coefficient of w^{-m-2}, giving

$$[\alpha_n, L_m] = \sum_{j\geq 0}\binom{n}{j}(\alpha_j\omega)_{m+n-j+1} = n\alpha_{m+n} \ , \tag{5.14}$$

where we have used the fact that the terms $\alpha_j\omega$ in the sum vanish unless $j = 1$. □

To check **VOA2**, first note that we have $L_0|0\rangle = 0$. Next specializing (5.13) to $m = 0$ shows that each α_{-n} increases the eigenvalue by n, since $L_0\alpha_{-n}b = [L_0, \alpha_{-n}]b + \alpha_{-n}L_0 b = \alpha_{-n}(L_0 + n)b$. By induction it follows that indeed $L_0 a = \text{wt }(a)a$.

Finally we need to show **VOA1**, and establish the value of the central charge c_V:

Lemma 5.3.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m+n,0} \tag{5.15}$$

Proof. Again read off from (3.16) (beware of the shift in moding!)

$$[L_m, L_n] = [\omega_{m+1}, L_n] = \sum_{j\geq 0}\binom{m+1}{j}(\omega_j\omega)_{m+n-j+2} = \sum_{j\geq 0}\binom{m+1}{j}(L_{j-1}\omega)_{m+n-j+2} \tag{5.16}$$

First note that for $n \geq -1$ we can write

$$L_n\omega = \frac{1}{2}([L_n, \alpha_{-1}]\alpha_{-1} + \alpha_{-1}[L_n, \alpha_{-1}])|0\rangle = \frac{1}{2}(\alpha_{n-1}\alpha_{-1} + \alpha_{-1}\alpha_{n-1})|0\rangle \qquad (5.17)$$

Evaluating the different terms we get:

- $(L_{-1}\omega)_{m+n+2} = \mathrm{Res}_z z^{m+n+2}\partial Y(\omega, z) = \mathrm{Res}_z z^{m+n+2}\sum(-k-2)L_k z^{-k-3} = (-m-n-2)L_{m+n}$
- $(L_0\omega)_{m+n+1} = 2\omega_{m+n+1} = 2L_{m+n}$
- $L_1\omega = \frac{1}{2}(\alpha_0\alpha_{-1} + \alpha_{-1}\alpha_0)|0\rangle = 0$
- $L_2\omega = \frac{1}{2}(\alpha_1\alpha_{-1} + \alpha_{-1}\alpha_1)|0\rangle = \frac{1}{2}|0\rangle$
- $L_n\omega = 0$ for $n > 2$

Combining all the pieces we get

$$[L_m, L_n] = (-m-n-2)L_{m+n} + (2m+2)L_{m+n} + \frac{1}{2}\frac{1}{6}m(m^2-1)\delta_{m+n,0} \ , \qquad (5.18)$$

which establishes (5.15). □

5.2 Virasoro VOA

The Virasoro algebra $\mathcal{L}$ is the Lie algebra with basis $\{L_m : m \in \mathbb{Z}\} \cup \{\mathbf{c}\}$ with Lie bracket

$$[L_m, L_n]_{\mathcal{L}} := (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m+n,0}\mathbf{c} \qquad (5.19)$$

and $\mathbf{c}$ being a central element. It has a grading

$$\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_{(n)} \qquad (5.20)$$

where

$$\mathcal{L}_{(0)} := \mathbb{C}L_0 \oplus \mathbb{C}\mathbf{c} \ , \qquad \mathcal{L}_{(n)} := \mathbb{C}L_{-n} \text{ for } \ n \neq 0 \ . \qquad (5.21)$$

We also define

$$\mathcal{L}_{(\leq 1)} := \bigoplus_{n\leq 1} \mathcal{L}_{(n)} \ , \qquad \mathcal{L}_{(\geq 2)} := \bigoplus_{n\geq 2} \mathcal{L}_{(n)} \ . \qquad (5.22)$$

As in the Heisenberg case, we want U to be given by the universal enveloping algebra $\mathcal{U}(\mathcal{L})$. To construct a corresponding module, let $\ell \in \mathbb{C}$ be a complex number, and define the $\mathcal{L}_{(\leq 1)}$ module $\mathbb{C}|0\rangle_\ell$ by

$$\mathbf{c}|0\rangle_\ell = \ell|0\rangle_\ell \ , \qquad L_n|0\rangle_\ell = 0 \quad \forall n \geq -1 \ . \qquad (5.23)$$

We then define

$$V_{Vir}(\ell, 0) := \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{U}(\mathcal{L}_{(\leq 1)})} \mathbb{C}|0\rangle_\ell \ . \qquad (5.24)$$

By the Poincaré-Birkhoff-Witt theorem, as a vector space we have

$$V_{Vir}(\ell, 0) = \mathcal{U}(\mathcal{L}_{(\geq 2)}) \simeq \mathcal{S}(\mathcal{L}_{(\geq 2)}) \ . \tag{5.25}$$

That is, $V_{Vir}(\ell, 0)$ is spanned by vectors of the form

$$L_{-n_1} \cdots L_{-n_k} |0\rangle_\ell \qquad n_1 \geq \cdots \geq n_k \geq 2 \ . \tag{5.26}$$

The weight of this vector is given by $\sum_{i=1}^k n_i$. Note that from the commutation relations (5.19) the grading is exactly given by the eigenvalues of L_0.

We now want to construct a VA and VOA structure on $V := V_{Vir}(\ell, 0)$ by applying theorem 3.9. We define $T := L_{-1}$ and $|0\rangle := |0\rangle_\ell$. We define

$$\omega(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \ . \tag{5.27}$$

This is clearly a field, as any vector in V has finite weight, and there are no vectors of negative weight. Next we want to show that it is self-local:

Exercise 5.1. *Show:* $(z-w)^4[\omega(z), \omega(w)] = 0$.

Because of

$$[L_{-1}, \omega(z)] = \sum_{n \in \mathbb{Z}} (-n-1) L_{n-1} z^{-n-2} = \partial\omega(z) \tag{5.28}$$

1. in theorem 3.9 is satisfied. 2.–4. are satisfied by construction, so that $V_{Vir}(\ell, 0)$ is indeed a vertex algebra.

