Introduction to String Theory

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Chapter 1

Motivation

There are two main motivations for string theory: As an extension (”UV completion”) of quantum field theory, and as a theory of quantum gravity. In this course we will address both aspects, but we will mainly concentrate on the latter.

The original hope for string theory was that it would predict new phyiscs beyond the standard model. The first task is then of course to recover known physics from string theory. Another question is: Is string theory unique? The answer is mixed. It is indeed believed that string theory is unique. It is also believed however that it has an extremely large number of different vacuum solutions: one famous estimate is $N \sim 10^{500}$. The details of the low-energy effective theory depend crucially on the vacuum. String theory is thus not very predictive at energy scales we can measure ($E \sim 10^3$ GeV.) Thus, does string theory reproduce the standard model? The answer is somewhat mixed. All ingredients needed for the standard model appear from string theory: It produces scalars, fermions, gauge fields, interaction terms, but no actual string model has been constructed that reproduces the exact standard model. Still, a great many new quantum field theories have been discovered through string theory, and it is generally believed that the standard model should show up somewhere in this space of low-energy theories.

There is a good chance you have already heard some facts about string theory. One of the goal of this class is to address and explain these statements:

String theory lives in 26 (or 10) dimensions

Indeed, it turns out that string theory is anomalous unless it is in 26 dimensions, or 10 dimensions if we consider superstring theory, *i.e.* string theory with fermions. This is the result of a fairly subtle effect of quantization having to do with anomalies. We will also address how to get around this constraint to describe the 4 dimensional spacetime that we all know and love.

String theory is a theory of quantum gravity

Quantum gravity, that is the quantum version of Einstein’s theory of general relavity, is a notoriously difficult subject. In principle the procedure seems straightforward.

The degrees of freedom of the theory are given by the curvature tensor $R_{\mu\nu\sigma\rho}$, which has $D^2(D^2-1)/12$ independent components. The Einstein equation fixes the Ricci tensor, *i.e.* $D(D+1)/2$ components. The remaining components are thus not fixed by the matter distribution, and contain information on gravitational waves. By the usual wave-particle duality of quantum mechanics, one can quantize the wave degrees of freedom. The resulting spin 2 particle is called the *graviton*. Indeed we will find that string theory automatically leads to gravitons.

Experimentally, such gravitational waves have apparently been detected recently at LIGO. Detecting actual quantum gravitational effects seems to be completely out of reach at the moment. Simple dimensional analysis shows that quantum gravity effects should become significant only at the Planck length

$$\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35} m \ , \tag{1.1}$$

or equivalently at the Planck energy

$$E_{Pl} \sim 10^{19} GeV \ . \tag{1.2}$$

For comparison, the LHC reaches around $E \sim 1.6 \cdot 10^4 GeV$, that is 15 orders of magnitude less than the Planck energy. This pretty much makes direct detection of quantum gravity effects hopeless, so that currently quantum gravity remains a theorist's game.

While we are at it, let us also take the convention now that we choose units such that[1]

$$c = \hbar = 1 \ . \tag{1.3}$$

String theory is UV finite

It turns out that it is very hard to construct a consistent theory of quantum gravity due to issues with *renormalization*.

In quantum field theory, the Feynman rules tell us to sum and integrate over all possibile diagrams of how a process can happen. A typical integral of say a process in ϕ^4 theory is given by a momentum space integral.

$$\int d^4p \frac{1}{p^2} \ . \tag{1.4}$$

Since for large p the integrand goes like pdp, the integral diverges badly as $p \to \infty$. To render it finite, instead of integrating all the way up to infinity, we need *regularize* the integral by introducing a so-called *ultraviolet cutoff* Λ_{UV},

$$\int_{|p|<\Lambda_{UV}} d^4p \frac{1}{p^2} \ . \tag{1.5}$$

[1]If a real emergency arises, we will also choose the convention $2 = \pi = i = -1 = 1$.

The problem with this approach is that the result of course depends on Λ_{UV}, which is an unphysical quantity. To make physical observables independent of Λ_{UV}, we can add additional terms to the Lagrangian, so-called *counterterms*, which depend on Λ_{UV} in such a way that in the final physical result all the Λ_{UV} dependence cancels out. This strategy is called *renormalization*, since effectively it changes the values of the coupling constants of the theory.

In practice the coefficients of the counterterms are fixed by performing measurements. There are two types of theories. If we only have to introduce a finite number of counterterms, then we can fix them all by a finite number of measurements. Such a theory is therefore called *renormalizable*.

If we need to introduce an infinite number of counterterms, then we would have to perform an infinite number of measurements to fix their coefficients, which is of course hopeless. Such theories are called *non-renormalizable*, and they do not predict anything because they have an infinite number of free parameters that need to be fixed. It turns out that the naive attempt to quantize general relativity leads to a non-renormalizable theory. Fundamentally, this is the reason why quantum gravity is such a difficult subject from a theoretical point of view.

Note that we only needed to worry about this renormalization procedure because the initial theory was not UV-finite. String theory circumvents the problem of non-renormalizability by being UV-finite in the first place. This is a crucial property, and we will discuss it in detail.

String theory has $N \sim 10^{500}$ different vacua and is therefore not predictive

Even though string theory is believed to be unique, it does have a great many different vacua. This makes it very hard to make concrete low-energy predictions that could be observed say at the LHC. We will at least sketch where those many possible vacua come from.

1.1 Prerequisites

This course requires knowledge of quantum mechanics. If you know about the harmonic oscillator, fermions and spin, you should be good. We also require some special relativity. You should also know classical mechanics (Lagrangians, actions, Poisson brackets et al.), and classical field theory such as Electrodynamics for instance. Knowing some general relativity is an advantage, but not necessary — you mainly just need to know how to contract indices. Knowing some quantum field theory is nice, but it is not necessary. You may miss out on some of the motivation for what we are doing if you don't know any QFT, but you should still be able to follow most of the course.

1.2 Suggested Literature

Some suggested textbooks:

- B. Zwiebach,"A first course in string theory," Cambridge, UK: Univ. Pr. (2009)
- M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vols 1+2," Cambridge Monogr.Math.Phys..
- J. Polchinski, "String theory. Vol. 1+2'
- R. Blumenhagen, D. Lüst and S. Theisen, "Basic concepts of string theory,"'

We are mostly following Green-Schwarz-Witten and Blumenhagen-Lüst-Theisen. For an introduction to 2d CFT, good references are

- P. H. Ginsparg, "Applied Conformal Field Theory," hep-th/9108028.
- P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory."

Chapter 2

The Classical String

2.1 The Nambu-Goto action

Consider first a point particle in D dimensions. Its trajectory carves out a *worldline*, *i.e.* a one dimensional line L in D dimensional Minkowski *spacetime* $\mathbb{R}^{1,D-1}$,

$$X^\mu(\tau) \qquad \mu = 0, \dots D-1 \qquad : L \to \mathbb{R}^{1,D-1} \ . \tag{2.1}$$

We can parametrize this trajectory by a parameter τ. We can for instance take τ to be the proper time of the particle, but any other parametrization will work just as well.

The basic idea of string theory is to replace particles by strings, *i.e.* extended one-dimensional objects. For concreteness take a closed string. It now carves out a 2d *worldsheet* Σ in $\mathbb{R}^{1,D-1}$. Again we want to parametrize this worldsheet. To this end, at a fixed time, parametrize it along its length by the worldsheet coordinate $0 \leq \sigma < \pi$. To describe its evolution, pick the other worldsheet coordinate $-\infty < \tau < \infty$. Its worldsheet Σ is thus a cylinder in spacetime parametrized by

$$X^\mu(\sigma, \tau) \qquad \mu = 0, \dots D-1 \qquad : \Sigma \to \mathbb{R}^{1,D-1} \ . \tag{2.2}$$

For convenience we will often label the two worldsheet coordinates σ and τ simply by σ^a, $a = 1, 2$. (We will use lowercase Roman letters $a, b \dots$ to denote worldsheet indices, and lowercase Greek letters $\mu, \nu \dots$ to denote spacetime indices.)

We want to write down an action that describes the dynamics of this string. A natural guess is to take the action to be proportional to the area of Σ embedded in $\mathbb{R}^{1,D-1}$. As you will remember from your calculus class, the area is obtained from the Gram matrix

$$\gamma_{ab} := \partial_a X^\mu \partial_b X_\mu \tag{2.3}$$

by integrating

$$S = -TA = \int d\tau d\sigma \sqrt{-\det \gamma_{ab}} \tag{2.4}$$

where T is a dimensionful constant, the *string tension.* This is the so-called "Nambu-

Goto action". We will often write the string tension in terms of α',

$$T = (2\pi\alpha')^{-1} \tag{2.5}$$

or in terms of the *string length*

$$\ell_s = \sqrt{2\alpha'} = 1/\sqrt{\pi T} \ . \tag{2.6}$$

The inner product in the definition (2.3) is with respect to the Minkowski metric $\eta_{\mu\nu}$,

$$a \cdot b = a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu \ . \tag{2.7}$$

For the Minkowski metric we take the mostly plus convention

$$\eta = diag(-1, 1, \ldots, 1) \ , \tag{2.8}$$

which may differ by an overall sign from what you are familar with, but which has the advantage that its spacelike part is equal to the standard Euclidean metric. As usual we define $\eta^{\mu\nu}$ to be the inverse matrix to $\eta_{\mu\nu}$, and we raise and lower indices by contracting with the metric.

Let us now discuss the symmetries of (2.4). By the above considerations it clearly has spacetime Poincare symmetry:

$$X^\mu \mapsto A^\mu_\nu X^\mu + b^\mu \tag{2.9}$$

where $A^\mu_\nu \in SO(1, D-1)$ leaves (2.4) invariant.

Moreover by construction it is also invariant under reparametrization of the world-sheet,

$$\sigma^a \mapsto \tilde{\sigma}^a(\sigma, \tau) \tag{2.10}$$

The infinitesimal version of this is $\sigma^a \mapsto \sigma^a + \xi^a(\sigma, \tau)$ which leads to

$$\delta X^\mu(\sigma, \tau) = \xi^a \partial_a X^\mu(\sigma, \tau) \ . \tag{2.11}$$

Physically, only the spacetime configuration matters, not the parametrization of the world sheet. Reparametrization invariance can be thought of as an (infinite dimensional) gauge symmetry of the theory, *i.e.* a redundancy in its description.

Having a gauge redundancy is well known for instance from *e.g.* electrodynamics. When working with gauge theories, we can do two things: We can either work with with gauge degrees of freedom, but require that all physical observables be invariant under gauge transformations. Alternatively we can fix the gauge and only work with the remaining physical degrees of freedom. This second approach has the disadvantage that the gauge fixed theory looks often much more complicated and less symmetric than the underlying gauge theory — in a sense this is the reason why we introduced the gauge symmetry in the first place. These issues are somewhat uninteresting for classical theories, but they become very important for quantization.

2.2 The Polyakov action

The problem with the Nambu-Goto action is that it contains a square root and is therefore highly non-linear. This will become a problem when we want to quantize the string. We therefore want to introduce a linear action. To this end, we introduce an auxiliary field: the worldsheet metric $h_{ab}(\sigma, \tau)$. It is a symmetric 2×2 matrix of signature $sig(h) = (-1, 1)$. The Polyakov action is then given by

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{ab}(\sigma, \tau) \partial_a X^\mu \partial_b X_\mu \tag{2.12}$$

where $h = \det h_{ab}$. As mentioned above, we have introduced an additional gauge symmetry to the problem in order to make the action nicer. We claim that (2.12) is equivalent to (2.4). Indeed, it is very easy to obtain the equation of motion for X from (2.12):

$$\frac{\delta S}{\delta X_\mu} = T \partial_a (\sqrt{-h} h^{ab} \partial_b X^\mu) = 0 \ . \tag{2.13}$$

On the other hand note that S only depends on h, and not its derivatives. The equations of motion for h are thus non-dynamical:

$$T_{ab} := \frac{4\pi}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}} = -\frac{1}{\alpha'} \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu \right) \overset{!}{=} 0 \tag{2.14}$$

In terms of the Gram matrix this is

$$\gamma_{ab} = \frac{1}{2} h_{ab} h^{cd} \gamma_{cd} \tag{2.15}$$

so for the determinant we get

$$\sqrt{-\gamma} = \frac{1}{2} \sqrt{-h} h^{cd} \gamma_{cd} \tag{2.16}$$

Plugging this into (2.12), we indeed recover (2.4), so that the two actions are equivalent, just as claimed.

Let us again investigate the symmetries of the action. (2.12) has the following symmetries:

1. spacetime Poincare symmetry:

$$X^\mu \mapsto A^\mu_\nu X^\mu + b^\mu \tag{2.17}$$

where $A^\mu_\nu \in SO(1, D-1)$.

2. Reparametrization invariance now requires the metric to be rescaled as well:

$$\begin{aligned} \delta X^\mu &= \xi^a \partial_a X^\mu && (2.18) \\ \delta h^{ab} &= \xi^c \partial_c h^{ab} - \partial_c \xi^a h^{cb} - \partial_c \xi^b h^{ac} && (2.19) \\ \delta(\sqrt{h}) &= \partial_a(\xi^a \sqrt{h}) && (2.20) \end{aligned}$$

3. Finally, there is a new symmetry: Weyl symmetry. Note that (2.14) allows for a position dependent rescaling of the metric,

$$\delta h^{ab} = \Lambda(\sigma, \tau) h^{ab} \ . \tag{2.21}$$

Also note that this is special to the case of $d = 2$ for the worldsheet! This symmetry will be crucial in what follows.

2.3 Fixing the worldsheet metric

Now we use the symmetries to fix a gauge choice for h. Note that h is symmetric and therefore has three independent components. From (2.18) we have two free choices for ξ^a, and (2.21) gives one free choice for Λ. We can use this to completely fix h. The simplest choice is of course $h_{ab} = \eta_{ab}$, *i.e.* to pick the worldsheet Minkowski metric. The then action becomes

$$S = -\frac{T}{2} \int d^2\sigma \eta^{ab} \partial_a X^\mu \partial_b X_\mu = -\frac{T}{2} \int d^2\sigma \left((X')^2 - (\dot{X})^2 \right) \tag{2.22}$$

where we took $X' = \partial_\sigma X$ and $\dot{X} = \partial_\tau X$. In this form the string action looks very familiar: it contains the usual kinetic term $\dot{X}^2$ and the potential term X'^2 coming from the elastic energy. The only slightly unusual thing is that we have a relativistic string in spacetime rather than a more familiar non-relativistic string in D space dimensions.

As expected (2.22) leads to the standard wave equation

$$\Box X^\mu = (\partial_\sigma^2 - \partial_\tau^2) X^\mu = 0 \ . \tag{2.23}$$

The constraint equation (2.14) gives

$$\begin{aligned} T_{01} = T_{10} &= -\frac{1}{\alpha'} \dot{X}^\mu X'_\mu = 0 && (2.24) \\ T_{00} = T_{11} &= -\frac{1}{\alpha'} \frac{1}{2} (\dot{X}^2 + X'^2) = 0 && (2.25) \end{aligned}$$

We can obtain the Poisson brackets:

$$\begin{aligned} \{X^\mu(\sigma), X^\nu(\sigma')\} &= \{\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\} = 0 && (2.26) \\ \{\dot{X}^\mu(\sigma), X^\nu(\sigma')\} &= T^{-1} \delta(\sigma - \sigma') \eta^{\mu\nu} && (2.27) \end{aligned}$$

To solve (2.23) we define the usual lightcone coordinates

$$\sigma^+ = \tau + \sigma \qquad \sigma^- = \tau - \sigma \ , \tag{2.28}$$

leading to the most general solution

$$X^\mu(\sigma,\tau) = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+) \tag{2.29}$$

under the peridocity condition

$$X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau) \ . \tag{2.30}$$

In lightcone coordinates, (2.14) become

$$T_{++} = \frac{1}{2}(T_{00}+T_{01}) \ = \ -\frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu = 0 \tag{2.31}$$

$$T_{--} = \frac{1}{2}(T_{00}-T_{01}) \ = \ -\frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu = 0 \ . \tag{2.32}$$

Since we are dealing with a string, it is very natural to describe it in terms of its oscillation modes. We can thus write down the most general solution of (2.23) as the Fourier expansion:

$$X^\mu_R \ = \ \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in\sigma^-} \tag{2.33}$$

$$X^\mu_L \ = \ \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in\sigma^+} \tag{2.34}$$

We have chosen the prefactors for future convenience. Namely, with these definitions the Fourier coefficients α and $\tilde{\alpha}$ are dimensionless. We want X^μ to be real. It follows that the Fourier coefficients have to satsify the reality conditions

$$x^\mu = (x^\mu)^* \qquad p^\mu = (p^\mu)^* \qquad \alpha^\mu_n = (\alpha^\mu_{-n})^* \qquad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^* \ . \tag{2.35}$$

The Poisson brackets are given by

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\} \ = \ im\delta_{m+n}\eta^{\mu\nu} \tag{2.36}$$

$$\{\alpha^\mu_m, \tilde{\alpha}^\nu_n\} \ = \ 0 \tag{2.37}$$

$$\{p^\mu, x^\nu\} \ = \ \eta^{\mu\nu} \tag{2.38}$$

For convenience we will in the future define α_0 and $\tilde{\alpha}_0$ as

$$\alpha^\mu_0 = \tilde{\alpha}^\mu_0 := \frac{1}{2}\ell_s p^\mu \ . \tag{2.39}$$

Generators of spacetime Poincaré transformations: momentum and angular momenta:

$$P^\mu = T\int_0^\pi d\sigma \dot{X}^\mu(\sigma,0) = \pi T(l\alpha_0^\mu + l\tilde{\alpha}_0^\mu) = p^\mu \tag{2.40}$$

and

$$J^{\mu\nu} = T\int_0^\pi d\sigma X^\mu(\sigma,0)\dot{X}^\nu(\sigma,0) - X^\nu(\sigma,0)\dot{X}^\mu(\sigma,0) = l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} \tag{2.41}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{2.42}$$

and

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu) \tag{2.43}$$

and analog for $\tilde{E}^{\mu\nu}$. Next we want to implement the constraint equations (2.14). To this end let us compute the Fourier modes of T_{++} and T_{--}. Note that we have

$$\partial_- X = \sqrt{2\alpha'}\sum_n \alpha_n^\mu e^{-2in\sigma^-} \ , \qquad \partial_+ X = \sqrt{2\alpha'}\sum_n \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \ . \tag{2.44}$$

We can now compute the Fourier modes of T_{--} and T_{++} at $\tau = 0$,

$$L_m \ := \ -\frac{1}{4\pi}\int_0^\pi e^{-2im\sigma}T_{--}d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \tag{2.45}$$

$$\tilde{L}_m \ := \ -\frac{1}{4\pi}\int_0^\pi e^{2im\sigma}T_{++}d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \tag{2.46}$$

Imposing (2.14) thus amounts to setting $L_m = \tilde{L}_m = 0$. Note that *worldsheet* time and position translations $\tau \mapsto \tau + a$, $\sigma \mapsto \sigma + b$ are generated by $L_0 + \tilde{L}_0$ and $L_0 - \tilde{L}_0$ respectively, whereas *spacetime* translations are generated by P^μ (2.40).