To see that it is also a VOA, note that the grading conditions are automatically satisfied. We define $\omega = L_{-2}|0\rangle_\ell$, which is automatically $\omega \in V_{(2)}$, since

$$L_0 \omega = L_0 L_{-2} |0\rangle = [L_0, L_{-2}] |0\rangle = 2\omega \ . \tag{5.29}$$

By construction the modes of $\omega(z)$ satisfy the Virasoro algebra with central charge $c_V = \ell$. The translation operator and the grading are also satisfied. In conclusion we h ave that $V_{Vir}(\ell, 0)$ is a VOA with central charge ℓ.

5.3 Tensor Products

Let $V^1, \ldots, V^r$ be VAs. The tensor product VA is constructed from the tensor product

$$V = V^1 \otimes \cdots \otimes V^r \ . \tag{5.30}$$

Its state field map is given by

$$Y(a^{(1)} \otimes \cdots \otimes a^{(r)}, z) = Y(a^{(1)}, z) \otimes \cdots \otimes Y(a^{(r)}, z) \tag{5.31}$$

and the vacuum state is

$$|0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle \tag{5.32}$$

and the translation operator is

$$T = \sum_{i=1}^{r} I_{V^1} \otimes \cdots \underbrace{\otimes T \otimes}_{i} \cdots \otimes I_{V^r} \ . \tag{5.33}$$

VA1 follows immediately. **VA2** can be shown using the Leibniz rule for the derivative. To show **VA3**, we take the order of locality $K := \max(K_1, \ldots, K_r)$, and then use

$$(z-w)^K Y(a^{(i)}, z) Y(b^{(i)}, w) = (z-w)^K Y(b^{(i)}, w) Y(a^{(i)}, z) \tag{5.34}$$

repeatedly to establish locality. We therefore have

Proposition 5.4. *Let $V^1, \ldots, V^r$ be vertex algebras. Then the tensor product vertex algebra $(V, Y, |0\rangle, T)$ as defined above is indeed a vertex algebra.*

Now suppose that V^i is also a VOA with central charge c_i. Then the tensor product is also a VOA. Its grading is given by

$$V_{(n)} = \bigoplus_{n_1+\ldots+n_r=n} V^1_{(n_1)} \otimes \ldots \otimes V^r_{(n_r)} \tag{5.35}$$

and the conformal vector is

$$\omega = \sum_{i=1}^{r} |0\rangle \otimes \cdots \underbrace{\otimes \omega \otimes}_{i} \cdots |0\rangle \ , \tag{5.36}$$

so that the Virasoro modes are given by

$$L_n = \sum_{i=1}^{r} I_{V^1} \otimes \cdots \underbrace{\otimes L_n \otimes}_{i} \cdots \otimes I_{V^r} \ . \tag{5.37}$$

A straightforward computation shows that the L_n satisfy the Virasoro algebra with central charge $c = \sum_i c_i$. It is also straightforward to check **VOA3** and **VOA2**. We therefore have:

Proposition 5.5. *The tensor product of finitely many VOA is a VOA whose central charge is the sum of the central charges.*

6 Modules

Operational definition: A *module* of a VA V is a space W with a map $V \to \mathcal{E}(W)$ 'such that all VA axioms that make sense hold'. To make this more precise, it is better to use the Jacobi axiom for a VA — that is, we will use **VA1** and **VA4**. What structure do we want to maintain? We do not want to require the existence of a vacuum in the module W, and it therefore makes no sense to require creativity. We still want to maintain that the field corresponding to the vacuum be the identity though. We therefore define:

Definition 6.1. *Let V be a vertex algebra. A V-module is a vector space W with linear map Y_W from $V \to \mathcal{E}(W)$*

$$Y_W(\cdot, z) : V \to \mathcal{E}(W) \subset (\mathrm{End}(W))[[z, z^{-1}]] \qquad a \mapsto Y_W(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

such that

1. $Y_W(|0\rangle, z) = I_W$
2. $$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(a, z_1)Y_W(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_W(b, z_2)Y_W(a, z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W(Y(a, z_0)b, z_2) \quad (6.1)$$

Note that this is essentially the Jacobi identity. We did have to modify the right hand side to make sense however, note the roles of the Y and Y_W.

One could hope to replace the Jacobi identity by weak commutativity and the existence of a translation operator. Unfortunately that does not work for modules. The reason is that we need some notion of associativity. It is however possible to replace the Jacobi axiom by weak commutativity and associativity, *i.e.* by locality and the requirement that for large k

$$(z + w)^k Y_W(a, z + w)Y_W(b, w)v = (z + w)^k Y_W(Y(a, z)b, w)v \; . \qquad (6.2)$$

Definition 6.2. *Let V be a vertex operator algebra. A V-module is a module W which is a V-module for V viewed as a vertex algebra such that*

$$W = \bigoplus_{h\in\mathbb{C}} W_h$$

where

$$W_h = \{w \in W : L_0 w = hw\}$$

and the following grading restrictions hold: $\dim W_h < \infty$ *for all* $h \in \mathbb{C}$ *and* $W_h = 0$ *for all* h *whose real part is sufficiently negative.*

We will not work with modules very much. Nonetheless, here are a few useful definitions and remarks

- The Virasoro modes L_n^W of $Y_W(\omega, z)$ satisfy the Virasoro algebra with the same central charge c. To see this, write (6.1) as

$$[L_m, L_n] = \sum_{j\geq 0} \binom{m+1}{j} (L_{j-1}\omega)_{m+n-j+2} \tag{6.3}$$

and then evaluate the right-hand side in a similar way as in the proof of Lemma 5.3, but this time using the known commutation properties of L_n rather than α.

- A *submodule* of a module is a subspace U such that (U, Y_W) is itself a V-module $\Leftrightarrow Y_W(a,z)b \in U[[z]][z^{-1}]\ \forall a \in V, b \in U \Leftrightarrow a_n b \in U \forall a \in V, b \in U, n \in \mathbb{Z}$.
- A V-module W is called *irreducible* if it contains no proper submodules.
- An example of an irreducible module for the Heisenberg VOA V is $M(1,\alpha)$, $\alpha \in \mathbb{C}$, which can be constructed very similarly to V itself: Define

$$M(1,\alpha) := \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{h}_{\geq 0})} (\mathbb{C}|\alpha\rangle)\ , \tag{6.4}$$

where the difference to the original construction of the Heisenberg VOA is in the action of α_0 on $|\alpha\rangle$, namely

$$\alpha_0|\alpha\rangle = \alpha|\alpha\rangle\ . \tag{6.5}$$

The grading is then given by

$$M(1,\alpha) = \bigoplus_{n\in\mathbb{N}} M(1,\alpha)_{(n+\frac{1}{2}\alpha^2)}\ , \tag{6.6}$$

as can be seen by the action of L_0 on $|\alpha\rangle$.