Let us pause for a moment and take stock. We want to interpret our oscillating string as a particle. The intuition is that the string is very small, and thus appears pointlike. The position of the particle is given by the position of the center of mass of the string, x^μ. The momentum of this particle is given by P^μ, and its angular momentum is given by $J^{\mu\nu}$. We know that the mass M of the particle can be obtained from

$$P^\mu P_\mu = -M^2 \ . \tag{2.47}$$

Using the constraint equations, we can compute this from

$$0 = L_0 + \tilde{L}_0 = \frac{1}{4}\ell_s^2 p^2 + \sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) \tag{2.48}$$

Using $P^\mu = p^\mu$ we thus get for the mass of the particle

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (|\alpha_n|^2 + |\tilde{\alpha}_n|^2) \ . \tag{2.49}$$

The mass M is thus determined by the oscillation spectrum. Moreover from the constraint equation $L_0 - \tilde{L}_0 = 0$ we find that

$$\sum_{n=1}^{\infty} |\alpha_n|^2 = \sum_{n=1}^{\infty} |\tilde{\alpha}_n|^2 \ , \tag{2.50}$$

that is the total right moving excitation is equal to the left-moving excitation.

Similarly the spin of the particle is given by $E^{\mu\nu} + \tilde{E}^{\mu\nu}$. A string with a given oscillation spectrum $\alpha_n, \tilde{\alpha}_n$ gives rise to a particle of mass and spin determined by (2.49) and (2.43). Its position and momentum are given by the center of mass position and the total momentum of the string.

Chapter 3

Quantization

3.1 A first naive attempt

There is a standard recipe for quantization. Starting with a set of classical observables on some phase space, we find a Hilbert space $\mathcal{H}$ where the observables are implemented by self-adjoint linear operators. The commutator of these operators is given by replacing the Poisson bracket of the corresponding observables,

$$\{\cdot,\cdot\} \to i[\cdot,\cdot] \ . \tag{3.1}$$

(There is of course a factor $\hbar$ which as usual we set to 1). Complex conjugation of observables becomes taking adjoints,

$$\cdot^* \to \cdot^\dagger \ . \tag{3.2}$$

This recipe is usually ambiguous due to issues of operator ordering etc. Very often however there is a relatively natural way of quantizing.

In our case, the very simple form of the Poisson brackets (2.36-2.38) suggests that the string oscillation modes α should be the starting point for quantizing the string. (2.36-2.38) thus become the commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu} \tag{3.3}$$

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0 \tag{3.4}$$

$$[p^\mu, x^\nu] = -i\eta^{\mu\nu} \tag{3.5}$$

and (2.35) become

$$x^\mu = (x^\mu)^\dagger \qquad p^\mu = (p^\mu)^\dagger \qquad \alpha^\mu_n = (\alpha^\mu_{-n})^\dagger \qquad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^\dagger \ . \tag{3.6}$$

What about the Hilbert space $\mathcal{H}$? (3.5) are simply the standard commutation relations of a particle in D dimensions. Ignoring the usual issues of non-normalizable states, we can take $\mathcal{H}$ to be spanned by eigenstates $|p^\sigma\rangle$ of p^σ. Not surprisingly, the oscillator modes satisfy the commutation relations of an infinite family of harmonic oscillators.

We can therefore proceed in the same fashion as for the harmonic oscillator: we take $|p^\sigma\rangle$ to be the ground state that is annihilated by all positive modes,

$$\alpha_n^\mu|p^\sigma\rangle = \tilde{\alpha}_n^\mu|p^\sigma\rangle = 0 \qquad \forall\mu, n > 0 \ . \tag{3.7}$$

The Hilbert space $\mathcal{H}$ is then spanned by the action of all negative modes, that is by states of the form

$$\alpha_{-m_1}^{\mu_1}\cdots\alpha_{-m_i}^{\mu_i}\tilde{\alpha}_{-n_1}^{\nu_1}\cdots\tilde{\alpha}_{-n_j}^{\nu_j}|p^\sigma\rangle \qquad m_k, n_k > 0, \nu_k, \mu_k = 1,\ldots,D \quad \forall k \ . \tag{3.8}$$

The question is, is this space really a Hilbert space with a positive definite inner product? The answer to this is no. To see this we compute the norms

$$|\alpha_{-m}^\mu|p^\sigma\rangle|^2 = \langle p^\sigma|\alpha_m^\mu\alpha_{-m}^\mu|p^\sigma\rangle = \langle p^\sigma|[\alpha_m^\mu, \alpha_{-m}^\mu]|p^\sigma\rangle = m\eta^{\mu\mu} \ . \tag{3.9}$$

Crucially, the spacetime Minkowski metric $\eta^{\mu\nu}$ is not positive definite: We have $\eta^{00} = -1$. This means that the states generated by the time direction modes α^0 and $\tilde{\alpha}^0$ have negative norm. $\mathcal{H}$ is thus not an acceptable Hilbert space. This has far-reaching consequences.

Taking a step back, it is actually not too surprising that our first naive approach failed. The problem is that we have chosen a Hilbert space $\mathcal{H}$ which is too big. The Hilbert space $\mathcal{H}^{phys}$ of actual *physical states* should be smaller, and hopefully only contain states of positive norms!

So what went wrong? One way to think about the problem is to note that we haven't imposed the constraint equation (2.14). We could therefore try to define $\mathcal{H}^{phys}$ as the space of states on which $T = 0$, and hope that all such states have positive norm. This is called *covariant quantization.*

There is another way to think about this. Let us return for a moment to the symmetries of the Polyakov action. It turns out that even by choosing the worldsheet metric $h = \eta$ has not completely fixed the choice of coordinates. In fact if we can combine a reparametrization with a Weyl symmetry by choosing ξ^a such that

$$\partial^a\xi^b + \partial^b\xi^a = \Lambda\eta^{ab} \ . \tag{3.10}$$

This means that we have not completely fixed the gauge, which means that the space of states $\mathcal{H}$ is too big, since two states which differ by a gauge transformation are physically the same. The second approach is thus to fix the remaining gauge choice, thereby eliminating unphysical degrees of freedom, and then only quantizing the actual physical degrees of freedom, thus ending up with $\mathcal{H}^{phys}$. For reasons that will become obvious in just a bit, this is called *lightcone quantization.* For the moment we will choose the second approach. We will return to the first approach later. It turns out that both approaches give the same result.

3.2 Worldsheet conformal symmetry : classical

Let us first discuss the residual gauge symmetry (3.10) in more detail. Such a coordinate transformation leaves the metric invariant up to a (position dependent) rescaling Λ. In particular this means that it is angle-preserving or *conformal*. This remaining *conformal symmetry* plays a crucial role. In fact we will devote several lectures to analyzing it systematically. For the moment we will only discuss some aspects.

For the gauge fixed Polyakov action (2.22), we can immediately see (3.10) by going to lightcone coordinates. We have $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ and $\eta_{++} = \eta_{--} = 0$, so that (3.10) in terms of lightcone coordinates we find the conditions

$$\partial^- \xi^- = \partial^+ \xi^+ = \partial_+ \xi^- = \partial_- \xi^+ = 0 \qquad \Lambda = \partial_- \xi^- + \partial_- \xi^+ \ , \tag{3.11}$$

that is for any pair $\xi^\pm(\sigma^\pm)$ we can find a corresponding Λ to make it into a conformal transformation. This means we can arbitrarily reparametrize left- and right-moving coordinates independently

$$\sigma^\pm \mapsto \tilde{\sigma}^\pm(\sigma^\pm) \tag{3.12}$$

Writing (2.22) in lightcone coordinates

$$S = 2T \int d\sigma^+ d\sigma^- \partial_- X^\mu \partial_+ X_\mu \tag{3.13}$$

we immediately see that this is a symmetry. Actually this symmetry is generated by the modes L_m and $\tilde{L}_m$ of the energy-momentum tensor. To see this, let us introduce coordinates $z_\pm = e^{2i\sigma^\pm}$. One can then show that

$$\{L_m, X\} = i z_-^{m+1} \partial_{z_-} X \qquad \{\tilde{L}_m, X\} = i z_+^{m+1} \partial_{z_+} X \tag{3.14}$$

i.e. they generate the coordinate transformations $z_\pm \mapsto z_\pm + i z_\pm^{m+1}$. Using the Jacobi identity then immediately shows that the L_m satisfy the *Witt algebra*

$$\{L_m, L_n\} = i(m-n)L_{m+n} \ . \tag{3.15}$$

How can we deal with the conformal transformations (3.12)? From the perspective of string theory, those conformal transformations are residual gauge transformations. One way to deal with gauge symmetries is to impose that observables are invariant under them. This is exactly what the constraints

$$L_m = 0 \ , \qquad \tilde{L}_m = 0 \tag{3.16}$$

impose. Another way is to simply fix the residual gauge symmetry. To this end, write

$$\tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-)) \qquad \tilde{\sigma} = \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) - \tilde{\sigma}^-(\sigma^-)) \tag{3.17}$$

This asserts that we can write $\tilde{\tau}$ as an arbitrary solution of the wave equation $\partial_-\partial_+\tilde{\tau} =$

0. What about $\tilde{\sigma}$? Once we fix $\tilde{\tau}(\sigma,\tau)$, the functions $\tilde{\sigma}^{\pm}(\sigma^{\pm})$ are fixed completely up to a constant which we can shift between the two. This means that $\tilde{\sigma}$ is fixed up to a constant shift,

$$\tilde{\sigma} \mapsto \tilde{\sigma} + c \; . \tag{3.18}$$

This shows that once we fix $\tilde{\tau}$, the worldsheet coordinates are almost completely fixed. We will have to implement by hand invariance of physical observables under the remaining gauge symmetry (3.18).

How do we fix $\tilde{\tau}$? Note that it satisfies the same equation as any component X^{μ}. We can thus fix it to be proportional to such a component. Let us introduce (partial) light-cone coordinates in spacetime: Define

$$X^{\pm} = \frac{X^0 \pm X^1}{\sqrt{2}} \; . \tag{3.19}$$

The inner product of vectors in light cone coordinates is

$$V \cdot W = \sum_{i=2}^{D-1} V^i W^i - V^+ W^- - V^- W^+ \; . \tag{3.20}$$

So called *light-cone gauge* on the worldsheet is then to choose $\tau = X^+(\sigma,\tau)/p^+ +$ *constant*, so that in the new light-cone worldsheet coordinates

$$X^+(\tau,\sigma) = x^+ + \ell_s^2 p^+ \tau \; . \tag{3.21}$$

This way we have fixed the oscillation modes α^+ and $\tilde{\alpha}^+$ to zero. To deal with the oscillator modes α^- and $\tilde{\alpha}^-$, note that the constraints equations (2.24-2.25) can be written as

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \tag{3.22}$$

which become

$$\sum_{i=2}^{D-1} (\partial_{\pm} X^i)^2 = 2\partial_{\pm} X^+ \partial_{\pm} X^- = \ell_s^2 p^+ \partial_{\pm} X^- \; . \tag{3.23}$$

From this we can extract the oscillator modes α^- and $\tilde{\alpha}^-$ as

$$\alpha_m^- = \frac{1}{\ell_s p^+} \sum_{i=2}^{D-1} \sum_n \alpha_{m-n}^i \alpha_n^i \qquad \tilde{\alpha}_m^- = \frac{1}{\ell_s p^+} \sum_{i=2}^{D-1} \sum_n \tilde{\alpha}_{m-n}^i \tilde{\alpha}_n^i \; . \tag{3.24}$$

In conclusion, fixing the worldsheet coordinates to light cone gauge has eliminated the oscillator modes $\alpha^{0,1}$ as degrees of freedom. This looks very promising: remember that the problem with unitarity came from the α^0, which now have been eliminated. We would therefore expect that the remaining $D-2$ modes span a positive definite Hilbert space.

3.3 Worldsheet conformal symmetry : quantum

Let us now turn to the quantum version of the theory. We have to be quite careful here, since quantization is a fairly subtle business. In particular there is the risk of *anomalies*: symmetries which are present in a classical theory, but which are broken once we quantize it. Checking for such anomalies unfortunately tends to involve long and tedious computations. The results however are often very simple, and will turn out to be crucial for our understanding of string theory!

Where do such anomalies come from in our case? As pointed out above, quantization is usually ambiguous. If we have a classical observable AB, it is not clear whether the quantum observable is given by $\hat{A}\hat{B}$, $\hat{B}\hat{A}$, or some linear combination thereof. Note that in the case of α, the ambiguity is not too bad: the two orderings differ at most by a constant. Still, it is possible that the choice of an ordering (or possibly all choices of ordering!) will break a symmetry of the classical theory and thus lead to an anomaly.

Let us see what happens for the Witt algebra. How shall we define the ordering for L_m? A very natural ordering is *normal ordering* $:\cdot:$,

$$:\alpha_m^\mu \alpha_n^\nu := \begin{cases} \alpha_m^\mu \alpha_n^\nu & : n \geq m \\ \alpha_n^\nu \alpha_m^\mu & : n < m \end{cases} \tag{3.25}$$

and similarly for the normal ordered product of multiple modes. In particular this means that annihilation operators are always to the right of creation operators.

Let us for the moment simply take normal ordering as the correct prescription so that

$$L_m = \frac{1}{2}\sum_n :\alpha_{m-n}\cdot\alpha_n: \tag{3.26}$$

Let us now compute the quantum algebra of the L_m. First we compute

$$\begin{aligned}
[L_m, \alpha_n^\nu] &= \frac{1}{2}\Big(\sum_{k\geq m/2}[\alpha_{m-k}\cdot\alpha_k, \alpha_n^\nu] + \sum_{k<m/2}[\alpha_k\cdot\alpha_{m-k}, \alpha_n^\nu]\Big) \\
&= \frac{1}{2}\Big(\sum_{k\geq m/2} k\alpha_{m-k}^\nu \delta_{k+n} + (m-k)\alpha_k^\nu \delta_{m-k+n} + \sum_{k<m/2}(m-k)\alpha_k^\nu\delta_{m-k+n} + k\alpha_{m-k}^\nu \delta_{k+n}\Big) \\
&= \frac{1}{2}\Big(\sum_k k\alpha_{m-k}^\nu\delta_{k+n} + (m-k)\alpha_k^\nu\delta_{m-k+n}\Big) = -n\alpha_{m+n}^\nu \ .
\end{aligned} \tag{3.27}$$

We see that ordering didn't play any role here. Let us now turn to

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\Big(\sum_{k\geq n/2}[L_m, \alpha_{n-k}\cdot\alpha_k] + \sum_{k<n/2}[L_m, \alpha_k\cdot\alpha_{n-k}]\Big) \\
&= \frac{1}{2}\Big(\sum_{k\geq n/2}(k-n)\alpha_{n-k+m}\cdot\alpha_k - k\alpha_{n-k}\cdot\alpha_{m+k} + \sum_{k<n/2}(-k\alpha_{k+m}\cdot\alpha_{n-k} + (k-n)\alpha_k\cdot\alpha_{m+n-k}\Big)
\end{aligned} \tag{3.28}$$

So far, the result looks very similar to the classical Witt algebra: we can already see the bilinear modes which will lead to L_{m+n}. For this however we need to normal order the modes. It is useful to distinguish two cases. Assume first that $m \neq n$. In this case we immediately see that all the α commute, so that we can trivially establish normal ordering to get

$$\begin{aligned}\frac{1}{2}(\sum_k (k-n) : \alpha_{n+m-k} \cdot \alpha_k : -k : \alpha_{n-k} \cdot \alpha_{k+m} :)& \\ = \frac{1}{2}(\sum_k (k-n) : \alpha_{n+m-k} \cdot \alpha_k : +(m-k) : \alpha_{n+m-k} \cdot \alpha_k :)&\end{aligned} \tag{3.29}$$

where we have shifted $k \mapsto k-m$. We indeed recover the classical result

$$[L_m, L_n] = (m-n)L_{m+n} \qquad m \neq n \ . \tag{3.30}$$

For $m = -n$, we have to more careful, since restoring normal ordering will lead to additional terms coming from the non-vanishing commutators. Assuming $m > 0$,

$$\begin{aligned}[L_m, L_{-m}] = \frac{1}{2}(\sum_{k \geq -m/2} (k+m)\alpha_{-k} \cdot \alpha_k - k\alpha_{-m-k} \cdot \alpha_{m+k}& \\ + \sum_{k < -m/2} (-k\alpha_{k+m} \cdot \alpha_{-m-k} + (k+m)\alpha_k \cdot \alpha_{-k})&\end{aligned} \tag{3.31}$$

Picking the first and last term gives

$$\begin{aligned}\frac{1}{2}(\sum_{k\geq 0} (k+m)\alpha_{-k} \cdot \alpha_k + \sum_{0>k\geq -m/2} (k+m)\alpha_{-k} \cdot \alpha_k + \sum_{k<-m/2} (k+m)\alpha_k \cdot \alpha_{-k})& \\ = \frac{1}{2}(\sum_k (k+m) : \alpha_{-k}\alpha_k : -D \sum_{0>k\geq -m/2} k(k+m))&\end{aligned} \tag{3.32}$$

where we have used the commutator

$$\alpha_k \cdot \alpha_{-k} - \alpha_{-k} \cdot \alpha_k = kD \ . \tag{3.33}$$

This follows from (3.3) by appropriate contractions. Note that due to the contraction every dimension gives the same contribution k, regardless of the sign of the commutator in (3.3). For the second and third term, we shift $k \mapsto k-m$ to get

$$\begin{aligned}\frac{1}{2}(\sum_{k\geq m/2} (m-k)\alpha_{-k} \cdot \alpha_k + \sum_{m/2>k\geq 0} (m-k)\alpha_k \cdot \alpha_{-k} + \sum_{k<0} (m-k)\alpha_k \cdot \alpha_{-k})& \\ = \frac{1}{2}(\sum_k (m-k) : \alpha_{-k} \cdot \alpha_k : +D \sum_{m/2>k\geq 0} k(-k+m))&\end{aligned} \tag{3.34}$$

After a shift of the first sum $k \mapsto k - m$, in total we thus get

$$m \sum_k : \alpha_{-k} \cdot \alpha_k : + D \sum_{m-1 \geq k \geq 0} k(m-k) = 2mL_0 + \frac{D}{12} m(m^2 - 1) \ . \tag{3.35}$$

After a this long and tedious computation, it turns out that we have indeed picked up a term different from the classical result! We can obtain the case $m < 0$ by antisymmetry of the commutator. In total the quantum algebra of the L_m is thus given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12} m(m^2 - 1)\delta_{m+n,0} \ . \tag{3.36}$$

This is called the *Virasoro algebra*, and it will play a crucial role in what follows. It is the extension of the Witt algebra by a central element D, the *central charge*. As it is different from the Witt algebra, it seems like that conformal symmetry is anomalous.

We may wonder if the conformal anomaly is indeed physical, or if we simply chose a bad ordering prescription. Note that the ordering of the L_m for $m \neq 0$ is unambiguous, since the contributing α commute. The only ambiguity is in the definition of L_0. A different ordering would simply shift L_0 by a constant. If we thus define a differently ordered algebra $L'_n := L_n + a\delta_{n,0}$, the commutation relation would be

$$[L'_m, L'_n] = (m-n)L'_{m+n} + \frac{D}{12} m(m^2 - 1) - a\delta_{n,0} \ . \tag{3.37}$$

This shows that the central charge D is independent of the choice of ordering, and is thus an actual physical anomaly of the quantum theory.

At first sight this may sound worrisome, since it implies that the quantum theory is no longer conformal. The situation is not that bad however, and the conformal anomaly is a very mild anomaly. In particular (3.36) is a *central extension* of the Witt algebra. This is a very natural phenomenon in quantum mechanics. Since states in quantum mechanics are given not by vectors in a Hilbert space but rather by rays, symmetries are not representations, but rather *projective representations* of the symmetry group. Considering those is equivalent to considering central extensions of the symmetry group. In hindsight the apparence of the central term is thus not too surprising.

Note however that for the purposes of string theory, the central term is potentially problematic. This is essentially because we take conformal symmetry to be a gauge symmetry. In particular when taking the covariant approach to quantization, we want to impose the constraints $L_m = 0$ after quantization. More precisely these means that we want all physical states to be annhiliated by all the L_m. This leads to immediate problems. We cannot for instance ask for the vacuum to be annihilated by all L_m due to the central term, since we know that

$$[L_m, L_{-m}]|0\rangle = \frac{D}{12} m(m^2 - 1)|0\rangle \neq 0 \ . \tag{3.38}$$

The best we can hope for is to demand that physical states are annihilated by say all positive modes $L_m, m > 0$. It is not clear however that this is strong enough to eliminate all negative norm states. We will return to this question later on. For the moment we will return to light cone quantization.

3.4 Poincaré anomaly

We will now try to quantize the theory after having fixed light cone gauge. Nothing changes for the transverse modes α^i_m , $i = 2, \dots D-1$. The α^+ have been eliminated completely. The subtle part are the α^-, which are fixed by the transverse modes through (3.24). For quantization, we have to specify the ordering of (3.24) though. As usual, the α^-_m , $m \neq 0$ have no ordering ambiguity. We will thus choose

$$\alpha^-_m = \frac{2}{\ell_s p^+}(\frac{1}{2}\sum_{i=2}^{D}\sum_n :\alpha^i_{m-n}\alpha^i_n: - a\delta_{n,0}) = \frac{2}{\ell_s p^+}(L^\perp_m - a\delta_{m,0}) \tag{3.39}$$

where $L^\perp$ are the Virasoro generators coming from the transverse coordinates only. They satisy (3.36) with central charge $D-2$ rather than D.

What could go wrong after quantization? By going to lightcone coordinates, we break manifest Poincaré symmetry. Classically, it is clear that this does not pose any problems, and the generators give the Poincaré algebra. Quantization may lead to *anomalies*. The problem is in the ordering prescription, *i.e.* in the inherent ambiguity of quantization. Once we have broken a manifest symmetry, there is no guarantee that we can find an ordering prescription that is compatible with the symmetry of the system! Let us therefore check whether the generators $J^{\mu\nu}$ still satisfy the Poincaré algebra.

We could of course evaluate all the commutators by brute force to check. Let us try to be more efficient though. First note that all completely transverse generators J^{ij} are unaffected by light-cone gauge fixing. Their commutators are thus simply the classical expressions, and won't pick up any anomalies. Next, any generators involving the $+$ direction will be unproblematic too, since we have eliminated the modes α^+. The only problematic commutators are thus of the form $[J^{i-}, J^{j-}]$, which classically vanish.

When computing anomalies, it is useful to first constrain the form of a possible anomaly. The commutator potentially contains terms quartic and quadratic in α. It cannot contain a constant term, since it must transform non-trivially under $SO(1, D-1)$, and the oscillator modes are the only objects carrying $SO(1, D-1)$ indices. The quartic terms cannot lead to an anomaly, since they are obtained in the same way as classically, *i.e.* they have no ordering ambiguities. Therefore the commutator can only contain quadratic term. Finally it is clear from the form of E^{ij} that the total level has to be zero. Together with antisymmetry and the fact that E^{i-} annihilates the vacucum this means that

$$[J^{i-}, J^{j-}] = -\frac{1}{(\ell_s p^+)^2}\sum_{m=1}^{\infty}\Delta_m(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m) + \text{right movers} . \tag{3.40}$$

We thus only need to determine the c-function Δ_m. To do this we evaluate the matrix element

$$\Xi^{ij} = -(\ell_s p^+)^2 \langle 0|\alpha^i_m [J^{i-}, J^{j-}] \alpha^j_{-m}|0\rangle \ . \tag{3.41}$$

Note that the right movers $\tilde{E}^{\mu\nu}$ do not give any contribution to Ξ. To obtain their contribution, we would have to repeat the computation with the analogue matrix element. From (3.40) we obtain indeed

$$\Xi^{ij} = m^2 \Delta_m \ . \tag{3.42}$$

Let us now compute Δ_m by actually evaluating the commutator for the matrix element. The commutator will thus only give contributions of the following form:

$$[J^{i-}, J^{j-}] = [l^{i-}, l^{j-}] + [E^{i-}, E^{j-}] + [l^{i-}, E^{j-}] + [E^{i-}, l^{j-}] \tag{3.43}$$

Let us now evaluate those contributions separately.

- $[l^{i-}, l^{j-}]$: We have $[l^{i-}, l^{j-}] = 0$, since there are no oscillator modes involved. This term does not give a contribution.

- $[E^{i-}, E^{j-}]$: It is useful to write the contribution in terms of an inner product of states. Define

  $$i\ell_s p^+ E^{j-} \alpha^j_{-m}|0\rangle = |A_j\rangle - |B_j\rangle \ , \tag{3.44}$$

  and also

  $$i\ell_s p^+ E^{i-} \alpha^j_{-m}|0\rangle = |C_{ij}\rangle \ , \tag{3.45}$$

  so that

  $$-(\ell_s p^+)^2 \langle 0|\alpha^i_m [E^{i-}, E^{j-}] \alpha^j_{-m}|0\rangle = \langle A_i|A_j\rangle + \langle B_i|B_j\rangle - \langle A_i|B_j\rangle - \langle B_i|A_j\rangle - \langle C_{ji}|C_{ij}\rangle \ . \tag{3.46}$$

  Using (2.43) and (3.39) we find that we should define those states as

  $$|A_j\rangle := 2\sum_{l=1}^{\infty} \frac{1}{l} (\alpha^j_{-l} L^\perp_l) \alpha^j_{-m}|0\rangle = 2m \sum_{l=1}^{m} \frac{1}{l} \alpha^j_{-l} \alpha^j_{l-m}|0\rangle \tag{3.47}$$

  $$|B_j\rangle := 2\sum_{l=1}^{\infty} \frac{1}{l} (L^\perp_{-l} \alpha^j_l) \alpha^j_{-m}|0\rangle = 2L^\perp_{-m}|0\rangle \tag{3.48}$$

  and

  $$|C_{ij}\rangle = 2\sum_{l=1}^{\infty} \frac{1}{l} (\alpha^i_{-l} L^\perp_l) \alpha^j_{-m}|0\rangle = 2m \sum_{l=1}^{m} \frac{1}{l} \alpha^i_{-l} \alpha^j_{l-m}|0\rangle \qquad i \neq j \ . \tag{3.49}$$

  To evaluate (3.46), we compute

  $$\langle A_i|A_j\rangle = 0 \quad (i \neq j) \ , \qquad \langle B_i|B_j\rangle = 4\langle 0|[L^\perp_m, L^\perp_{-m}]|0\rangle \tag{3.50}$$

and

$$\langle A_i|B_j\rangle = \langle B_i|A_j\rangle = 4m\sum_{l=1}^{m}(m-l) = 2m^2(m-1) \tag{3.51}$$

and

$$\begin{aligned}\langle C_{ji}|C_{ij}\rangle &= 4m^2\sum_{k,l=1}^{m}\frac{1}{kl}\langle 0|\alpha^i_{m-k}\alpha^j_k\alpha^i_{-l}\alpha^j_{l-m}|0\rangle \\ &= 4m^2\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\langle 0|\alpha^i_{m-k}\alpha^i_{k-m}\alpha^j_k\alpha^j_{-k}|0\rangle = 4m^2(m-1)\end{aligned} \tag{3.52}$$

In total we thus get

$$-(\ell_s p^+)^2\langle 0|\alpha^i_m[E^{i-},E^{j-}]\alpha^j_{-m}|0\rangle = 4\langle 0|[L^\perp_m,L^\perp_{-m}]|0\rangle - 8m^2(m-1) \tag{3.53}$$

- $[l^{i-},E^{j-}]$: Introducing

$$|D_{ij}\rangle := i\ell_s p^+ l^{i-}\alpha^j_{-m}|0\rangle = i\ell_s p^+(x^ip^- - x^-p^i)\alpha^j_{-m}|0\rangle \tag{3.54}$$

we have

$$-(\ell_s p^+)^2\langle 0|\alpha^i_m[l^{i-},E^{j-}]\alpha^j_{-m}|0\rangle = \langle D_{ii}|A_j\rangle - \langle D_{ii}|B_j\rangle - \langle C_{ji}|D_{ij}\rangle \tag{3.55}$$

We can then use that p^i commutes with all oscillator modes and annihilates $|0\rangle$ to get

$$|D_{ij}\rangle = \frac{4i}{\ell_s}x^i(L^\perp_0 - a)\alpha^j_{-m}|0\rangle = \frac{4i}{\ell_s}(m-a)x^i\alpha^j_{-m}|0\rangle\ . \tag{3.56}$$

Using this expression we get $\langle A|D\rangle = \langle C|D\rangle = 0$, since there is an odd number of oscillator modes. For $\langle B|D\rangle$ however we can get an even number of oscillator modes from the term $p^j\alpha^j_m$ in $L^\perp_m$, which leads to

$$\langle B_i|D_{jj}\rangle = \frac{8i}{\ell_s}(m-a)\langle 0|L^\perp_m x^j\alpha^j_{-m}|0\rangle = 4(m-a)\langle 0|\alpha^j_m\alpha^j_{-m}|0\rangle = 4(m^2-ma) \tag{3.57}$$

where we have used $[L^\perp_m,x^j] = -i\frac{1}{2}\ell_s\alpha^j_m$. In total we thus get

$$-(\ell_s p^+)^2\langle 0|\alpha^i_m[l^{i-},E^{j-}]\alpha^j_{-m}|0\rangle = -4m^2+4ma\ . \tag{3.58}$$

- $[E^{i-},l^{j-}]$: A completely analogous computation gives

$$-(\ell_s p^+)^2\langle 0|\alpha^i_m[E^{i-},l^{j-}]\alpha^j_{-m}|0\rangle = \langle A_i|D_{jj}\rangle - \langle B_i|D_{jj}\rangle - \langle D_{ji}|C_{ij}\rangle = -4m^2+4ma\ . \tag{3.59}$$

Collecting all the contributions we thus get

$$\frac{1}{4}\Xi^{ij} = \langle 0|[L^\perp_m, L^\perp_{-m}]|0\rangle - 2m^2(m-1) - 2m^2 + 2am \tag{3.60}$$

so that

$$\frac{1}{4}m^2\Delta_m = -2m^3 + \frac{D-2}{12}m(m^2-1) + 2am \tag{3.61}$$

We have indeed found an anomaly. The bad news is that this time, it is a serious one: it's not just a central extension. It seems like we broke spacetime Poincaré symmetry. The good news is that under very specific conditions we can make it vanish: if we choose

$$D = 26\ , \qquad a = 1\ , \tag{3.62}$$

then $\Delta_m = 0$. A completely analogue computation for the right moving matrix element $\tilde{\Xi}$ gives of course again $D = 26$ and $\tilde{a} = 1$. This implies that bosonic string theory is only consistent in 26 dimensions.[1] This may come as a disappointment, since we would have liked $D = 4$. There are ways around this which we discuss in the next section.

3.5 Spectrum

For the moment let us just go with this and reap the reward of our long computations.

The Hilbert space is now spanned by the 24 left moving transverse modes α^i_{-m} and 24 right moving transverse modes $\tilde{\alpha}^i_{-m}$. Since their commutator is

$$[\alpha^i_m, \alpha^j_n] = m\delta_{m+n}\delta^{ij} \tag{3.63}$$

all states now have manifestly positive norm. There is one last detail to take care of: we still left the residual gauge symmetry (3.18), $\sigma \mapsto \sigma + c$ unfixed. Physical states in our Hilbert space are thus given by states which are invariant under that symmetry. On the physical states this symmetry is generated by $L^\perp_0 - \tilde{L}^\perp_0$, as can be seen from

$$[L^\perp_0 - \tilde{L}^\perp_0, X^i(\tau,\sigma)] = \partial_\sigma X^i(\tau,\sigma)\ . \tag{3.64}$$

We thus impose the *level-matching condition*

$$(L^\perp_0 - \tilde{L}^\perp_0)|\phi\rangle = 0\ . \tag{3.65}$$

Define the *level operators*

$$N = \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} \alpha^i_{-n}\alpha^i_n \qquad \tilde{N} = \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \tag{3.66}$$

[1] You may have heard that it string theory needs 10, not 26 dimensions. This is indeed the case for superstring theory. We will return towards the end of the semester.