- In physics modules of V are essentially what is called *primary fields*.

7 Correlation functions

7.1 Matrix Elements

Let us define the *restricted dual* V' of a graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$ as

$$V' := \bigoplus_{n\in\mathbb{Z}} (V_{(n)})^*\ , \tag{7.1}$$

i.e. the finite linear combinations of homogeneous dual elements. Denote by $\langle\cdot,\cdot\rangle$ the usual pairing between elements of V' and V. Since every homogeneous component of V is finite dimensional, there are no subtleties in defining dual elements. Using such pairings, we can define *matrix elements*. A matrix element is the formal power series in $\mathbb{C}[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$ given by

$$\langle v', Y(u^1, z_1)\dots Y(u^n, z_n)w\rangle\ . \tag{7.2}$$

It turns that we can interpret this formal power series as a meromorphic function in $z_1, \ldots, z_n$! To see this, let us first restrict to the case $n = 2$.

Let $\mathbb{C}(z_1, z_2)$ be the field of all rational functions in z_1, z_2. For our purposes it is enough to consider a subalgebra of this. Set $S := \{z_1, z_2, z_1 \pm z_2\}$. We then define $\mathbb{C}[z_1, z_2]_S$ as the subalgebra of $\mathbb{C}(z_1, z_2)$ generated by $z_1^{\pm}, z_2^{\pm}$, and $(z_1 \pm z_2)^{-1}$. To connect functions in $\mathbb{C}[z_1, z_2]_S$ to formal power series, define the linear map ι_{12},

$$\iota_{12} : \mathbb{C}[z_1, z_2]_S \to \mathbb{C}[[z_1, z_1^{-1}, z_2, z_2^{-1}]] \tag{7.3}$$

such that $\iota_{12}(f(z_1, z_2))$ is the formal Laurent series expansion of $f(z_1, z_2)$ involving only finitely many negative powers in z_2. Analogously, we define ι_{21}.

Example: Binomial expansion convention: $\iota_{12}((z_1 - z_2)^n) = (z_1 - z_2)^n$, $\iota_{21}((z_1 - z_2)^n) = (-z_2 + z_1)^n$.

Theorem 7.1. *The formal series*

$$\langle v', Y(u, z_1)Y(v, z_2)w\rangle$$

lies in the image of the map ι_{12}:

$$\langle v', Y(u, z_1)Y(v, z_2)w\rangle = \iota_{12} f(z_1, z_2)$$

where f is a uniquely determined rational function in z_1, z_2 of the form

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{(z_1 - z_2)^k z_1^l z_2^m}$$

for some $g \in \mathbb{C}[z_1, z_2]$. Moreover

$$\langle v', Y(v, z_2)Y(u, z_1)w\rangle = \iota_{21} f(z_1, z_2)\ .$$

Proof. Take k such that

$$(z_1 - z_2)^k \langle v', Y(u, z_1)Y(v, z_2)w\rangle = (z_2 - z_1)^k \langle v', Y(v, z_2)Y(u, z_1)w\rangle$$

LHS as only finitely many negative powers of z_2. It also only has finitely many positive powers of z_1. To see this, note that from (5.9) only finitely many terms of $Y(v, z_2)w$ will have weight smaller than wt v' − wt $u + 1$. Since the only non-vanishing terms must have weight wt v', this means there are only a finitely many terms u_n with n negative.

For the RHS vice versa. Therefore this is equal to some Laurent polynomial $h \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. Then $f(z_1, z_2) := h(z_1, z_2)/(z_1 - z_2)^k$ satisfies the required properties. To see this, note that

$$\langle v', Y(u, z_1)Y(v, z_2)w\rangle = (z_1 - z_2)^{-k}(z_1 - z_2)^k \langle v', Y(u, z_1)Y(v, z_2)w\rangle = (z_1 - z_2)^{-k} \iota_{12} h(z_1, z_2)$$

Here we used the fact that because of $Y(v, z_2)w \in V[[z_2]][z_2^{-1}]$ there are only finitely many negative powers of z_2 in the series. The triple product in the middle thus exists and we can use associativity to obtain the first equality. Similarly we obtain the last statement in the theorem by multiplying with $(-z_2+z_1)^{-k}$. To see uniqueness, note that the matrix element fixes the Laurent expansion of f in the annulus $0<|z_2|<|z_1|$. Uniqueness then follows from uniqueness of the analytic continuation. □

Exercise 7.1. *Compute the free boson 2-pt matrix element* $\langle|0\rangle', a(z)a(w)|0\rangle\rangle$.

Note that the matrix element (and therefore also $f(z_1,z_2)$) is homogeneous. To see this, note that for the term $\langle v', u_m v_n w\rangle z_1^{-m-1} z_2^{-n-1}$ not to vanish we need to have $\text{wt } v' = \text{wt } w + \text{wt } u - n - 1 + \text{wt } v - m - 1$ such that its degree is $\text{wt } v' - \text{wt } v - \text{wt } u - \text{wt } w$.

We can generalize this to matrix elements with more than two fields inserted. In that case we multiply by $\prod_{1\leq i<j\leq n}(z_i-z_j)^{k_{ij}}$ and repeat the argument recursively, using the fact that for terms with given powers of z_i, $i>k$, there are only finitely negative powers of z_k. $f(z_1,\ldots,z_n)$ is then a rational functions with poles at $z_i=z_j,0,\infty$.

7.2 Cluster decomposition

Most VOAs of physical interest satisfy an additional property: They are of *CFT type*, which in physics is called *cluster decomposition.* A VOA is said to be of CFT type if $V_{(0)}$ is spanned by the vacuum $|0\rangle$ only, and there are no states of negative weight, that is

$$V = \mathbb{C}|0\rangle \oplus \bigoplus_{n>0} V_{(n)} \ . \tag{7.4}$$

Note that all examples of VOAs we have constructed so far satisfy this property. (7.4) has some immediate consequences. First of all, it immediately gives an upper bound on the truncation level. Using lemma 5.1, we have

$$a_n b = 0 \qquad \forall n \geq \text{wt } a + \text{wt } b \ . \tag{7.5}$$

From proposition 3.8 it then immediately follows that the order of locality is at most $\text{wt } a + \text{wt } b$. The matrix element f then has poles of at most order $\text{wt } u + \text{wt } v$ at $z_1 = z_2$.