They are related to the usual number counting operators N_{in} as

$$N = \sum_{n=1}^{\infty} \sum_{i=2}^{D-1} n N_{in} \ . \tag{3.67}$$

The Virasoro modes are related to the number operators by

$$L_0^\perp = \frac{1}{8}\ell_s^2 p^i p_i + N \qquad \tilde{L}_0^\perp = \frac{1}{8}\ell_s^2 p^i p_i + \tilde{N} \ . \tag{3.68}$$

Using

$$p^- = \frac{1}{\ell_s}(\alpha_0^- + \tilde{\alpha}_0^-) = \frac{2}{\ell_s^2 p^+}(L_0^\perp + \tilde{L}_0^\perp - 2) \tag{3.69}$$

we can obtain the mass as

$$\begin{aligned} M^2 = -P^2 = -p^i p_i + 2p^+ p^- &= -p^i p_i + \frac{4}{\ell_s^2}(\frac{1}{4}\ell_s^2 p^i p_i + N + \tilde{N} - 2) \\ &= \frac{2}{\alpha'}(N + \tilde{N} - 2) \ , \end{aligned} \tag{3.70}$$

and the level-matching condition (3.65) becomes

$$N = \tilde{N} \ . \tag{3.71}$$

As in the classical case, the mass of the particles is determined by the oscillator excitations, which of course are quantized now. In fact, (3.70) is almost exactly the same as (2.49), the only differences being that there is a shift of the energy, and that only 24 rather than all 26 oscillation modes can be excited. Let us analyze the spectrum level by level.

- $N = \tilde{N} = 0$: The lightest state is given by

$$|p^\mu\rangle \ . \tag{3.72}$$

  Its mass is

$$M^2 = -\frac{4}{\alpha'} \ . \tag{3.73}$$

  Because it has negative mass squared, this is has spacelike momentum, *i.e.* it is a tachyon. The usual interpretation of a tachyon is that the theory is not stable, or more precisely, that we're expanding around an unstable vacuum. Take for instance a scalar field with a Mexican hat potential,

$$\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{3.74}$$

  with $V(\phi) = -m^2\phi^2 + \lambda\phi^4$, $\lambda > 0$. If we try to expand around the vacuum $\phi = 0$, we find that ϕ has negative mass squared. From the potential it is clear that this point is unstable, and we should expand around the bottom of the potential, where we indeed find positive mass.

In any case, having a tachyon in the spectrum is a problem. It is one of the reason why we will consider the superstring later on, the other reason being that there are no fermions in the spectrum.

- $N = \tilde{N} = 1$: The next lowest states are given by

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|p^\mu\rangle \tag{3.75}$$

and have mass 0. Clearly they are in the representation $24 \otimes 24$ of $SO(24)$, where 24 is the fundamental representation. This actually provides a consistency check for our results: For massive particles, the little group is $SO(D-1)$, and only for massless particle is it $SO(D-2)$. If we had chosen a different normal ordering constant a, then these states would not be massless.

To elaborate on this, as a start consider a scalar particle in D dimensions, given by a state $|p^\mu\rangle$. Under a Poincaré transformation Λ it transforms as

$$|p^\mu\rangle \mapsto |\Lambda^\nu_\mu p^\mu\rangle =: |\Lambda p\rangle \ . \tag{3.76}$$

A non-scalar particle will carry some aditional index α, and it will transform as

$$|\alpha, p\rangle \mapsto U(\Lambda)^\alpha_\beta|\alpha, \Lambda p\rangle \tag{3.77}$$

where $U(\Lambda)$ is some unitary representation matrix of the Poincaré group. Let us now restrict to transformations which keep p^μ fixed, *i.e.* the stabilizer subgroup. This group is called the *little group*. Since for such transformations clearly $|\alpha, p\rangle \mapsto U(\Lambda)^\alpha_\beta|\alpha, p\rangle$, it follows that particle states of a fixed momentum form a (possibly projective) representation of the little group. It is clear that the little group does not depend on the direction of the momentum we choose. For a massive particle, we can thus always go to the rest frame and choose $p^\mu = (m, 0, \ldots 0)$. The little group is then $SO(D-1)$. For $D = 4$ for instance this explains why the spin of massive particles is given by (projective) representations of $SO(3)$. For a massless particle we can always choose $p^\mu = (1, -1, 0, \ldots, 0)$. A subgroup of the little group is then $SO(D-2)$. This explains why in $D = 4$, massless particles such as photons have only two polarizations, even though they are spin 1 particles.

Decomposing (3.75) into irreducible representations of $SO(24)$ we get a symmetric traceless representation (graviton), antisymmetric (2-form gauge boson), and a scalar. Why do we identify the symmetric traceless representation with the graviton? Gravitational waves come from the linearized version of the Einstein equation. More precisely, we take a situation where the metric $g_{\mu\nu}(x)$ is only slightly perturbed away from Minkowski space,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad |h| \ll 1 \ , \tag{3.78}$$

where $h_{\mu\nu}(x)$ satisfies a wave equation. Gravitons are then the quantization of the perturbation $h_{\mu\nu}(x)$. By definition, the metric $g_{\mu\nu}$ (and therefore also $h_{\mu\nu}$ is symmetric. Moreover by taking the trace

$$D = g^{\mu}_{\mu} = \eta^{\mu}_{\mu} + h^{\mu}_{\mu} = D + h^{\mu}_{\mu} \tag{3.79}$$

we find that h is indeed traceless, $h^{\mu}_{\mu} = 0$. Naively, one might expect that h is in the symmetric traceless representation of $SO(1, D-1)$. Note however that just as in electrodynamics there remain gauge transformations, in this case coordinate transformations. Fixing a gauge then reduces the degrees of freedom to $SO(D-2)$. For $D = 4$ for example this means that gravitational waves have exactly two physical degrees of freedom, corresponding to "plus" polarization and "cross" polarization.

- $N = \tilde{N} = 2$: At level 2, the states are given by

$$\alpha^i_{-2}\tilde{\alpha}^j_{-2}|p^\mu\rangle\ , \alpha^i_{-1}\alpha^k_{-1}\tilde{\alpha}^j_{-2}|p^\mu\rangle\ ,\ \alpha^i_{-2}\tilde{\alpha}^j_{-1}\tilde{\alpha}^l_{-1}|p^\mu\rangle\ ,\ \alpha^i_{-1}\alpha^k_{-1}\tilde{\alpha}^j_{-1}\tilde{\alpha}^l_{-1}|p^\mu\rangle \tag{3.80}$$

Those states have mass

$$M^2 = \frac{4}{\alpha'}\ . \tag{3.81}$$

String theory thus predicts a great many massive states. How can this be compatible with experiment? If we choose α' very small, then all those states will have mass much higher than anything that can be measured experimentally. The standard model is thus an effective description. As $SO(24)$ representations, they decompose as

$$(24 \oplus Sym(24 \otimes 24)) \otimes (24 \oplus Sym(24 \otimes 24)) \tag{3.82}$$

It turns out that $(24 \oplus Sym(24 \otimes 24))$ is a symmetric traceless representation of $SO(25)$. Our results are thus consistent with $SO(25)$ as the little group for those massive states.

Chapter 4

2d Conformal Field Theories

4.1 Conformal symmetries

When discussing the bosonic string, we discovered that even after gauge fixing, the Polyakov action

$$S = \frac{T}{2}\int d\sigma^+ d\sigma^- \partial_- X^\mu \partial_+ X_\mu \tag{4.1}$$

still has the residual conformal symmetry

$$\sigma^\pm \mapsto \tilde{\sigma}^\pm(\sigma^\pm) \ . \tag{4.2}$$

We have also seen that this conformal symmetry of the worldsheet plays a crucial role, since it allows us to fix light-cone gauge. Let us therefore investigate conformal field theories in two dimensions more carefully. As usual, we will start out with classical considerations first, and then switch to the quantum versions. Note the following very important conceptual difference: In this chapter we take conformal symmetry to be an actual physical theory, whereas in string theory we considered it to be a gauge symmetry!

Let us briefly discuss theories in arbitrary dimension d. As a reminder about Noether's theorem, an (infinitesimal) coordinate transformation ϵ:

$$x^\mu \mapsto x^\mu + \epsilon^\mu(x) \tag{4.3}$$

which is a symmetry of the theory leads just like any other symmetry to some conserved Noether current

$$\partial_\mu J^\mu_\epsilon(x) = 0 \ . \tag{4.4}$$

As usual, we can obtain a corresponding Noether charge by integrating over a codimension 1 surface Σ,

$$Q_\epsilon = \int_\Sigma \hat{n}_\mu J^\mu_\epsilon(x) \ . \tag{4.5}$$

Since from (4.4) the divergence of J^μ_ϵ vanishes, we can deform the Σ without changing

Q_ϵ. The Noether current for a coordinate transformation is actually given by the energy momentum tensor,

$$J^\mu_\epsilon(x) := T^{\mu\nu}\epsilon_\nu(x) \ . \tag{4.6}$$

Let us now discuss what this implies for T. Translation invariance is $\epsilon^\mu(x) = \epsilon^\mu$, which gives energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$. Lorentz invariance with $\epsilon^\sigma(x) = \frac{1}{2}(\eta^{\mu\sigma}x^\nu - \eta^{\nu\sigma}x^\mu)$ gives $T^{\mu\nu} = T^{\nu\mu}$.

Finally, let us turn to conformal transformations, *i.e.* transformations that leave the metric invariant up to rescaling,

$$\eta^{\mu\nu} \mapsto \Lambda(x)\eta^{\mu\nu} \ . \tag{4.7}$$

For such transformations $\epsilon^\mu(x)$ has to satisfy

$$\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu = \Lambda\eta^{\mu\nu} \ , \tag{4.8}$$

which yields

$$\partial_\mu T^{\mu\nu}\epsilon_\nu(x) = \frac{1}{2}T^{\mu\nu}\eta_{\mu\nu} = \frac{1}{2}T^\mu_\mu \ . \tag{4.9}$$

The theory is thus conformal if the energy momentum tensor satisfies the properties

$$\partial_\mu T^{\mu\nu} = 0 \ , \qquad T^{\mu\nu} = T^{\nu\mu} \ , \qquad T^\mu_\mu = 0 \ . \tag{4.10}$$

4.2 Conformal invariance in 2d

Let us now specialize to two dimensions. We are interested in theories on the cylinder. We picked coordinates $\tau \in (-\infty, \infty)$ along the cylinder, and $\sigma \in [0, \pi)$ around the cylinder.

To work with 2d CFTs, it is better to pick different coordinates. First let us go to the Euclidean coordinates by switching to Euclidean time

$$t = i\tau \ . \tag{4.11}$$

More precisely, we continue the real time τ analytically to purely imaginary time. In physics this is often called Wick rotation. This means in particular that the 2d Minkowski metric η^{ab} turns into the standard Euclidean metric δ^{ab}. There are several other differences, but we won't go into them too much except where needed.

It is convenient to switch to different coordinates:

$$z = e^{2(t+i\sigma)} \qquad \bar{z} = e^{2(t-i\sigma)} \ . \tag{4.12}$$

Here and in the future we use Wirtinger notation for complex functions. In particular we will denote by $f(z)$ a complex function that is holomorphic (or meromorphic) in z, *i.e.* $\partial_{\bar{z}} f(z) = 0$, by $f(\bar{z})$ a complex function that is anti-holomorphic (anti-meromorphic) in z, *i.e.* $\partial_z f(\bar{z}) = 0$, and by $f(z, \bar{z})$ a complex function that is neither.

In fact it is often useful to take the point of view that z and $\bar{z}$ are two independent complex coordinates obtained by continuing the real coordinates $\mathrm{Im}(z)$ and $\mathrm{Re}(z)$ to the complex plane, and then in the end specialize to the submanifold $\bar{z} = z^*$, where * denotes honest complex conjugation.

Due to the periodicity of σ (4.12) is a bijection from the cylinder to $\mathbb{C} - \{0\}$, the complex plane with the origin removed. In fact, it is better to think of it as the Riemann sphere $\hat{\mathbb{C}}$ with two punctures at 0 and ∞. As we will see, the natural arena for 2d CFTs are punctured Riemann surfaces, for which the Riemann sphere is the simplest example. For general such a coordinate transformation would be very problematic, since we wouldn't expect it to be invariant under (4.12). Note however that (4.12) is a conformal transformation, under which our CFT is invariant. Although there are some subtleties due to the quantum conformal anomaly to which we will return, conformal invariance is the reason we can do this.

In two dimensions, (4.10) implies that T has only two indendent components. In our complex coordinates in fact (4.10) imply

$$T_{z\bar{z}} = T_{\bar{z}z} = 0 \tag{4.13}$$

and

$$\partial_{\bar{z}} T_{zz} = 0, \qquad \partial_z T_{\bar{z}\bar{z}} = 0 \tag{4.14}$$

so that $T(z) := T_{zz}(z)$ is holomorphic and $\bar{T}(\bar{z}) := T_{\bar{z}\bar{z}}(\bar{z})$ is antiholomorphic. Previously we had denoted the right-moving energy momentum tensor by $\tilde{T}$, but we see that here the bar notation makes sense. Conformal transformations are then given by

$$z \mapsto z + \epsilon(z) \qquad \bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z}) \tag{4.15}$$

where $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ are two arbitrary (anti-)holomorphic functions. This is the well-known statement that holomorphic and anti-holomorphic transformations are angle-preserving. All this also motivates why we switched coordinates: we can now use the powerful tools of complex analsys.

Each holomorphic transformation thus leads to a Noether current and a corresponding Noether charge which we obtain by integrating over a codimension 1 surface. In the case at hand we thus then define the Noether charge

$$Q_\epsilon = \oint_C dz T(z)\epsilon(z) \tag{4.16}$$

where we choose the contour to enclose 0. Note that the detailed shape of the contour is irrelevant since the integrand is meromorphic, at least as long as we do not cross any poles. In particular we can define L_n to be the Noether charge corresponding to $z \mapsto z + z^{n+1}$ such that

$$T(z) = \sum_n L_n z^{-n-2} \ . \tag{4.17}$$

To obtain the algebra that the Noether charges L_m satisfy, we can use their action on functions:

$$\{L_m, \phi(z)\} = z^{m+1}\partial_z\phi(z) \tag{4.18}$$

since the charge generates a coordinate transformation $z \mapsto z + \epsilon z^{m+1}$. Using the Jacobi identity then gives

$$\{L_m, L_n\} = (m-n)L_{m+n} \ . \tag{4.19}$$

As expected, we recover the *Witt algebra* (3.15), which generates conformal transformations in 2d. A completely analog argument for anti-holomorphic transformations gives the same algebra for the $\bar{L}_m$.

Let us discuss a few special generators. First, $P := L_0 - \bar{L}_0$ generates translations around the cylinder. On the Riemann sphere, this corresponds to transformations

$$z \mapsto e^{i\alpha}z \qquad \bar{z} \mapsto e^{-i\alpha}\bar{z} \qquad \alpha \in \mathbb{R}\ , \tag{4.20}$$

i.e. rotations around the origin. On the other hand $H = L_0 + \bar{L}_0$ generates time translations on the cylinder. On the Riemann sphere this corresponds to

$$z \mapsto az \qquad \bar{z} \mapsto a\bar{z} \qquad a \in \mathbb{R}\ , \tag{4.21}$$

that is dilations. Those are the translation generators on the cylinder. Once we are on the sphere, the translations in z and $\bar{z}$ are generated by L_{-1} and $\bar{L}_{-1}$.

More generally, it is useful to distinguish global conformal transformations which are actually globally well-defined, and transformations which are not globally defined. As we know, the only global holomorphic bijections of $\hat{\mathbb{C}}$ are Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}\ . \tag{4.22}$$

These are generated by the subalgebra of the two Witt algebras generated by L_{-1}, L_0, L_1 and $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$. This of course contains dilations, translations, but also inversions which are generated by L_1 and $\bar{L}_1$.

Since the latter transformations are not necessarily globally defined, we will most of the time not take consider the group that they form, but rather only the algebra of their generators.

4.3 Quantum conformal

Let us now turn to the quantum version of a 2d conformal field theory. Just as when quantizing a classical mechanical system, quantizing a classical field theory requires various ingredients.

The Hilbert space $\mathcal{H}$:

The first step is to fix a Hilbert space $\mathcal{H}$. Starting with a theory on the cylinder, ket states $|\phi\rangle$ are naturally identified with incoming states at $t \to -\infty$. The Hilbert space $\mathcal{H}$ thus naturally lives at $t = -\infty$. Next we want to define a hermitian conjugation $\dagger$ on this space. Note that because we switched to Euclidean space, time is now imaginary. Hermitian conjugation $\dagger$ now maps $t \mapsto -t$. It is thus very natural to identify bra states $\langle\phi|$ with outgoing states at $t \to \infty$.

What happens if instead of the cylinder we want to consider the theory on the Riemann sphere $\hat{\mathbb{C}}$ using the map (4.12)? The ket states are naturally mapped to the $z = 0$, and the bra states to $z = \infty$. Hermitian conjugation becomes the map

$$z \mapsto 1/z^* \ , \tag{4.23}$$

which leaves the surface $|z| = 1$ (corresponding to $t = 0$) invariant. This is actually quite important for quantizing the theory. Remember that the first step in quantization is to obtain equal time Poisson brackets, which we turn into equal time commutators. This means that we need to specify an equal time surface in our space. On the cylinder, we did this by choosing the circle $t = 0$. After mapping to $\hat{\mathbb{C}}$, this becomes the circle $|z| = 1$. We are thus quantizing with respect to the unit circle rather than the more standard $t = 0$. This is called *radial quantization*.

Fields

Next we need an algebra of operators acting on $\mathcal{H}$. In a quantum field theory we actually want not just operators, but *fields*, *i.e.* operator valued functions (or, more precisely, distributions) on space time

$$\phi(z, \bar{z}) \ , \tag{4.24}$$

i.e. operators ϕ which are functions of the position. This may sound familiar from quantum mechanics. Quantum mechanics is quantum field theory in 1+0 dimensions, *i.e.* the only coordinate is the time t. A 'field' in quantum mechanics is then simply an operator in the Heisenberg picture,

$$A(t) = e^{iHt} A e^{-iHt} \ . \tag{4.25}$$

A classical field is an observable which depends not only on time but also on its spatial position, and its quantization is thus a time and position dependent operator. The generalization of the Heisenberg picture (4.25) for quantum fields is simply

$$\phi(x^\mu) = e^{iP_\mu x^\mu} \phi(0) e^{-iP_\mu x^\mu} = e^{iHt + iP_j x^j} \phi(0) e^{-iHt - iP_j x^j} \tag{4.26}$$

where H is the Hamilton operator and P_j are the spatial momentum operators.

It is important to note that fields are highly intricate objects which satisfy many non-trivial relations. For instance they have to satisfy *locality*. In Minkowski space this means that the commutator of two spacelike separated fields must vanish,

$$[\phi(x^\mu), \psi(y^\mu)] = 0 \qquad \text{if } (x-y)^2 > 0 \ . \tag{4.27}$$

This is simply the statement that because of causality, the measurement at a point x cannot affect measurements at a spacelike separated point y. In Euclidean space this statement is even more powerful, since two different points are always spacelike separated, so that

$$[\phi(x^\mu), \psi(y^\mu)] = 0 \qquad \text{if } x \neq y \ . \tag{4.28}$$

To understand (4.28) correctly, it is important to remember that fields are distributions, and not functions. (4.28) does *not* imply that the commutator of ϕ and ψ vanishes, but simply that it is a distribution with support $y = x$. We should think of it as a sum of delta distributions $\delta(x-y)$ and their derivatives. This is actually a very important point, and we will return to it later, computing the commutator explicitly in some cases.

In 2d CFTs such fields $\phi(z,\bar{z})$ are often also called *vertex operators*. On $\hat{\mathbb{C}}$ (4.26) becomes

$$\phi(z,\bar{z}) = e^{zL_{-1}} e^{\bar{z}\bar{L}_{-1}} \phi(0) e^{-\bar{z}\bar{L}_{-1}} e^{-zL_{-1}} \ . \tag{4.29}$$

In conformal field theories, there is a one-to-one correspondence between states of the Hilbert space and fields in the theory, the so-called *operator-state correspondence*,

$$\phi(z,\bar{z}) \leftrightarrow |\phi\rangle \tag{4.30}$$

The state $|\phi\rangle$ is obtained from $\phi(z,\bar{z})$ by acting on the vacuum at the origin,

$$\phi(0)|0\rangle = |\phi\rangle \tag{4.31}$$

Symmetries

Next we want to impose symmetries in the theory. Since we are considering conformal field theories, the most naive guess would be to require that the Hilbert space $\mathcal{H}$ forms a representations of the two copies of the Witt algebra (4.19) L_m and $\bar{L}_m$. However, as already discussed above, this is too strong a requirement. Even though we have been talking about states in the Hilbert space $\mathcal{H}$, this is strictly speaking not correct. Two vectors in $\mathcal{H}$ correspond to the same physical state if they are related by a phase,

$$|\phi\rangle \sim e^{i\alpha}|\phi\rangle \ . \tag{4.32}$$

What does this mean for the represenation ρ of a symmetry G? Usually we require ρ to be an actual representation of the group, so that $\rho(gh) = \rho(g)\rho(h)$ for all $h, g \in G$. In view of (4.32) we would be happy to have the weaker requirement

$$\rho(gh) = \rho(g)\rho(h)e^{ic(g,h)} \ , \tag{4.33}$$

where $c(g,h)$ is a phase. A ρ which satisfies (4.33) is called a *projective representation* of G. The upshot is that in quantum mechanics we should consider projective representations of the symmetries of the system.

Mathematically it turns out that instead of considering projective representations of a group (or an algebra), we can instead consider proper representation of a central extension of that group (or algebra). A *central extension* is an extension of the original group by elements that commute with all the elements of the group.

You are probably already familiar with this from the discussion of spin in quantum mechanics. The rotation symmetry group of (non-relativistic) space is $SO(3)$, so the Hilbert space should carry a representation of $SO(3)$. All representations of $SO(3)$ however have integer spin, whereas electrons for instance have spin $1/2$, which is a representation of $SU(2)$. The resolution to this seeming puzzle is of course that $SU(2)$ is a central extension of $SO(3)$

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1 \tag{4.34}$$

by a central element $\mathbb{Z}_2$. Actually you may not have noticed this, because we usually consider representations of the Lie algebra rather than the Lie group, which for $SO(3)$ is automatically the Lie algebra $su(2)$. For finite dimensional semisimple Lie algebras there are no central extensions, so we never need to worry projective representations. This is no longer true for infinite dimensional Lie algebras such as Witt algebra. In fact, the Witt algebra has one central extension: the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1) \ . \tag{4.35}$$

The central element c commutes with all generators, and is therefore often just written as a number. It is called the *central charge* of the theory. We of course already encountered (4.35) when we quantized the bosonic string, where we found that the central charge was D.

From this discussion it follows that $\mathcal{H}$ decompose into representations of two copies of L_m and $\bar{L}_m$ of (4.35). We call the eigenvalue of L_0 the (left-moving) *weight* of a state, and analogous for $\bar{L}_0$. Because of

$$[L_0, L_m] = -mL_m \ , \tag{4.36}$$

the L_m with positive and negative m act as lowering and raising operators respectively. Since we want to interpret $H = L_0 + \bar{L}_0$ as the energy of a state, we need the weights to be bounded from below. This means we want irreducible highest weight representations[1] with a highest weight state $|\phi\rangle$ such that

$$L_m|\phi\rangle = 0 \quad L_0|\phi\rangle = h|\phi\rangle \qquad \bar{L}_m|\phi\rangle = 0 \quad \bar{L}_0|\phi\rangle = \bar{h}|\phi\rangle \qquad \forall m > 0 \tag{4.37}$$

[1]Those representations should properly be called *lowest* weight representations, and $|\phi\rangle$ the *lowest* weight state. Unfortunately the terminology highest weight is firmly established in the literature.

$h, \bar{h}$ are called the conformal weights of ϕ. This standard notation is a bit unfortunate — in particular $\bar{h}$ is *not* the complex conjugate of h. The highest weight state $|\phi\rangle$ and its operator $\phi(z, \bar{z})$ are called *primary field*. All other states of the theory can be obtained by acting with suitable combinations of L_{-m} on a primary field. Such states and their operators are called *descendant fields*. The general philosophy is that in a CFT we only need to understand the primary fields. Everything else follows from them by the conformal symmetry of the theory.

One special highest weight representation is the vacuum $|0\rangle$. It should be invariant under all coordinate transformations which are well-defined at the origin. From (4.18) it follows that it thus satisfies

$$L_m|0\rangle = \bar{L}_m|0\rangle = 0 \qquad \forall m > -2 \ . \tag{4.38}$$

It is thus a primary field, which in addition is also annihilated by L_0 (*i.e.* it has $h = \bar{h} = 0$) and by translations L_{-1}. Its corresponding field is simply the identity operator $\mathbf{1}$.

What about the other primary fields? Naively one could expect that primary fields should be invariant under conformal transformations. This is however too strong a condition: the most we can expect is covariant. After all, in ordinary quantum field theory, there are fields of non-zero spin such as fermions and currents which transform non-trivially under Lorentz transformations. In fact, under conformal transformations, primary fields transform as

$$\phi(z', \bar{z}') = (\partial_z z')^{-h}(\partial_{\bar{z}}\bar{z}')^{-\bar{h}}\phi(z, \bar{z}) \ . \tag{4.39}$$

To see this, for simplicity assume that $\phi(z)$ is holomorphic. Note that L_{-1} is the standard translation generator, so that

$$\phi(z) = e^{zL_{-1}}\phi(0)e^{-zL_{-1}} \ . \tag{4.40}$$

In particular that means that

$$\phi(z)|0\rangle = e^{zL_{-1}}|\phi\rangle \ . \tag{4.41}$$

Let us now consider the transformation behavior under a conformal transformation which is non-singular at 0, *i.e.* $z \mapsto z + \epsilon z^{n+1}$, $n \geq 0$, which we know should be generated by L_n. Using the identities

$$[L_n, L_{-1}^k] = (n+1)!\binom{k}{n+1}(L_{-1})^{k-n} + (n+1)!\binom{k}{n}(L_{-1})^{k-n}L_0 + \dots \tag{4.42}$$

where $\dots$ indicate operators that annihilate primary fields. This yields

$$\begin{aligned}[L_n, \phi(z)]|0\rangle = [L_n, e^{zL_{-1}}]|\phi\rangle &= (z^{n+1}e^{zL-1}L_{-1} + z^n(n+1)e^{zL_{-1}}L_0)|\phi\rangle \\ &= (z^{n+1}\partial_z\phi(z) + h(n+1)z^n\phi(z))|0\rangle \ . \end{aligned}\tag{4.43}$$

Locality implies that this holds in general, not just while acting on the vacuum. We can then use translation and inversion to argue that (4.43) actually holds for all n. In total we thus get

$$[L_n, \phi(z,\bar{z})] = (z^{n+1}\partial_z\phi(z,\bar{z}) + h(n+1)z^n\phi(z,\bar{z})) \tag{4.44}$$

Taking into account the anti-holomorphic Virasoro algebra and integrating the generators, we get the transformation property (4.39).

Correlation functions

Finall, the observable physical data of quantum field theories that we are interested in are correlations functions. They are usually defined as time ordered products of operators wedged between vacua. We started out with a theory on a cylinder, which we mapped to the plane. Equal time surfaces on the cylinder get mapped to circles centered on the origin. Time ordering becomes radial ordering, that is

$$R(\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)) = \begin{cases} \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2) & : |z_1| > |z_2| \\ \phi_2(z_2,\bar{z}_2)\phi_1(z_1,\bar{z}_1) & : |z_2| > |z_1| \end{cases} \tag{4.45}$$

and similar for multiple operators. Correlation functions are then defined as

$$\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle := \langle 0|R(\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n))|0\rangle\ . \tag{4.46}$$

In conformal field theory, the form of the correlation functions is strongly constrained by the conformal symmetry. Take the two-point function of two primary fields,

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = g(z_1,\bar{z}_1,z_2,\bar{z}_2)\ . \tag{4.47}$$

By translation invariance we know that $g = g(z_1 - z_2, \bar{z}_1 - \bar{z}_2)$. Using a rescaling transformation $z \mapsto \lambda_1 z, \bar{z} \mapsto \lambda_2 z$ we find that g has to be homogeneous. Finally invariance under special conformal transformations fixes it up to a constant

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}\delta_{h_1,\bar{h}_1}\delta_{h_2,\bar{h}_2}}{(z_1-z_2)^{2h_1}(\bar{z}_1-\bar{z}_2)^{2\bar{h}_1}}\ . \tag{4.48}$$

Similarly three point functions of primary fields are fixed up to an constant C_{123} as

$$\begin{aligned}\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle &= \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_1-z_3)^{h_1+h_3-h_2}(z_2-z_3)^{h_2+h_3-h_1}} \\ &\times \frac{1}{(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2-\bar{h}_3}(\bar{z}_1-\bar{z}_3)^{\bar{h}_1+\bar{h}_3-\bar{h}_2}(\bar{z}_2-\bar{z}_3)^{\bar{h}_2+\bar{h}_3-\bar{h}_1}}\ .\end{aligned} \tag{4.49}$$

The correlation functions of descendant fields look a bit more complicated, but can always be obtained from the corresponding primary correlation functions. This is why we stated before that it is enough to understand the primary fields.

Operator Product Expansion (OPE)

To compute correlation functions, a very useful tool is the so-called operator-product expansion (OPE). The idea is that the radially ordered product of two fields at different points x and y can be expanded can be expanded as a (position-dependent) linear combinations of fields at y,

$$R(\phi(x)\psi(y)) = \sum_j f_j(x-y)\varphi_j(y) \ . \tag{4.50}$$

The idea here is that if we move to fields close to each other, we can replace them by an infinite sum of other fields in the theory. (The non-trivial part of this statement is that the operators $\phi_j(y)$ appearing in the sum are not just operators, but actual fields $\phi_j(y)$.) In view of the operator-state correspondence, (4.50) is not too surprising: Placing ψ at $y=0$ and acting on $|0\rangle$ we get

$$\phi(x)\psi(0)|0\rangle = \phi(x)|\psi\rangle = \sum_j f_j(x)|\varphi_j\rangle \tag{4.51}$$

where we have used that the second expression is simply some x-dependent vector in $\mathcal{H}$, which we can certainly write as a sum of basis vectors $|\varphi_j\rangle$ with x-dependent coefficients $f_j(x)$. By applying the operator-state correspondence to $|\psi\rangle$ and $|\varphi_j\rangle$, we recover (4.50). Although operator product expansions can be defined in any QFT, they turn out to be much more powerful for CFTs as we will see below.

4.4 Meromorphic fields and contour deformation

Let us discuss a special class of fields: so-called *meromorphic fields*. As we have seen, in general the correlation functions of a field $\phi(z,\bar{z})$ are non-meromorphic functions of $z,\bar{z}$. For some special fields, the z dependence is actually meromorphic. The above considerations then immediately imply that such fields must have $\bar{h}=0$. Of course a similar story holds for anti-meromorphic fields.

In fact we have already encountered such a field: the energy momentum tensor $T(z)$. The expansion (4.17) shows that it depends meromorphically on z. On the other hand, $\bar{T}(\bar{z})$ is an anti-meromorphic field. In fact it belongs to the vacuum representation $|0\rangle$. To see this note that from (4.31) we obtain the corresponding state

$$|T\rangle = T(0)|0\rangle = L_{-2}|0\rangle \ . \tag{4.52}$$

A general meromorphic primary field $\phi(z)$ of weight $(h,0)$ can be expanded in modes

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \ . \tag{4.53}$$

We can of course recover the modes using the contour integral

$$\phi_n = \oint_C dz \phi(z) z^{h+n-1} , \tag{4.54}$$

where the contour C circles around 0. The corresponding state is given

$$|\phi\rangle = \phi(0)|0\rangle = \phi_{-h}|0\rangle . \tag{4.55}$$

Note that in order for this to work, *i.e.* for $\phi(z)$ to be non-singular at $z = 0$, we need $\phi_n|0\rangle = 0$ for $n > -h$. Plugging (4.53) into (4.44), we can obtain the commutation relation between the Virasoro modes and the modes of the primary fields as

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n} . \tag{4.56}$$

Plugging in $m = 0$ we see that ϕ_n lowers the weight by n.

Comparing (4.56) with the Virasoro algebra (4.35), we find that $T(z)$ is *not* a primary field due to the appearance of the central term! In fact we could have noticed that already, since (unless $c = 0$) $|T\rangle$ is not annihilated by L_2:

$$L_2|T\rangle = L_2 L_{-2}|0\rangle = \frac{c}{2}|0\rangle . \tag{4.57}$$

$|T\rangle$ is thus not a highest weight state. Note that even though $T(z)$ is not primary, at least $|T\rangle$ is annihliated by L_1. This means that it transforms like a primary field under global $SL(2,\mathbb{C})$ transformations. Under more general transformations, in fact we have

$$(\partial_z z')^2 T'(z') = T(z) - \frac{c}{12}\{z', z\} , \tag{4.58}$$

where the correction term to the standard transformation of a primary field is given by the *Schwarzian derivative*

$$\{f, z\} = \frac{2\partial_z^3 f \partial_z f - 3\partial_z^2 f \partial_z^2 f}{2\partial_z f \partial_z f} . \tag{4.59}$$

Let us now briefly return to Hermitian conjugation. We mentioned that this corresponds to mapping $z \mapsto 1/z^*$. For a primary field we thus want to define Hermition conjugation on the real surface $\bar{z} = z^*$ as

$$(\phi(z, \bar{z}))^\dagger := \bar{z}^{-2h} z^{-2\bar{h}} \phi(1/\bar{z}, 1/z) . \tag{4.60}$$

This definition is obvious except for the prefactor we have introduced. One way to justify it is to check that this gives a well-defined inner product between the bra and ket states. Another justification is that it leads to very simple expressions for the hermitian conjugates of the modes $(\phi^\dagger)_n$: For a meromorphic primary on the real surface the conjugated modes are defined as

$$\phi(z)^\dagger = \sum_{m\in\mathbb{Z}} (\phi^\dagger)_m \bar{z}^{-m-h} . \tag{4.61}$$

On the other hand from (4.60) we have

$$\phi(z)^\dagger = \bar{z}^{-2h} \sum_{m\in\mathbb{Z}} \phi_m \bar{z}^{h+m} = \sum_{m\in\mathbb{Z}} \phi_{-m} \bar{z}^{-h-m} \tag{4.62}$$

so that $\phi^\dagger_m = \phi_{-m}$. Using this it follows immediately that for unitary theories, all weights h and $\bar{h}$ are non-negative:

$$0 \leq |L_{-1}|\phi\rangle|^2 = \langle\phi|L_1 L_{-1}|\phi\rangle = \langle\phi|2L_0|\phi\rangle = 2h . \tag{4.63}$$

(In fact, cluster decomposition tells us that the unique state with $h = 0$ is the vacuum.)

If there are multiple meromorphic fields in the theory, their modes form an algebra whose commutators we can compute. It turns out that knowing their OPE is equivalent to knowing their commutators, and writing down the OPE is often a more efficient way.