Let us define *correlation functions* F through the matrix elements with $w=|0\rangle$ and $v'=\langle 0|$, where $\langle 0|$ is defined as the dual to $|0\rangle$. We define

$$\iota_{12\ldots n} F((u^1,z_1),\ldots,(u^n,z_n)) = \langle 0|Y(u^1,z_1)\ldots Y(u^n,z_n)|0\rangle \tag{7.6}$$

where we used the physics notation for the dual pairing. From the discussion above we know that F is a rational function with poles at $z_i=z_j,0,\infty$. It turns out to be useful to consider F as function on the Riemann sphere, or more precisely, on $\hat{\mathbb{C}}^n$. Since it is rational, it is a meromorphic function.

For VOA of CFT type we can say slightly more about the positions of the poles. Note that because $Y(u^n,z_n)|0\rangle = u^n + O(z_n)$, the matrix element and therefore also F only contains non-negative powers of z_n. That is, there is no pole at $z_n=0$. Similar arguments show that there are also no poles at $z_i = 0$. Next consider the behavior at $z_1 \to \infty$. Because $\rangle 0|Y(u^1, z_1)b\rangle = z_1^{-\text{wt } u^1 - \text{wt } b}\langle 0|u^1_{\text{wt } b + \text{wt } u^1 - 1}b\rangle$ we see that F falls off at least as fast of $z_1^{-\text{wt } u^1}$. (This is where the name cluster decomposition comes from.) There is therefore no pole at $z_1 = \infty$, and because of similar arguments not at $z_i = \infty$ either. The only poles of F are therefore at $z_i = z_j$. In total we can write the n-point correlation function as

$$F((u^1, z_1), \ldots, (u^n, z_n)) = \frac{g(z_1, \ldots, z_n)}{\prod_{i<j}(z_i - z_j)^{\text{wt } u^i + \text{wt } u^j}} \tag{7.7}$$

where $g(z_1, \ldots, z_n)$ is a homogeneous polynomial such that F has degree $-\sum_i \text{wt } u^i$.

7.3 Möbius transformations

In section 4 we motivated the appearance of the Virasoro algebra as coming from conformal transformations. Let us now work out what that means for fields and correlation functions. We want to see now how L_{-1}, L_0, L_1 produce the Möbius transformations.

From proposition 3.1 we already know that

$$e^{\lambda L_{-1}} Y(a, z) e^{-\lambda L_{-1}} = Y(a, z + \lambda) \ . \tag{7.8}$$

Next we can integrate (5.7) to

$$\lambda^{L_0} Y(a, z) \lambda^{-L_0} = Y(\lambda^{L_0} a, \lambda z) \ . \tag{7.9}$$

One way to see this is to integrate $[L_0, a_n] = -(\text{wt } a - n - 1)a_n$ to $\lambda^{L_0} a_n \lambda^{-L_0} = \lambda^{\text{wt } a - n - 1} a_n$. Finally from (3.16) we have the action of L_1,

$$[L_1, Y(a, z)] = z^2 \partial Y(a, z) + 2z Y(L_0 a, z) + Y(L_1 a, z) \ , \tag{7.10}$$

which integrates to

Proposition 7.2.

$$e^{\lambda L_1} Y(a, z) e^{-\lambda L_1} = Y\left(e^{\lambda(1-\lambda z)L_1}(1 - \lambda z)^{-2L_0} a, z(1 - \lambda z)^{-1}\right) \tag{7.11}$$

Proof. (Sketch.) Define the formal power series $A(\lambda)$ via

$$e^{\lambda L_1} Y(a, z) e^{-\lambda L_1} = Y\left(A(\lambda) a, z(1 - \lambda z)^{-1}\right) \ , \tag{7.12}$$

where $A(\lambda)$ has constant term $\mathbf{1}$. Obtain ODE

$$\frac{dA(\lambda)}{d\lambda} = z^2 \partial_z A(\lambda) + 2z A(\lambda) L_0 + A(\lambda) L_1$$

This has a unique solution with constant term $\mathbf{1}$. □

Let us now apply such operators $U \in \{e^{\lambda L_0}, e^{\lambda L_{-1}}\}$ in correlation functions. First note that

$$\langle 0|Y(u^1, z_1)\dots Y(u^n, z_n)|0\rangle = \langle 0|U^{-1}Y(u^1, z_1)\dots Y(u^n, z_n)U|0\rangle \tag{7.13}$$

This follows on the one hand from the fact that for such U, $U|0\rangle = |0\rangle$. On the other hand, all such U only raise the weight of states. Since $\langle 0|$ picks out the vacuum, this can only come from $U^{-1}|0\rangle = |0\rangle$, which establishes the claim.

We can use this to derive further properties of correlation functions. For instance we have

$$\begin{aligned}\langle 0|Y(u^1, z_1)\dots Y(u^n, z_n)|0\rangle \\ = \langle 0|e^{-\lambda L_{-1}}(e^{\lambda L_{-1}}Y(u^1, z_1)e^{-\lambda L_{-1}})\dots(e^{\lambda L_{-1}}Y(u^n, z_n)e^{-\lambda L_{-1}})e^{\lambda L_{-1}}|0\rangle \\ = \langle 0|Y(u^1, z_1+\lambda)\dots Y(u^n, z_n+\lambda)|0\rangle\ ,\end{aligned} \tag{7.14}$$

which translates to

$$F((u^1, z_1), \dots, (u^n, z_n)) = F((u^1, z_1+\lambda), \dots, (u^n, z_n+\lambda)) \tag{7.15}$$

That is, the correlation function is *translation invariant*. This is of course compatible with the position of the poles, and further restricts the form of the polynomial g. Similarly, using $U = e^{\lambda L_0}$, we recover our result from above,

$$F((u^1, \lambda z_1), \dots, (u^n, \lambda z_n)) = \lambda^{-\sum_i \mathrm{wt}\ u^i} F((u^1, z_1), \dots, (u^n, z_n))\ . \tag{7.16}$$

Exercise 7.2. *What is the most general form of a 2-pt function of two fields of weight* wt a *and* wt b*?*

8 Lattice VOAs

8.1 Lattices

Consider $\mathbb{R}^d$ with inner product $(\cdot,\cdot)$. Choose a basis $\{e_i\}$ of $\mathbb{R}^d$. The lattice L is given by

$$L := \left\{ \sum_i n_i e_i : n_i \in \mathbb{Z} \right\}\ . \tag{8.1}$$

Note that L is an abelian group under addition of vectors. We say L is *even* if $(\mu,\mu) \in 2\mathbb{Z}$ for all $\mu \in L$. In that case L is automatically *integral*, that is $(\mu,\nu) \in \mathbb{Z}$ for all $\mu,\nu \in L$.