Let us take two meromorphic fields $A(z)$ and $B(w)$ of weight h_A and h_B. Due to scale invariance, their operator product expansion must be of the form

$$R(A(z)B(w)) = \sum_{h=0}^{\infty} \frac{C_h(w)}{(z-w)^{h_A+h_B-h}} , \tag{4.64}$$

where $C_h(w)$ is a meromorphic field (not necessarily primary) of weight h. Note that we've made use of unitarity so that that sum starts with the vacuum at $h = 0$.

Let us now smear the distribution $A(z)$ by some holomorphic test function $f(z)$ and compute the commutator of that mode

$$A[f] := \oint dz f(z) A(z) \tag{4.65}$$

with $B(w)$. To compute $[A[f], B(w)]$, we need to evaluate

$$\oint_{C_1} f(z) dz A(z) B(w) - \oint_{C_2} f(z) dz B(w) A(z) . \tag{4.66}$$

We choose the contour C_1 as a circle around the origin with radius $r_1 > |w|$, whereas C_2 is a circle around the origin with radius $r_2 < |w|$. Note that with this choice of contours, the operators in (4.66) are already radially ordered, so that we can write

$$[A[f], B(w)] = \oint_{C_1} f(z)dzR(A(z)B(w)) - \oint_{C_2} f(z)dzR(B(w)A(z)) =$$
$$\left(\oint_{C_1} - \oint_{C_2}\right) dzf(z)R(A(z)B(w)) = \oint_{|w-z|=\epsilon} dzf(z)R(A(z)B(w)) \ , \quad (4.67)$$

where we have deformed the two contours into a single small contour around the point w, assuming that there are no other poles (which is the case if there are no other fields nearby). As promised we find that the commutator is indeed non-vanishing if the OPE $R(A(z)B(w))$ contains singular terms. More precisely, taking (4.64) we find that

$$[A[f], B(w)] = \sum_{n=0}^{h_A+h_B-1} \frac{\partial_w^n f(w)}{n!} C_{h_A+h_B-n-1}(w) \ . \quad (4.68)$$

To make the connection to (4.28), write (4.68) as

$$[A(z), B(w)] = \sum_{n=0}^{h_A+h_B-1} \frac{\delta^{(n)}(z-w)}{n!} C_{h_A+h_B-n-1}(w) \ . \quad (4.69)$$

Here δ is the 'holomorphic' delta distribution: For holomorphic f it satisfies $\oint dzf(z)\delta(z) = f(0)$ with the contour circling around 0. (This is of course just the standard Cauchy integral written in the language of distributions.)

The upshot is that from (4.68) we see that all the information on the commutators of the modes is contained in the singular part of the OPE and vice versa. We will therefore in the future usually only indicate the singular part of the OPE and denote this by $\sim$. The OPE of a $T(z)$ with a primary field $\phi(w)$ for instance is given by

$$T(z)\phi(w) \sim \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial_w\phi(w) + \dots \ . \quad (4.70)$$

To check that this is indeed equivalent to (4.44), simply evaluate (4.68) with $f(z) = z^{n+1}$. (Note that it is enough that $f(z)$ is holomorphic at w, which is certainly the case here.) The OPE of $T(z)$ with itself is given by

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \dots \ . \quad (4.71)$$

Using appropriate smearing functions, one can indeed check that this leads to the Virasoro algebra (4.35) for the modes L_m.

Chapter 5

Examples

5.1 Free boson

In the previous chapter we set up the framework for general 2d CFTs. Let us now discuss examples. We have already encountered one example when discussing the bosonic string: namely, the free boson, whose action is given by

$$S = \frac{1}{4\pi\alpha'}\int dz d\bar{z} \partial_z X \partial_{\bar{z}} X \ . \tag{5.1}$$

Let us now discuss this theory from our new-found conformal perspective. In our complex coordinates, the expansion of (2.29) in Fourier modes becomes

$$X(z,\bar{z}) = x - i\frac{\alpha'}{2}p\log|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n}\right) \ . \tag{5.2}$$

We now want to compute OPEs and correlation functions in this theory.

Let us start by considering the product of two fields

$$X(z_1,\bar{z}_1)X(z_2,\bar{z}_2) \ , \qquad |z_1| > |z_2| \ . \tag{5.3}$$

From our general discussions of OPEs, we expect that this product diverges if we send z_1 to z_2 due to the singular part of the OPE. How does this divergence come about in this concrete example? Note from (5.2) that there is an infinite number of creation operators to the left of annihilation operators. Once we try to set $z_1 = z_2$, these will lead to diverging sums.

If we want to avoid this issue, we can use the normal ordered product $: \cdot :$. The normal ordering prescription is the same as in (3.25). For concreteness, we will group p with the annihilation operators, and x with the creation operators. Because all the annihilation operators are to the right of the creation operators, the normal ordered product of two fields $: X(z_1,\bar{z}_1)X(z_2,\bar{z}_2) :$ does not have a singularity if we send $z_1 \to z_2$.

We can thus use this to define new fields as

$$: XX : (z, \bar{z}) := \lim_{z_1 \to z} : X(z_1, \bar{z}_1) X(z, \bar{z}) : \ . \tag{5.4}$$

The normal ordered product is related to the ordinary product by (where we take $|z_1| > |z_2|$ to agree with the radially ordered product)

$$: X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) := X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) - [X^a(z_1, \bar{z}_1), X^c(z_2, \bar{z}_2)] \ , \tag{5.5}$$

where we have decomposed X into into its annihilation operator part X^a and its creation operator part X^c, $X = X^a + X^c$. Using the commutation relations of the α and $\tilde{\alpha}$ we can evaluate the commutator as

$$\begin{aligned}
[X^a(z_1, \bar{z}_1), X^c(z_2, \bar{z}_2)] &= \frac{\alpha'}{2}(-\log|z_1|^2 + \sum_{n>0} \frac{1}{n}(z_2/z_1 + \bar{z}_2/\bar{z}_1)) \\
&= \frac{\alpha'}{2}(-\log|z_1|^2 - \log(1 - z_2/z_1) - \log(1 - \bar{z}_2/\bar{z}_1)) = -\frac{\alpha'}{2}\log|z_1 - z_2|^2 \ .
\end{aligned} \tag{5.6}$$

The singularity for $z_1 \to z_2$ is indeed contained in the commutator term. Note that this commutator term is simply the Wick contraction

$$\overbracket{X(z_1, \bar{z}_1) X}(z_2, \bar{z}_2) = [X^a(z_1, \bar{z}_1), X^c(z_2, \bar{z}_2)] = -\frac{\alpha'}{2}\log|z_1 - z_2|^2 \ . \tag{5.7}$$

In fact, since we are dealing with a free field, we can use Wick's theorem to compute all correlation functions from just (5.7), by rewriting any product as a normal ordered product plus Wick contractions in the usual manner. We will often use this in the following.

For most purposes the field $X(z, \bar{z})$ is not the best field to use since it has a logarithm in its expansion. It is often better to consider the fields

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n z^{-n-1} \qquad \bar{\partial} X(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n \bar{z}^{-n-1} \tag{5.8}$$

Note that these fields are (anti-)meromorphic. Let us now compute their OPE. Again assuming for simplicity $|z_1| > |z_2|$ we have

$$\begin{aligned}
\partial X(z_1) \partial X(z_2) = \partial\overbracket{X(z_1) \partial X}(z_2) + : \partial X(z_1) \partial X(z_2) :\sim \partial\overbracket{X(z_1) \partial X}(z_2) \\
= -\frac{\alpha'}{2} \frac{1}{(z_1 - z_2)^2} \ ,
\end{aligned} \tag{5.9}$$

where in the first line we have used that normal ordered products are non-singular, and in the second line we obtained the Wick contraction by taking derivatives from (5.7). This shows why Wick's theorem is so powerful for computing singular parts of teh OPE: Since normal ordered terms are always regular, we can always neglect them.

It is straightforward to check that for the free boson, the energy momentum tensor is given by

$$T(z) = -\frac{1}{\alpha'} : \partial X \partial X : (z) \qquad \bar{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X \bar{\partial} X : (\bar{z}) \ . \tag{5.10}$$

Let us now check that ∂X is indeed a primary field by computing the OPE with T and comparing with (4.70). Only keeping contractions we obtain

$$\begin{aligned} T(z)\partial X(0) \sim -\frac{1}{\alpha'} \Big(: \partial X(z) \partial X(z) : \partial X(0) + \ : \partial X(z) \partial X(z) : \partial X(0) \Big) \\ = \frac{\partial X(z)}{z^2} \sim \frac{1}{z^2} \partial X(0) + \frac{1}{z} \partial^2 X(0) \ , \end{aligned} \tag{5.11}$$

where in the last line we have Taylor expanded around 0. Comparing to (4.70) shows that ∂X is a primary field of weight $h = 1$. It is straightforward to see that $\bar{T}(\bar{z})\partial X(0) \sim 0$, so that $\bar{h} = 0$. In fact $\partial X(z)$ is the operator corresponding to the state $\alpha_{-1}|0\rangle$, which can be easily seen by evaluating

$$\partial X(0)|0\rangle = \alpha_{-1}|0\rangle \ . \tag{5.12}$$

Similarly $\bar{\partial} X$ is a primary of weight (0,1) whose state is given $\tilde{\alpha}_{-1}|0\rangle$.

From this, naively we might guess $X(z, \bar{z})$ has dimension $(0, 0)$. If X is indeed dimensionless, this suggests that it makes sense to consider fields of the form

$$: e^{ikX(z,\bar{z})} : \tag{5.13}$$

Naively we would expect this to have weight 0 as well. Let us check if this is indeed the case. In a first step, we have

$$\partial X(z) : e^{ikX(0,0)} :\sim -\frac{ik\alpha'}{2z} : e^{ikX(0,0)} : \tag{5.14}$$

so that we get

$$\begin{aligned} T(z) : e^{ikX(0,0)} :\sim \partial X(z) \frac{1}{z} ik : e^{ikX(0,0)} : + \frac{\alpha' k^2}{4} \frac{1}{z^2} : e^{ikX(0,0)} : \\ = \frac{\alpha' k^2}{4} \frac{1}{z^2} : e^{ikX(0,0)} : + \frac{1}{z} \partial : e^{ikX(0,0)} : , \end{aligned} \tag{5.15}$$

where in the first term in the first line comes from a single contraction (5.14), and the second term from the double contraction, and we have again Taylor expanded in the second line. From this and the analogue computation with $\bar{T}(\bar{z})$ we see that (5.13) is indeed a primary field, but due to quantum effects, it does have weight

$$(h, \bar{h}) = (\tfrac{1}{4}\alpha' k^2, \tfrac{1}{4}\alpha' k^2) \ . \tag{5.16}$$

In fact, (5.13) is the operator corresponding to $|k\rangle$,

$$:e^{ikX(0,0)}:|0\rangle = |k\rangle\ . \tag{5.17}$$

Indeed, we have

$$p:e^{ikX(0,0)}:|0\rangle = pe^{ikx}|0\rangle = [p, e^{ikx}]|0\rangle = k:e^{ikX(0,0)}:|0\rangle\ . \tag{5.18}$$

Finally we can use this to compute correlation functions. As usual, we want to write everything normal ordered. We have for instance

$$\begin{aligned}
:e^{ik_1X(z_1,\bar z_1)}::e^{ik_2X_2(z_2,\bar z_2)}: &= e^{ik_1X^c(z_1,\bar z_1)}e^{ik_1X^a(z_1,\bar z_1)}e^{ik_2X^c(z_2,\bar z_2)}e^{ik_2X^a(z_2,\bar z_2)}\\
&= e^{-k_1k_2[X^c(z_1,\bar z_1),X^a(z_2,\bar z_2)]}e^{ik_1X^c(z_1,\bar z_1)}e^{ik_2X^c(z_2,\bar z_2)}e^{ik_1X^a(z_1,\bar z_1)}e^{ik_2X^a(z_2,\bar z_2)}\\
&= |z_1-z_2|^{\alpha' k_1k_2} :e^{ik_1X(z_1,\bar z_1)}e^{ik_2X_2(z_2,\bar z_2)}:
\end{aligned} \tag{5.19}$$

where we used the Baker-Campell-Hausdorff formula $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]}$ which is valid since the commutator is a c-number and commutes with everything. We can use this to compute the two point functions

$$\begin{aligned}
\langle 0|:e^{ik_1X(z_1,\bar z_1)}::e^{ik_2X_2(z_2,\bar z_2)}:|0\rangle &= |z_1-z_2|^{\alpha' k_1k_2}\langle 0|:e^{ik_1X(z_1,\bar z_1)}e^{ik_2X_2(z_2,\bar z_2)}:|0\rangle\\
&= |z_1-z_2|^{\alpha' k_1k_2}\langle -k_2|k_1\rangle = \frac{\delta(k_1+k_2)}{|z_1-z_2|^{\alpha' k_1^2}}
\end{aligned} \tag{5.20}$$

5.2 Free boson on a circle

So far we interpreted the free boson as a spacetime coordinate in flat space, *i.e.* we allowed X to take arbitrary values in $\mathbb{R}$. Let us now put the boson on a circle of radius R. This means we that we need to identify

$$X \sim X + 2\pi RL \qquad L\in\mathbb{Z}\ , \tag{5.21}$$

i.e. X now take values in $\mathbb{R}/(2\pi R\mathbb{Z})$. How does that affect the theory?

In the original theory, states could have any momentum $p\in\mathbb{R}$. Now that we have the identification (5.21), p has to be quantized since we need

$$e^{ipX} = e^{ip(X+2\pi RL)} \qquad \forall L\in\mathbb{Z}\ . \tag{5.22}$$

It follows that

$$p = \frac{M}{R} \qquad M\in\mathbb{Z}\ . \tag{5.23}$$

If we were dealing with a theory of particles, this would be all there is. Since we are dealing with closed strings, a new possibilty opens up. On flat space, the closed string condition was simply that the string was closed, $X(\tau,\sigma+\pi) = X(\tau,\sigma)$. On the circle however a closed string can wind L times around the circle, *i.e.*

$$X(\tau,\sigma+\pi) = X(\tau,\sigma) + 2\pi RL \qquad L \in \mathbb{Z} \ . \tag{5.24}$$

How does this affect the mode expansion? We have

$$X(\tau,\sigma) = x + 2\alpha' \frac{M}{R}\tau + 2RL\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n e^{-2in\sigma^-} + \tilde{\alpha}_n e^{-2in\sigma^+} \right) \tag{5.25}$$

We can again split this into a left- and right-moving part as

$$X_R(\sigma^-) \;=\; x_R + \alpha' p_R \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in\sigma^-} \tag{5.26}$$

$$X_L(\sigma^+) \;=\; x_L + \alpha' p_L \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \tag{5.27}$$

Here we have introduced new variables x_L, x_R, p_L, p_R such that $x_L + x_R = x$, $p_L + p_R = 2p$. We choose their commutators such that

$$[x_L, p_L] = [x_R, p_R] = i \ , \qquad [x_R, p_L] = [x_L, p_R] = 0 \ . \tag{5.28}$$

What is the spectrum of p_L and p_R? From $p_L + p_R = 2p = 2\frac{M}{R}$ and $\alpha'(p_L - p_R) = 2RL$ we obtain

$$p_L = \frac{M}{R} + \frac{RL}{\alpha'} \ , \qquad p_R = \frac{M}{R} - \frac{RL}{\alpha'} \ . \tag{5.29}$$

The ground state of this theory are now no longer parametrized by a single momentum p, but rather by two momenta p_L, p_R. For the ground state it is straightforward to check that $|p_L, p_R\rangle$ has weight

$$(h, \bar{h}) = (\frac{1}{4}\alpha' p_L^2, \frac{1}{4}\alpha' p_R^2) \ , \tag{5.30}$$

and its corresponding vertex operator is given by

$$: e^{ip_L X_L + ip_R X_R} : (z, \bar{z}) \ , \tag{5.31}$$

where the spectrum of p_L, p_R is parametrized by the momentum number M and the winding number L through (5.29).

Chapter 6

Compactifications

6.1 String compactifications and the Poincaré anomaly

Let us now return to string theory and the problem of 26 dimensions. From experience we know that the real world should have 4 dimensions, but bosonic string theory requires 26. How could we go around this? The basic idea goes back to Kaluza-Klein, and has to do with compact dimensions. Suppose one of the 26 dimensions was actually not flat space, but a very very small circle. This is called *compactifying* this dimension. Technically the world would still have 26 dimensions, but if the circle is small enough, we could not distinguish it from a point, so that effectively our world would look like it only has 25 dimensions. Following this line of thought, we could replace 22 of the 26 dimensions by circles. Spacetime would then look four dimensional, even though secretly it still has 26 dimensions.

Let us pursue this line of thought. For such a setup, the gauge-fixed Polyakov action will decompose into two parts, $S = S^{(e)} + S^{(i)}$. $S^{(e)}$ is the external part, which is the action of 4 free bosons in flat Minkowski space. $S^{(i)}$ is the internal part. For the setup above, it is for instance the action of 22 free bosons on the torus $\mathbb{T}^{22}$.

$$S = S^{(e)} + S^{(i)} = \frac{1}{4\pi\alpha'} \sum_{\mu=0}^{3} \int d\sigma^+ d\sigma^- \partial_- X^\mu \partial_+ X_\mu + S^{(i)}[X^4, \dots X^{26}] \tag{6.1}$$

The constraint equations are again

$$0 = T_{ab} = T^{(e)}_{ab} + T^{(i)}_{ab} \ . \tag{6.2}$$

The energy momentum tensor has two contributions coming from S_e and S_i. The contribution from the external part is

$$T^{(e)}_{++} = \sum_{\mu=0}^{3} \partial_+ X^\mu \partial_+ X_\mu \qquad T^{(e)}_{--} = \sum_{\mu=0}^{3} \partial_- X^\mu \partial_- X_\mu \tag{6.3}$$

What about $T^{(i)}$? The only assumption we want to make at the moment is that $S^{(i)}$ is a conformal field theory. This means that $T^{(i)}$ is symmetric, conserved, and traceless, so that we can write it in terms of the two components $T^{(i)}_{++}$ and $T^{(i)}_{--}$ for which $\partial_- T^{(i)}_{++} = 0$ and $\partial_+ T^{(i)}_{--} = 0$.

Since both $S^{(i)}$ and $S^{(e)}$ are conformal field theories, (6.1) has worldsheet conformal symmetry as a residual gauge symmetry. As before we can reparametrize the worldsheet coordinate τ to $\tilde{\tau}$ where $\tilde{\tau}$ satisfies

$$\Box\tilde{\tau}(\sigma,\tau) = 0 \ . \tag{6.4}$$

Since X^+ satisfies the same equation, we can fix τ in such a way that $X^+ = x^+ + \ell_s^2 p^+ \tau$, thereby eliminating the modes α^+ and $\tilde{\alpha}^+$. Finally we can impose the constraint equations (6.2)

$$T^{(i)}_{\pm\pm} + \sum_{i=2}^{3} (\partial_\pm X^i)^2 = 2\partial_\pm X^+ \partial_\pm X^- = \ell_s^2 p^+ \partial_\pm X^- \ . \tag{6.5}$$

From this we can extract the oscillator modes α^- and $\tilde{\alpha}^-$ as

$$\alpha^-_m = \frac{2}{\ell_s p^+}\left(\frac{1}{2}\sum_{i=2}^{3}\sum_n \alpha^i_{m-n}\alpha^i_n + L^{(i)}_m\right) = \frac{2}{\ell_s p^+}(L^{(e)}_m + L^{(i)}_m) \tag{6.6}$$

$$\tilde{\alpha}^-_m = \frac{2}{\ell_s p^+}\left(\frac{1}{2}\sum_{i=2}^{3}\sum_n \tilde{\alpha}^i_{m-n}\tilde{\alpha}^i_n + \tilde{L}^{(i)}_m\right) = \frac{2}{\ell_s p^+}(\tilde{L}^{(e)}_m + \tilde{L}^{(i)}_m) \ , \tag{6.7}$$

where $L^{(i)}_m$ and $\tilde{L}^{(i)}_m$ are the Fourier modes of $T^{(i)}_{++}$ and $T^{(i)}_{--}$ respectively. Compared to the original case we thus have

$$L^\perp_m = L^{(e)}_m + L^{(i)}_m \ . \tag{6.8}$$

How would this work on the level of the quantum theory? On flat space, we saw that the closed bosonic string is described by 26 free bosons on flat space. Let us denote the Fock space of such a free boson by V. Fixing light cone gauge effectively removes two of them, so that the Hilbert space is essentially given by

$$\mathcal{H} = \bigotimes_{i=2}^{25} V \ . \tag{6.9}$$

In our setup, the Hilbert space factorizes into an internal and an external part,

$$\mathcal{H} = \mathcal{H}^{(e)} \otimes \mathcal{H}^{(i)} \ . \tag{6.10}$$

The external part consists of ground states $|p^\mu\rangle$, where $\mu = 0,1,2,3$, and their oscillator modes α^2 and α^3. The internal part is some general CFT whose spectrum is given by the eigenvalues of L_0^i and $\bar{L}_0^i$. For our original example, $\mathcal{H}^{(i)}$ would be the Hilbert space of 22 free bosons on the torus $\mathbb{T}^{22}$. Is this theory still anomaly free? Let us try to repeat the argument in section 3.4 and see what happens.

Since only the first 4 dimensions $\mu = 0, 1, 2, 3$ correspond to flat Minkowski space, we have no right to expect the spacetime symmetry to be the full $SO(1, 25)$ Lorentz group. The only thing we can require is that the Lorentz group $SO(1, 3)$ of the flat Minkowski space part is anomaly free. It is straightforward to repeat the argument in section 3.4 for J^{2-} and J^{3-}. To this end, note that the external and internal CFTs are orthogonal, so that

$$[L_m^{(i)}, L_n^{(e)}] = [L_m^{(i)}, \alpha_n^{2,3}] = 0 \ . \tag{6.11}$$

For the Poincaré anomaly to vanish, we recover the same condition

$$0 = \langle 0|[L_m^{\perp}, L_{-m}^{\perp}]|0\rangle - 2m^2(m-1) - 2m^2 + 2am \ . \tag{6.12}$$

Again, this is only satisfied if $a = 1$ and the central charge of the Virasoro algebra $L^{\perp}$ is $c = 24$. Since $L^{(e)}$ contributes central charge $c = 2$ to $L^{\perp}$, this means that the theory is anomaly free only if $L^{(i)}$ has central charge $c = 22$,

$$[L_m^{(i)}, L_n^{(i)}] = (m-n)L_{m+n}^{(i)} + 11m(m^2-1)\delta_{m,-n} \ . \tag{6.13}$$

With such an internal CFT the $L^{\perp}$ form a Virasoro algebra of total central charge 24 again. One way of doing this is for instance our original example, *i.e.* choosing $\mathcal{H}_i$ to be the theory of 22 free bosons compactified on circles, that is on a 22 dimensional torus $\mathbb{T}^{22}$.

6.2 Spectrum

Let us now compute the spectrum of our compactified bosonic string. First note that as before we still have one gauge degree of freedom that we have not fixed, namely the shift of σ by a constant. Imposing invariance under that shift again leads to the level matching condition

$$(L_0^{\perp} - \tilde{L}_0^{\perp})|\phi\rangle = 0 \ , \tag{6.14}$$

since also for a general CFT $L_0^i - \bar{L}_0^i$ is the operator that generates wordsheet translations. Next, in terms of the level operators $N^{(e)}$ of the modes $\alpha^{2,3}$ we have

$$L_0^{(e)} = \tfrac{1}{8}\ell_s^2(p_2^2 + p_3^2) + N^{(e)} \ . \tag{6.15}$$

We want to interpret states of this compactified theory as particles in 4-dimensional Minkowski space. This means that their momentum is given by $p^{\mu}, \mu = 0, 1, 2, 3$, and the mass is given by

$$M^2 = -p_2^2 - p_3^2 + 2p^+p^- \ . \tag{6.16}$$

Using

$$p^- = \frac{1}{\ell_s}(\alpha_0^- + \tilde{\alpha}_0^-) = \frac{2}{\ell_s^2 p^+}(L_0^\perp + \tilde{L}_0^\perp - 2) \tag{6.17}$$

we can write this as

$$M^2 = \frac{2}{\alpha'}(N^{(e)} + \tilde{N}^{(e)} + L_0^{(i)} + \tilde{L}_0^{(i)} - 2) \tag{6.18}$$

What does the spectrum of this compactified string look like? Just as in the flat space string, we can have excited modes coming from the remaining flat space directions $\alpha^{2,3}$. But now we also have contributions from the spectrum of the internal CFT, that is, from the eigenvalues of L_0^i and $\bar{L}_0^i$.

Although we set up our formalism in such a way that we can deal with arbitrary internal CFTs, let us go back to the free boson on a circle. For simplicity let us just concentrate on a single S^1 in the torus $\mathbb{T}^{22}$, say the one corresponding to $\mu = 25$. What is the spectrum of that CFT? Just as for the flat space directions, we of course have the bosonic modes α_{-n}^{25} which give make the same contribution to the mass. We can for instance consider the state

$$|p^\mu\rangle \otimes \alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}|0\rangle \ . \tag{6.19}$$

From (6.18) we see that this state has $M = 0$. Moreover it is invariant under $SO(1,3)$ transformations, *i.e.* it is a scalar particle. Compactifications on tori thus predict the existence of massless scalars.

Next, we also have ground states $|p_L^{25}, p_R^{25}\rangle$ corresponding to different momenta. These momenta are quantized by their momentum number M and winding number L,

$$p_L = \frac{M}{R} + \frac{2RL}{\ell_s^2} \ , \qquad p_R = \frac{M}{R} - \frac{2RL}{\ell_s^2} \ , \tag{6.20}$$

and their weights are given by

$$(h, \bar{h}) = (\frac{1}{4}\alpha' p_L^2, \frac{1}{4}\alpha' p_R^2) \ . \tag{6.21}$$

Consider for instance a state with no winding, $L = 0$. In that case we have $p_L^{25} = p_R^{25} = \frac{M}{R}$. Note that we don't really want to interpret p^{25} as a center of mass momentum any more. Although it corresponds to the momentum with which the particle moves along the circle, from a macroscopic point of view we cannot see this movement directly, because the circle is too small to be visible. The way we can measure it is through (6.16): Let us take a state of the form

$$|p^\mu\rangle \otimes \alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}|p_L^{25} = p_R^{25} = \frac{M}{R}\rangle \ . \tag{6.22}$$

From (6.16) we find that this state corresponds to a particle with mass $m^2 = M^2/R^2$. The mass of this particle thus depends on the radius of the circle on which we compactified the boson. In particular, if we choose the radius to be very small, this boson will be very heavy. This is the flip side of the statement that the circle is so small that we cannot see it: its associated particles with which we could probe its geometry are very heavy.

Chapter 7

Vertex Operators and scattering amplitudes

7.1 Covariant quantization

We constructed the string spectrum using lightcone quantization. The advantage of this is that it is a very physical construction that gives a manifestly positive definite Hilbert space. The disadvantage is that we have explicitly broken worldsheet conformal invariance by fixing the worldsheet coordinates. For many purposes such as computing scattering amplitudes conformal symmetry is extremely useful. It would therefore be useful to have a formalism that preserves it.

We already mentioned such a formalism: *covariant quantization*. The idea is to quantize without having imposed the constraint equations $L_m = 0$. As we saw the resulting space is not positive definite. We then impose the constraint by restricting to a subspace of physical states which are annihilated by the operators L_m.

As mentioned already, due to the central term in the Virasoro algebra we cannot impose $L_m|\phi\rangle$ for all m. The best we can do is impose it for all positive modes. We also want to impose the condition for L_0, but as usual need to allow for a ordering constant. The condition on physical states is thus

$$L_m|\phi\rangle = \bar{L}_m|\phi\rangle = 0 \qquad m > 0 \tag{7.1}$$

$$(L_0 - a)|\phi\rangle = (\bar{L}_0 - \tilde{a})|\phi\rangle = 0 \tag{7.2}$$

From our treatment of conformal field theory we immediately recognise these conditions: ϕ must be a primary field of weight $(a, \tilde{a})$. This is nicely compatible with conformal invariance. The question is then, is this enough to eliminate all states with non-positive norm $\langle\phi|\phi\rangle \leq 0$? It turns out that in general the answer to this is no. The only case where it works is if $D = 26$ and $a = \tilde{a} = 1$. In this way we have recover the same conditions as we found for lightcone quantization. In that case, our theory had a manifestly positive definite Hilbert space, but had Poincar'e symmetry for those special values; in covariant quantization, the theory is manifestly Poincar'e symmetric, but the Hilbert space is only positive definite for those special values. The two conditions are thus two sides of the same coin.

Let us briefly explain how this works for the lightest states. We will concentrate first on the left movers. For a fixed $\tilde{\alpha}^\nu$, the level (1,1) states are given by linear combinations

$$\zeta \cdot \alpha_{-1}\tilde{\alpha}^\nu_{-1}|k^\mu\rangle \ , \tag{7.3}$$

where ζ is a vector in $\mathbb{R}^{26}$. Note that the inner product between two such states is proportional to $(\zeta_1 \cdot \zeta_2)$. Imposing the L_0 condition in (7.1) first tells us that $k^2 = 0$, *i.e.* that such states correspond to massless particles. The conformal primary condition with L_1 gives

$$\zeta \cdot k = 0 \ . \tag{7.4}$$

k is null, so let us pick $k \sim (1, -1, 0, \ldots 0)$. We can then simply pick 24 ζ that correspond to unit vectors e_i for the $i = 2, \ldots 25$. Those states then simply correspond to the transverse excitations coming from the α^i, which we already know from lightcone gauge. There are however additional states that satisfy (7.4): since k is null, simply pick $\zeta = k$, *i.e.* longitudinal polarization. Even though it is a physical state, it has norm zero by the remark above. Note however that it is also orthogonal to all other physical states. The correct prescription is to identify it with a gauge degree of freedom, and mod out by such states, leading to a positive definite Hilbert space. This shows that the physical states indeed correspond to the transverse α^i_{-1} descendants, exactly as we saw directly in lightcone gauge. Obviously the same argument also works for the $\tilde{\alpha}^i_{-1}$. A more careful analysis shows that indeed lightcone quantization and covariant quantization give the same result also at higher level.

For our purposes, the bottom line is this: For any physical state we can always take $|\phi\rangle$ to satisfy (7.1), so that it is a conformal primary field of weight $(1, 1)$.

7.2 Scattering amplitudes

How do we scatter strings off each other? Let us first remind ourselves of how scattering amplitudes are computed in QFT with say a coupling constant λ. The Feynman rules tell us to take diagrams of a given topology. The topology determines with what power λ^n of the coupling constant the diagram contributes. For each topology, we then integrate over all physically different diagrams. For tree level diagrams, no such integration is necessary. For diagrams with loop, this means we need to integrate over all possible loop momenta.

How does this translate to string theory? In a string scattering process, say n_i strings come in as cylinders from $t = -\infty$. They interact, and in the end n_o strings go off to $t = \infty$ again. For a tree level process, we take the simplest topology: the diagram is thus simply a sphere with $n = n_i + n_o$ half-cylinders sticking out of it.

We now need to integrate over all such diagrams. A priori it may seem that there is an incredibly large number of diagrams to integrate over. This is however not the case. First remember that we only have to integrate over physically inequivalent diagrams. Conformal symmetry is a gauge symmetry of string theory, which means that if two such diagrams can be mapped to each other with a bijective conformal map, they are really conformally equivalent and should only be counted once. This observation reduces the number of different diagrams enormously.

Our diagram, let us call it $\Sigma_{0,n}$, is a *Riemann surface*. A Riemann surface is a complex manifold of complex dimension 1. The Riemann sphere $\hat{\mathbb{C}}$ is the prototype for a compact Riemann surface. Non-compact examples are Riemann spheres with punctures, *i.e.* with points removed. To Riemann surfaces Σ_1 and Σ_2 are called biholomorphic if there is holomorphic bijective function f

$$f : \Sigma_1 \to \Sigma_2 \ . \tag{7.5}$$

Biholomorphic Riemann surfaces are also conformally equivalent, since holomorphic maps are automatically also conformal maps because

$$dz d\bar{z} \mapsto f'(z)\bar{f}'(\bar{z}) dz d\bar{z} \ , \tag{7.6}$$

i.e. the metric remains invariant up to rescaling. This means that we only need to integrate over equivalence classes of surfaces. It turns out that the sphere with n half-cylinders is equivalent to the Riemann sphere $\hat{\mathbb{C}}$ with n punctures. We thus only need to integrate over all such spheres.

To get a rough picture of why they are equivalent, note that we can always map the semi-infinite tubes to discs with a puntcture in the middle. What happens at the puncture? In the original diagram we said that the state $|\phi\rangle$ lived all the way at the bottom of the tubes. The state is now mapped to the puncture. It is thus natural to insert the corresponding vertex operator $\phi(z)$ there.

We should now integrate over all such n-punctured spheres, which means integrating over the positions of the vertex operators $V(z,\bar{z})$. The actual operator is thus $dz d\bar{z} V(z,\bar{z})$. Since all incoming and outgoint states have to be physical, $V(z,\bar{z})$ is a primarly field of weight (1,1), which means that

$$dz d\bar{z} V(z,\bar{z}) \mapsto f'(z)\bar{f}'(\bar{z}) dz d\bar{z} f'(z)^{-1} \bar{f}'(\bar{z})^{-1} V(z,\bar{z}) \tag{7.7}$$

is indeed invariant under conformal transformations. The procedure we have outlined is thus consistent with conformal invariance.