8.2 Ingredients

8.2.1 Fock space

Let us now take the Heisenberg algebra $\mathfrak{h}$ in d dimensions — that is, d copies of the original Heisenberg algebra. We can repeat the construction in section 2.3, but this time simply with d commuting copies of α, that is

$$[\alpha^i_m, \alpha^j_n]_{\mathfrak{h}} = m\delta_{m+n,0}\delta_{i,j}\mathbf{k} \qquad i,j = 1,\dots d \ , \tag{8.2}$$

or for two general elements $h^1, h^2 \in \mathfrak{h}$

$$[h^1_m, h^2_n]_{\mathfrak{h}} = m\delta_{m+n,0}(h^1, h^2)\mathbf{k} \ . \tag{8.3}$$

This is simply the d fold tensor product of a single Heisenberg algebra. Now define the Fock space associated to a lattice L as

$$V_L := \mathcal{S}(\mathfrak{h}_-) \otimes \mathbb{C}[L] \tag{8.4}$$

Here $\mathbb{C}[L] = \bigoplus_{\mu \in L} \mathbb{C}|\mu\rangle$. This is a $\mathfrak{h}$-module in the usual way, where we define

$$h_{n>0}|\mu\rangle = 0 \ , \qquad h_0|\mu\rangle = (h,\mu)|\mu\rangle \ . \tag{8.5}$$

Similarly this is also a module of the Heisenberg VOA coming from $\mathfrak{h}$ — it is in fact an (infinite) direct sum of modules of the form (6.4). A basis of states in V can be written as $a|\mu\rangle$ where $\mu \in L$ and a is a monomial in $\mathfrak{h}_-$,

$$a|\mu\rangle = h^1_{-n_1} \cdots h^k_{-n_k}|\mu\rangle \ . \tag{8.6}$$

8.2.2 State-field map

We now want to turn this module into an actual VOA. That is, we need to define a state-field map

$$a|\mu\rangle \mapsto Y(a|\mu\rangle, z) \ . \tag{8.7}$$

It is clear how to turn the Heisenberg modes into fields. What is new is that we need to construct fields corresponding to the vectors $|\mu\rangle$. For this we first need to define an action of the Abelian group L on V. The most naive way of doing this is to define operators $e^\mu \in \mathrm{End}(V)$ as

$$\mu \mapsto e^\mu : e^\mu a|\nu\rangle = a|\mu + \nu\rangle \ , \tag{8.8}$$

which is a representation of L. As we will see, this action does not lead to local operators. Instead we need to define a projective representation

$$e^\mu a|\nu\rangle = \epsilon(\mu,\nu) a|\mu + \nu\rangle \tag{8.9}$$

where in general $\epsilon \in \mathbb{C}^*$. Associativity requires

$$\epsilon(\mu, \nu + \rho)\epsilon(\nu, \rho) = \epsilon(\mu, \nu)\epsilon(\mu + \nu, \rho) \tag{8.10}$$

As we will see, for locality to work we will need to require

$$\epsilon(\mu, \nu) = (-1)^{(\mu,\nu)}\epsilon(\nu, \mu) \ . \tag{8.11}$$

An explicit construction of such a cocycle: Choose basis $\mu_i, i = 1, \ldots, d$. On this basis

$$\epsilon(\mu_i, \mu_j) = \begin{cases} (-1)^{(\mu_i,\mu_j)} & i < j \\ 1 & i \geq j \end{cases} \tag{8.12}$$

and extend by linearity of L. In particular we have

$$e^\mu e^\nu = \epsilon(\mu, \nu)e^{\mu+\nu} \ . \tag{8.13}$$

Next we define an operator $\hat{z}^\mu \in \mathrm{End}(V)[[z, z^{-1}]]$ via

$$\hat{z}^\mu(a|\nu\rangle) := z^{(\mu,\nu)}a|\nu\rangle \tag{8.14}$$

Proposition 8.1.

$$[h_m, \hat{z}^\mu] = 0 \tag{8.15}$$
$$[h_m, e^\mu] = \delta_{m,0}(h, \mu)e^\mu \tag{8.16}$$
$$\hat{z}^\mu e^\nu = z^{(\mu,\nu)}e^\nu \hat{z}^\mu \tag{8.17}$$

Proof. Act with both sides on basis vectors $a|\mu\rangle$ and check that they agree. □

Next we define exponential operators

$$E^\pm(\mu, z) = \exp\left(\sum_{n\in\pm\mathbb{Z}_+} \frac{\mu_n}{n} z^{-n}\right) \in \mathrm{End}(V)[[z^\mp]] \tag{8.18}$$

They satisfy

Lemma 8.2.

$$[h_m, E^+(\mu, z)] = 0 \qquad m \geq 0 \tag{8.19}$$
$$[h_m, E^+(\mu, z)] = -(h, \mu)z^m E^+(\mu, z) \qquad m < 0 \tag{8.20}$$
$$[h_m, E^-(\mu, z)] = 0 \qquad m \leq 0 \tag{8.21}$$
$$[h_m, E^-(\mu, z)] = -(h, \mu)z^m E^-(\mu, z) \qquad m > 0 \tag{8.22}$$

Proof. Use

$$\left[h_m, \sum_{n\in\mathbb{Z}_+} \frac{\mu_n}{n} z^{-n}\right] = -(h,\mu) z^m \qquad m<0 \tag{8.23}$$

and similar for E^- together with

$$[A, e^B] = [A,B]e^B \qquad \text{if } [A,B] \text{ commutes with } B\ . \tag{8.24}$$

□

Lemma 8.3.

$$[E^\pm(\mu,z), E^\pm(\nu,w)] = 0 \tag{8.25}$$

$$E^+(\mu,z)E^-(\nu,w) = \left(1-\frac{w}{z}\right)^{(\mu,\nu)} E^-(\nu,w)E^+(\mu,z) \tag{8.26}$$

Proof. The first statement is obvious. For the second note that

$$\begin{aligned}\left[\sum_{m\in\mathbb{Z}_+} \frac{\mu_m z^{-m}}{m}, \sum_{n\in-\mathbb{Z}_+} \frac{\nu_n w^{-n}}{n}\right] &= -(\mu,\nu)\sum_{m\in\mathbb{Z}_+}\frac{1}{m}\left(\frac{w}{z}\right)^m \\ &= (\mu,\nu)\log\left(1-\frac{w}{z}\right)\ .\end{aligned} \tag{8.27}$$

We can now use the formal Baker-Campbell-Hausdorff formula for the case where all higher commutators vanish,

$$e^A e^B = e^B e^A e^{[A,B]} \qquad \text{if } [A,B] \text{ commutes with } A, B\ . \tag{8.28}$$

□

Now we are ready to define vertex operators in two special cases:

$$Y(|\mu\rangle, z) = E^-(-\mu,z)E^+(-\mu,z)e^\mu \hat{z}^\mu \tag{8.29}$$

and

$$Y(h_{-1}|0\rangle, z) = h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1} \tag{8.30}$$

Note that we have

$$Y(|\mu\rangle, z)|0\rangle = |\mu\rangle + O(z)\ , \tag{8.31}$$

8.2.3 Conformal vector

We define the conformal vector as in the tensor product of d Heisenberg VOAs,

$$\omega = \frac{1}{2}\sum_{i=1}^d \alpha^i_{-1}\alpha^i_{-1}|0\rangle \tag{8.32}$$

so that

$$Y(\omega, z) = \frac{1}{2}\sum_{i=1}^{d} : \alpha^i(z)\alpha^i(z) : \tag{8.33}$$

From proposition 5.5 we know that the modes L_m then form a Virasoro algebra with central charge d. We define $T := L_{-1}$ and indeed have

Proposition 8.4.

$$\begin{aligned} [L_{-1}, Y(h_{-1}|0\rangle, z)] &= \partial_z Y(h_{-1}|0\rangle, z) && (8.34) \\ [L_{-1}, Y(|\mu\rangle, z)] &= \partial_z Y(|\mu\rangle, z) && (8.35) \end{aligned}$$

Proof. The first identity follows as in the Heisenberg case. For the second one see proposition 6.5.2 in [2]. □

8.3 Construction

We now want to apply the construction theorem 3.9 to construct the lattice VA. We define $|0\rangle := |0\rangle$, and T as in 8.2.3. As the set of generators we pick $\{Y(\alpha^i_{-1}|0\rangle, z)\}_{i=1,\dots d} \cup \{Y(|\mu\rangle, z)\}_{\mu\in L}$. This is indeed a local set of fields:

Proposition 8.5.

$$\begin{aligned} (z-w)^2[Y(h^1_{-1}|0\rangle, z), Y(h^2_{-1}|0\rangle, w)] &= 0 && (8.36) \\ (z-w)[Y(h_{-1}|0\rangle, z), Y(|\nu\rangle, w)] &= 0 && (8.37) \\ (z-w)^K[Y(|\mu\rangle, z), Y(|\nu\rangle, w)] &= 0 && (8.38) \end{aligned}$$

with $K = \max(0, -(\mu, \nu))$.

Proof.
- (8.36): follows as in the Heisenberg case.
- (8.37):

$$[Y(h_{-1}|0\rangle, z), Y(|\nu\rangle, w)] = (h, \nu)w^{-1}\delta(z/w)Y(|\nu\rangle, w) \tag{8.39}$$

  This follows from expanding $h(z)$ and then using lemma 8.2 and proposition 8.1 (see Prop 6.5.2 in [2]).

- (8.38):

$$\begin{aligned} &Y(|\mu\rangle, z)Y(|\nu\rangle, w) \\ &\quad = z^{(\mu,\nu)}\left(1-\frac{w}{z}\right)^{(\mu,\nu)} E^-(-\mu, z)E^-(-\nu, w)E^+(-\mu, z)E^+(-\nu, w)e^{\mu}e^{\nu}\hat{z}^{\mu}\hat{w}^{\nu} \\ &\quad = \epsilon(\mu, \nu)\,(z-w)^{(\mu,\nu)}\, E^-(-\mu, z)E^-(-\nu, w)E^+(-\mu, z)E^+(-\nu, w)e^{\mu+\nu}\hat{z}^{\mu}\hat{w}^{\nu} \end{aligned} \tag{8.40}$$

On the other hand we have

$$\begin{aligned} &Y(|\nu\rangle, w)Y(|\mu\rangle, z) \\ &\quad = \epsilon(\nu,\mu)\,(w-z)^{(\mu,\nu)}\,E^-(-\mu,z)E^-(-\nu,w)E^+(-\mu,z)E^+(-\nu,w)e^{\mu+\nu}\hat{z}^\mu\hat{w}^\nu \\ &= (-1)^{(\mu,\nu)}\epsilon(\mu,\nu)\,(w-z)^{(\mu,\nu)}\,E^-(-\mu,z)E^-(-\nu,w)E^+(-\mu,z)E^+(-\nu,w)e^{\mu+\nu}\hat{z}^\mu\hat{w}^\nu \end{aligned} \tag{8.41}$$

If $(\mu, \nu) \geq 0$, then the two expressions are manifestly identical. Otherwise multiply with $(z-w)^K$ to get a non-negative exponent in the factor.

□

Let us now check that all the assumptions of theorem 3.9 are satisfied. 1. follows from proposition 8.4, and 2. is the same as in the Heisenberg case. 3. holds by construction of V, and 4. from (8.31) and the construction of the Fock space. This establishes that we have a vertex algebra, with Y defined as in theorem 3.9.

To check that it is a VOA, we use section 8.2.3. **VOA1** and **VOA3** follow immediately. We then use **VOA2** to actually define the grading. We namely have

$$L_0(h^1_{-n_1}\cdots h^r_{-n_r}|\mu\rangle) = \left(n_1 + \cdots + n_r + \frac{1}{2}(\mu,\mu)\right)(h^1_{-n_1}\cdots h^r_{-n_r}|\mu\rangle) \tag{8.42}$$

It follows directly that this grading is a good VOA grading.

In total we thus have

Theorem 8.6. *Let L be a positive definite even lattice of rank d. Then $(V_L, Y, |0\rangle, \omega)$ as defined above is a vertex operator algebra with central charge d.*

9 Characters, Modular Forms and Zhu's Theorem

9.1 Characters

Let V be a VOA of central charge c. We define the character of V as the series

$$\chi_V(\tau) = \mathrm{Tr}_V q^{L_0 - c/24} = q^{-c/24}\sum_n \dim V_{(n)} q^n \qquad q = e^{2\pi i \tau} \tag{9.1}$$

Exercise 9.1. *Let V be the Heisenberg VOA. Show:*

$$\chi_V(\tau) = \frac{1}{\eta(\tau)} = \frac{1}{q^{1/24}\prod_{n>0}(1-q^n)} \tag{9.2}$$

Exercise 9.2. *Let V, W be VOA. Show:*

$$\chi_{V\otimes W}(\tau) = \chi_V(\tau)\chi_W(\tau) \tag{9.3}$$

In particular, the character of the tensor product of 24 Heisenberg VOA is simply $1/\Delta(\tau)$. This hints that characters of VOAs have good modular transformation properties.

What about the character of a lattice VOA? Let L be a positive definite lattice. Then the *lattice theta function* $\Theta_L(\tau)$ is defined as

$$\Theta_L(\tau) = \sum_{\mu \in L} q^{\frac{1}{2}(\mu,\mu)} . \tag{9.4}$$

Exercise 9.3. *Let L be an even positive definite lattice, and V_L its associated VOA. Then*

$$\chi_{V_L}(\tau) = \frac{\Theta_L(\tau)}{\eta(\tau)^d} \tag{9.5}$$

Let L be a lattice. We define its *dual lattice* L^* as

$$L^* := \{\mu \in \mathbb{R}^d : (\mu, \nu) \in \mathbb{Z}\ \forall \nu \in L\} . \tag{9.6}$$

Note that if L is integral, then $L \subset L^*$. We say L is *self-dual* or *unimodular* if $L = L^*$. The lattice theta functions of self-dual lattices have very nice modular properties:

Theorem 9.1 (Hecke-Schoenberg)**.** *Let L an even positive definite self-dual lattice of rank $d = 2k$. Then $\Theta_L(\tau)$ is a modular form of weight k.*

It follows that the character χ_{V_L} is a modular function (possibly with a character), *i.e.* that it is modular invariant! Does this property hold for more general VOAs?

9.2 Rational VOAs and Zhu's Theorem

Roughly speaking we call a VOA *rational* if it only has a finite number of inequivalent irreducible modules. To make this definition precise, we need to introduce a few concepts. For details and additional references see [3] and [4].

Definition 9.1. *Let V be a VOA, and let M be a module of V as a vertex algebra, i.e. satisfying definition 6.1. We say it is a* weak V-module *if it has a grading $M = \bigoplus_{n\in\mathbb{N}} M_{[n]}$ such that*

$$v \in V_{(k)} \Rightarrow v_n^M : M_{[m]} \to M_{[m+k-n-1]} \tag{9.7}$$

We can now give a precise definition:

Definition 9.2. *A VOA V is* weakly rational *if*

1. *every V-module is completely reducible*
2. *the set $\Phi(V)$ of irreducible V-modules is finite*
3. *every irreducible weak V-module is a V-module*

Finally, we have

Definition 9.3. *We say V is C_2-cofinite if the subspace*

$$C_2(V) := \langle u_{-2}v : u, v \in V \rangle \tag{9.8}$$

has finite codimension in V, i.e. $V/C_2(V)$ is finite dimensional.

Theorem 9.2 (Zhu)**.** *Let V be a C_2 cofinite weakly rational VOA with $\Phi(V)$ its set of irreducible modules labeled by M, N. Then there is a representation ρ of $SL_2(\mathbb{Z})$ by complex matrices $\rho(A)$ indexed by M, N such that the characters obey*

$$\chi_M\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{N\in\Phi(V)} \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{MN} \chi_N(\tau) \ , \tag{9.9}$$

that is they form a $|\Phi(V)|$-dimensional representation of $SL_2(\mathbb{Z})$.

9.3 Modules of lattice VOAs

Let V_L be a lattice VOA. To construct its irreducible modules, let us start with the construction of Heisenberg modules again. That is, we choose $\lambda \in \mathbb{R}^d$ and associate it with a ground state $|\lambda\rangle$ such that for all $h \in \mathfrak{h}$

$$h_n|\lambda\rangle = 0 \qquad n > 0 \ , \qquad h_0|\lambda\rangle = (h, \lambda)|\lambda\rangle \ . \tag{9.10}$$

We then try to define the module map $Y_M : V_L \to M$ exactly as in the construction of V_L in section 8.2, that is in (8.29) and (8.30). There are however two issues:

- Since $\hat{z}^\mu|\lambda\rangle = z^{(\mu,\lambda)}|\lambda\rangle$ need to give an integral power of z for all $\mu \in L$, it follows that λ must be in the dual lattice L^*.
- Since $e^\mu|\lambda\rangle = |\lambda+\mu\rangle$, it is necessary that the module is a direct sum of all ground states of the form $\lambda + \mu$.

This suggests that all irreducible modules are of the form $V_{L+\lambda}$ given by

$$V_{L+\lambda} = \mathcal{S}(\mathfrak{h}_-) \otimes \mathbb{C}[L+\lambda] = \mathcal{S}(\mathfrak{h}_-) \otimes \bigoplus_{\mu\in L} \mathbb{C}|\lambda+\mu\rangle \tag{9.11}$$

where $\lambda \in L^*$ is in the dual lattice of L. Since two λ differing by a lattice vector give equivalent modules, the total number of irreducible modules is $|L^* : L|$, that is the order of the quotient of L^* by L as groups. All these arguments can be made more precise to give the following result:

Theorem 9.3. *Let L be an even lattice. Then V_L is weakly rational and C_2-cofinite, and its irreducible modules are the Fock spaces $V_{L+\lambda}$, $\lambda \in L^*$, giving a total of $|L^* : L|$ of them.*

Note that for integral lattices $|L^* : L|$ is always finite. Let $M = (e_1, \ldots, e_d)^T$ be the generator matrix of M whose rows are given by the generators e_i of the lattice. The volume of the unit cell is given by $vol(L) := (\det MM^T)^{1/2}$, which is also $|L^* : L|$.

If $vol(L) = 1$ or equivalently $L = L^*$ we say that L is *unimodular* or *self-dual*. In that case V_L only has a single module, namely V_L itself. Zhu's theorem 9.2 then tells us that $\chi_{V_L}(\tau)$ is a 1-dimensional representation of $SL_2(\mathbb{Z})$. Such a VOA is called *self-dual* or *holomorphic*. Let us furthermore assume that d is a multiple of 24. In that case $\chi_{V_L}(\tau)$ is a modular function without a character, that is it is invariant under $SL_2(\mathbb{Z})$ transformations. To see this, note that due to the moding $\rho(T) = 1$. Moreover we know that $\rho(S^2) = \rho((ST)^3) = 1$, from which the claim follows.

10 Monstrous Moonshine

10.1 Niemeier lattices

Let us consider even self-dual lattices in $d = 24$. By theorem 9.1 we know that their lattice theta function is a modular form of weight 12. Using the fact that any lattice L contains only one vector of length 0, we can therefore write it as

$$\Theta_L(\tau) = E_4(\tau)^3 + A\Delta(\tau) \ , \tag{10.1}$$

where E_4 is the Eisenstein series normalized such that $E_4(\tau) = 1 + \ldots$, $\Delta(\tau)$ is the discriminant function

$$\Delta(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2) = e^{2\pi i\tau}\prod_{n>0}(1 - e^{2\pi i\tau n}) \ , \tag{10.2}$$

and A is some integer. It turns out that there are exactly 24 even self-dual lattices in 24 dimensions: the so-called *Niemeier lattices* [5]. These lattices are interesting for multiple reasons. Among other things, they have large automorphism groups $Aut(L)$. An *automorphism* $g \in Aut(L)$ of a lattice L is an isometry that fixes the origin and maps L to itself. This means that the action of g on the coordinates can be represented as a matrix $O(g) \in O(d)$. On the other hand, when acting on the basis vectors of the lattice, g must act as a matrix $B(g) \in GL(d, \mathbb{Z})$, that is a $d \times d$ matrix with integral entries and determinant ± 1. That is,

$$B(g)M = MO(g) \ . \tag{10.3}$$

It then follows that $Aut(L)$ is finite. One very interesting Niemeier lattice is the *Leech* lattice. It is the only Niemeier lattice that has no vectors of length square 2. That is, its lattice theta function is given by

$$\Theta(\tau) = E_4(\tau)^3 - 720\Delta(\tau) = 1 + 196560q^2 + \ldots \tag{10.4}$$

It has a very interesting automorphism group: The Conway group Co_0.

10.2 Sporadic groups

Let G be a finite group. A subgroup $N < G$ is *normal* if $gNg^{-1} \subset N$. We write $N \triangleleft G$. The motivation for this definition is that then G/N is again a group. G is called *simple* if its only normal subgroups are G and the trivial group $\{e\}$. A first step towards classifying all finite groups is to classify all simple groups. Indeed, this has been achieved.

Theorem 10.1. *Any finite simple group is isomorphic to either*

- *a cyclic group of prime order* $\mathbb{Z}_p$
- *an alternating group* $A_n,\ n \geq 5$
- *a group of Lie type*

or one of 26 so-called sporadic *groups.*

The largest of the sporadic groups is the *Monster group* $\mathbb{M}$, which has order

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \cdot 10^{53} \quad (10.5)$$

Its smallest irreducible representations have dimensions $1, 196883, 21296876, \ldots$.

10.3 The Monster VOA $V^\natural$

Let V be a VOA. An *automorphism* of V is a (vector space) automorphism $\tilde{g} : V \to V$ satisfying

$$\tilde{g}Y(a,z)\tilde{g}^{-1} = Y(\tilde{g}a, z) \qquad \tilde{g}\omega = \omega\ . \quad (10.6)$$

Let us consider the Leech-VOA V_L. Not surprisingly, it is possible to lift an automorphism g of the underlying lattice L to automorphisms of V_L. The rough idea is to act on the Heisenberg modes as

$$\tilde{g} : \alpha^i \mapsto O(g)^i_j \alpha^j \quad (10.7)$$

and to act on the ground states as

$$\tilde{g} : |\mu\rangle \mapsto |O(g)\mu\rangle\ . \quad (10.8)$$

The only subtlety comes from the cocycles $\epsilon(\mu, \nu)$, which can require us to extend g. For the Leech VOA it turns out that the discrete part of $Aut(V_L)$ is $\mathbb{Z}_2^{24}.Co_0$, that is an extension of Co_0 by $\mathbb{Z}_2^{24}$. Co_0 itself is a $\mathbb{Z}_2$ extension of the simple sporadic group Co_1, $Co_0 = \mathbb{Z}_2.Co_1$.

The Leech-VOA has character

$$\chi_L(\tau) = j(\tau) - 720 = q^{-1} + 24 + 196884q + 21493760q^2 + \ldots \quad (10.9)$$

From what we have said, it is clear that this has Conway moonshine, that is its coefficients decompose into representations of Co_0. How do we get monstrous moonshine though? First note that $\mathbb{Z}_2^{24+1}.Co_1$ is a maximal subgroup of $\mathbb{M}$. That is, we are only missing a single generator. Next note that the constant term 24 is problematic, since there is no non-trivial monster representation of this dimension.

To address these issues, we want to *orbifold* V_L. To this end, we first consider the subspace $V_L^{\mathbb{Z}_2}$ of V_L which is invariant under the $\mathbb{Z}_2$ symmetry of the lattice $x \mapsto -x$. $V_L^{\mathbb{Z}_2}$ is again a VOA, but it is no longer holomorphic: It has four irreducible modules $M_+^1, M_-^1, M_+^{-1}, M_-^{-1}$. Here $M_\pm^1$ are the original V_L decomposed into states which have eigenvalue ± 1 under $\mathbb{Z}_2$. $M_\pm^{-1}$ are new modules, coming from the so-called *twisted sector*. The idea is now to adjoin some of these modules to $V^{\mathbb{Z}_2} = M_+^1$ to recover a holomorphic VOA. This can be done to define

$$V^\natural := M_+^1 \oplus M_+^{-1} \ . \tag{10.10}$$

It turns out that the symmetry of $V^\natural$ is enhanced to the monster symmetry, *i.e.* $Aut(V^\natural) \simeq \mathbb{M}$, and its character is given by

$$\chi_{V^\natural}(\tau) = j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots \ . \tag{10.11}$$

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