Actually we should not integrate over all insertions. The Riemann sphere is invariant under Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \ . \tag{7.8}$$

We can use this to fix the position of three of the insertions, say to $0, 1$ and ∞. The n-point tree amplitude is thus

$$\int dz_4 d\bar{z}_4 \cdots dz_n d\bar{z}_n \langle V_1(\infty) V_2(1) V_3(0) V_4(z_4,\bar{z}_4) \cdots V_n(z_n,\bar{z}_n)\rangle \tag{7.9}$$

7.3 The Shapiro-Virasoro amplitude

Let us now compute the simplest amplitude: the tree level scattering of 4 tachyons. Even though this is not a very physical process, it will still teach us several interesting things about string scattering. Let us first generalize (5.19) to an arbitrary number of tachyon vertex operators:

$$R\left(\prod_i : e^{ik_iX(z_i,\bar{z}_i)} :\right) = \left(\prod_{i<j} |z_i - z_j|^{\alpha' k_i\cdot k_j}\right) : \prod_i e^{ik_iX(z_i,\bar{z}_i)} : \tag{7.10}$$

It follows that the corresponding correlation function is given by

$$\langle 0| \prod_i : e^{ik_iX(z_i,\bar{z}_i)} : |0\rangle = \prod_{i<j} |z_i - z_j|^{\alpha' k_i\cdot k_j} \langle 0| \sum_i k_i\rangle = \prod_{i<j} |z_i - z_j|^{\alpha' k_i\cdot k_j} \delta(\sum_i k_i) \tag{7.11}$$

To compute the 4-tachyon scattering amplitude, we set $z_1 = \infty, z_2 = 1, z_3 = z, z_4 = 0$. More precisely, we send $z_1 \to \infty$ while including prefactors $z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}$ as in (4.60). In total we obtain

$$|z|^{\alpha' k_3\cdot k_4}|1-z|^{\alpha' k_2\cdot k_3} \tag{7.12}$$

Note that the limit $z_1 \to \infty$ indeed gives something finite, since

$$\lim_{z_1\to\infty} |z_1|^{\alpha' k_1^2}|z_1|^{\alpha' k_1\cdot k_4}|z_1 - z|^{\alpha' k_1\cdot k_3}|z_1 - 1|^{\alpha' k_1\cdot k_2} = 1 \tag{7.13}$$

since $\sum_i k_i = 0$. We thus have

$$A_4 = \frac{\kappa^2}{4\pi}\int d^2z|z|^{\alpha' k_3\cdot k_4}|1-z|^{\alpha' k_2\cdot k_3} \tag{7.14}$$

where κ is a coupling constant parametrizing the interaction strength of strings. In the exercises you will show that this integral can be evaluated in terms of Γ functions,

$$I = \int d^2z|z|^{-A}|1-z|^{-B} = B(1-\frac{A}{2}, 1-\frac{B}{2}, \frac{A+B}{2} - 1) \tag{7.15}$$

where

$$B(a,b,c) = \pi\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)} \; . \tag{7.16}$$

Let us now discuss this result, and compare it to what we would expect from a scattering amplitude in an ordinary quantum field theory.

In a field theory with 3-vertices, there are three topologically different diagrams that contribute at tree level to the scattering of 4 particles: the so-called s, t and u

channel. To each of this channel correspond a Mandelstam variable,

$$\begin{aligned} s = -(k_3+k_4)^2 &= -2k_3\cdot k_4 + m_3^2 + m_4^2 && (7.17)\\ t = -(k_2+k_4)^2 &= -2k_2\cdot k_4 + m_2^2 + m_4^2 && (7.18)\\ u = -(k_2+k_3)^2 &= -2k_2\cdot k_3 + m_2^2 + m_3^2\ . && (7.19)\end{aligned}$$

They satisfy the relation

$$s+t+u = \sum_i m_i^2\ , \tag{7.20}$$

where the m_i are the masses of the external particles. The Mandelstam variables correspond to the momentum squared of the intermediate particles in the respective channel. Poles of the amplitude then correspond to physical particles. To see this, note that the tree diagram with a particle of mass M say in the s channel contains the propagator

$$\sim \frac{1}{s-M^2}\ , \tag{7.21}$$

which leads to a pole for $s = M^2$. In this way we can read of the spectrum of the a theory from the poles of its scattering amplitudes.

In our case, since our external particles are tachyons, we have $m_i^2 = -\frac{4}{\alpha'}$. Comparing (7.14) and (7.15) we find

$$B(-\frac{1}{2}\alpha(s), -\frac{1}{2}\alpha(t), -\frac{1}{2}\alpha(u))\ , \tag{7.22}$$

where we defined $\alpha(s) = 2 + \frac{1}{2}\alpha' s$. Note that $\alpha(m^2)$ is exactly the leading Regge trajectory, *i.e.* the maximal spin of a particle of mass m.

Let us now analyse the poles of A_4. The Γ-function has poles for non-positive integers and no zeros. The poles of A_4 are thus exactly when $-\frac{1}{2}\alpha$ is a non-positive integer, *i.e.* if

$$\alpha = 0, 2, 4, \ldots\ . \tag{7.23}$$

Note that the poles of A_4 correspond precisely to the spectrum of the particles of the closed string. There is indeed an infinite number of them.

Chapter 8

Higher loop amplitudes

Let us now discuss loop amplitudes. The principle is the same as for tree amplitudes discussed in the previous chapter. Rather than $\hat{\mathbb{C}}$ with punctures, we will take a Riemann surface of a more complicated topology. It turns out that the topology of any Riemann surface is completely determined by its *genus* g, and the number of punctures n. We can think of the genus as the number of 'handles' of the surface: the sphere has genus 0, the torus has genus 1, and so on. From a QFT point of view, g corresponds to the number of loops in the diagram, and n of course to the number of external particles.

Let us consider a 1-loop n-particle amplitude so that $g = 1$. From the rules we learned in the previous section, we can obtain it by computing the CFT correlation function of n string vertex operators on a torus $\mathbb{T}$, and then integrating over all conformally inequivalent configurations. Schematically, it will look like

$$A_n^{g=1} \sim \underbrace{\int_{\mathcal{M}} d\mu \left(\prod_{i=1}^{n} \int_{\mathbb{T}} d^2 z_i \right)}_{\substack{\text{integrate over} \\ \text{inequivalent configurations}}} \underbrace{\langle V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n) \rangle_{\mathbb{T}}}_{\text{CFT correlator on } \mathbb{T}} \tag{8.1}$$

Here $\mathcal{M}$ is the space of all conformally inequivalent tori. Let us first discuss the 'string theoretic' part, *i.e.* the integral over inequivalent configurations. We will return to the computation of the CFT correlator in a bit. Just as in the $g = 0$ case, we can conformally map any $g = 1$ surface with n-punctures to a simpler surface, in this case, to a torus with n punctures. We thus need to integrate over conformally inequivalent tori.

8.1 The moduli space of tori

To describe a torus in the language of complex numbers, it is given by the quotient of the complex plane

$$\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}) \ , \tag{8.2}$$

that is by identifying

$$z \sim z + n\omega_1 + m\omega_2 \qquad m, n \in \mathbb{Z} \ , \tag{8.3}$$

where $\omega_{1,2}$ are the linearly independent periods of the torus. Again, we only care about the conformal equivalence class. In particular, we can rotate and rescale $\mathbb{C}$ such that $\omega_1 = 1$, and $\omega_2 =: \tau \in \mathbb{H}_+$. (If $\mathrm{Im}(\tau)$ should be negative, we can simply exchange ω_1 and ω_2 to achieve this.) We call τ the *modulus* of the torus.

A first guess would thus be to integrate (8.21) over all such moduli,

$$\int_{\mathbb{H}_+} \frac{d^2\tau}{\mathrm{Im}(\tau)} \left(\prod_{i=1}^{n} \int_{\mathbb{T}} d^2 z_i \right) \langle V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n) \rangle_{\mathbb{T}} \ . \tag{8.4}$$

Let us explain where the factor $\mathrm{Im}(\tau)$ in the denominator comes from. For this, let us briefly discuss the case of $g = 1$ with n insertions of string vertices $V(z_i, \bar{z}_i)$. Just as in the $g = 0$ case, we need to integrate over the insertions of the vertices. However, just as in the $g = 0$ case, the torus has global conformal symmetries. In this case they are simply given by translations. We can thus fix the position of one of the vertices,

$$\begin{aligned} &\int d^2 z_2 \ldots d^2 z_n \langle V_1(1) V_2(z_2, \bar{z}_2) \cdots V_n(z_n, \bar{z}_n) \rangle_{\mathbb{T}} = \\ &\qquad\qquad \frac{1}{\mathrm{Im}(\tau)} \int d^1 z_1 \ldots d^2 z_n \langle V_1(z_1, \bar{z}_1) \cdots V_n(z_n, \bar{z}_n) \rangle_{\mathbb{T}} \end{aligned} \tag{8.5}$$

In the second line, rather than fixing one of the positions, we integrate over all of them, and then divide by the volume of the translation group to correct for the overcounting. This volume is simply the volume of the torus, which is $\mathrm{Im}(\tau)$.

In (8.4) we are however still overcounting: there are conformally equivalent tori with different values of τ. To see this, consider the *modular group*

$$SL(2, \mathbb{Z}) : \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) \ , \qquad ad - bc = 1 \ , \tag{8.6}$$

acting on τ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \ . \tag{8.7}$$

Acting with $SL(2, \mathbb{Z})$ on τ leaves the torus conformally invariant. To see this, note that if we start out with the torus

$$z \sim z + n\tau + m \ , \tag{8.8}$$

after acting with $SL(2, \mathbb{Z})$ we can rescale coordinates to $\tilde{z} = (c\tau + d)z$ to get

$$\tilde{z} \sim \tilde{z} + (na + mc)\tau + (bn + dm) = \tilde{z} + \tilde{n}\tau + \tilde{m} \ . \tag{8.9}$$

One can show that through such a modular transformation one can always map any

$\tau \in \mathbb{H}_+$ to a point in the fundamental region $\mathcal{F}$ ('keyhole region'):

$$\mathcal{F} = \{\tau \in \mathbb{H}_+ : -\frac{1}{2} \leq \mathrm{Re}(\tau) < \frac{1}{2} \, , |\tau| > 1 \text{ or } -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq 0 \, , |\tau| = 1\} \tag{8.10}$$

This means that conformally inequivalent tori are in one-to-one correspondence with points in $\mathcal{F}$. Rather than integrating over the entire upper half-plane, the correct prescription is thus

$$\int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}(\tau)} \cdots \, . \tag{8.11}$$

8.2 CFT correlation functions on the torus

Let us now compute the CFT correlation function on the torus. First of all, what does such a torus look like? We can start out with a cylinder of circumference 1. We cut off a segment of length $y = \mathrm{Im}(\tau)$. We then twist the end of the segment by an angle $x = \mathrm{Re}(\tau)$, and glue the two ends together. This gives a torus of modulus τ.

What about the corresponding amplitude? In what follows, let us specialize to the case of $n = 0$, that is the vacuum amplitude on the torus,

$$Z(\tau) := \langle 1 \rangle_{\mathbb{T}} \, . \tag{8.12}$$

To evaluate this, let us go back to the description of the bosonic string on a cylinder. Suppose we start out with a state $|\phi\rangle$. We then propagate it through Euclidean time y. The corresponding generator is the Hamiltonian $H = L_0 + \bar{L}_0$, so that we get

$$e^{-2\pi y (L_0 + \bar{L}_0)} |\phi\rangle \, . \tag{8.13}$$

The twisting by angle x is generated by the momentum operator $P = L_0 - \bar{L}_0$, so that we get in total

$$e^{-2\pi y (L_0 + \bar{L}_0)} e^{2\pi i x (L_0 - \bar{L}_0)} |\phi\rangle = q^{L_0} \bar{q}^{\bar{L}_0} |\phi\rangle \, , \tag{8.14}$$

where we have introduced the new variable

$$q = e^{2\pi i \tau} \, . \tag{8.15}$$

To get the torus amplitude, we project to $\langle\phi|$ and sum over all (orthonormal) states ϕ,

$$\sum_\phi \langle\phi| q^{L_0} \bar{q}^{\bar{L}_0} |\phi\rangle = \mathrm{Tr}_{\mathcal{H}} q^{L_0} \bar{q}^{\bar{L}_0} \, . \tag{8.16}$$

The torus amplitude is thus given by the partition function, as is well-known from QFT.

Actually, (8.16) is not quite correct. The reason for this subtlety is that T is not quite a primary field, and therefore does not transform nicely under more complicated conformal transformations. As we argued before, naturally our CFT lives on the Riemann sphere $\mathbb{C}$. To map it to to a cylinder of circumference 2π, we use the map

$$z \mapsto w(z) = \log z \ . \tag{8.17}$$

As we mentioned before, due to the central charge c, $T(z)$ is not quite a primary field, but rather transforms as

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right] \tag{8.18}$$

with the Schwarzian derivative

$$\{f,z\} = \frac{2\partial_z^3 f \partial_z f - 3\partial_z^2 f \partial_z^2 f}{2\partial_z f \partial_z f} \ . \tag{8.19}$$

For the transformation (8.17), $\{w;z\} = \frac{1}{2}z^{-2}$. On the cylinder, the zero mode L_0^{cyl} is then given by

$$\begin{aligned} L_0^{cyl} = \int_0^{2\pi} dw T'(w) = \oint dz z\left(\frac{dw}{dz}\right)^2 T'(w) \\ = \oint dz z\left(T(z) - \frac{c}{24}z^{-2}\right) = L_0 - \frac{c}{24} \end{aligned} \tag{8.20}$$

The correct expression for the torus amplitude is thus not quite (8.16), but rather

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \ . \tag{8.21}$$

At first sight, the difference between (8.16) and (8.21) seems minor: it is only a shift in the vacuum energy from 0 to $-\frac{c}{24}$. In fact it will turn out to be crucial.

8.3 Modular invariance and UV finiteness

Does this prescription make sense? There are of course many different choices for the fundamental region, and the result should not depend on them. This is only the case if the integrand (8.11) transforms nicely under modular transformations. In particular, $d^2\tau Z(\tau)$ must be invariant under all modular transformations in $SL(2,\mathbb{Z})$. Using

$$\frac{d}{d\tau}\frac{a\tau+b}{c\tau+d} = \frac{a(c\tau+d) - c(a\tau+b)}{(c\tau+d)^2} = \frac{1}{(c\tau+d)^2} \ , \tag{8.22}$$

and

$$\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2} \ , \tag{8.23}$$

we find that $Z(\tau)$ must have the following modular transformation properties:

$$Z\left(\frac{a\tau+b}{c\tau+d}\right) = |c\tau+d|^2 Z(\tau) \ . \tag{8.24}$$

A function $f(\tau)$ which satisfies $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k_1}(c\bar{\tau}+d)^{k_2} f(\tau)$ is called a (non-holomorphic) modular form of weight (k_1, k_2). (8.24) thus tells us that $Z(\tau)$ has to be a modular form of weight (1,1).

Let us see how this works for a single boson on flat space. We want to evaluate the trace

$$\mathrm{Tr}_V q^{L_0-\frac{1}{24}} \bar{q}^{\bar{L}_0-\frac{1}{24}} \ . \tag{8.25}$$

As we saw earlier, for a fixed momentum k, the contribution of all oscillator modes is

$$|q|^{\alpha' k^2/2} \left| q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \right|^2 = e^{-\pi \mathrm{Im}(\tau)\alpha' k^2} |\eta(\tau)|^{-2} \tag{8.26}$$

where $\eta(\tau)$ is the Dedekind eta function. Integrating over all momenta k, we get

$$Z_1(\tau) = |\eta(\tau)|^{-2} \int dk e^{-\pi \mathrm{Im}(\tau)\alpha' k^2} = (\alpha' \mathrm{Im}(\tau))^{-1/2} |\eta(\tau)|^{-2} \ . \tag{8.27}$$

It turns out that the Dedekind eta function has nice modular transformation properties. Namely, $|\eta(\tau)|^2$ is a modular form of weight $(\frac{1}{2}, \frac{1}{2})$. This means that the partition function $Z_1(\tau)$ of a single free boson is invariant under modular transformations. The contribution of the 24 transverse directions to $Z(\tau)$ is $Z_1(\tau)^{24}$ and therefore modular invariant. As we are in lightcone gauge, we have eliminated the oscillator modes α_0 and α_1. We still have the contribution of the momenta p_0 and p_1 though, which give a factor of $(\alpha' \mathrm{Im}(\tau)^{-1})$.[1] In total we thus obtain

$$Z(\tau) = \frac{1}{\alpha' \mathrm{Im}(\tau)} (\alpha' \mathrm{Im}(\tau))^{-12} |\eta(\tau)|^{-48} \ , \tag{8.28}$$

which indeed is a modular form of weight (1,1) that satifies (8.24).

In total we thus obtain for the 0-point string amplitude at $g = 1$

$$A_0^{g=1} \sim \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}(\tau)} Z(\tau). \tag{8.29}$$

Is this amplitude finite? From quantum field theory, we would expect it to diverge. Schematically, a 1-loop integral in a QFT diagram with a massless particle looks something like

$$\int d^4 p \frac{1}{p^2} \ . \tag{8.30}$$

[1] The integral over p_0 is actually divergent. We will simply take its analytic continuation.

It diverges in the UV region, for $p \to \infty$. What happens for (8.29)? In the torus, the UV region corresponds to $\mathrm{Im}(\tau) \to 0$, that is very short and thick tori. The crucial observation is that for our choice of $\mathcal{F}$, in (8.29) we never even integrate over that critical region. We have therefore cut out any potential UV divergences. By construction, (8.29) is therefore UV finite. This is what makes string theory so much better behaved than most QFTs.

Does that mean that (8.29) is indeed finite? The answer to this is no. Even though there are no UV divergences, the integrand $Z(\tau)$ diverges exponentially for very long and thin tori, namely

$$Z(it) \to e^{4\pi t} \qquad \text{for } t \to \infty. \tag{8.31}$$

This divergence is due to the tachyon in the spectrum. Unlike particles with positive mass square, whose correlators decay exponentially over long distances, tachyons grow exponentially, thus leading to divergences. Note that this is an IR (long-distance) effect. Such divergences are known already from QFTs, and are far less serious than UV divergences. Still, this is the first time that the presence of the tachyon has become a problem for us. We will therefore turn to superstring theory, which does not have a tachyon any more.

Chapter 9

Superstring Theory

9.1 Free fermion

We saw several problems with the bosonic string. It leads to spacetime bosons only. Also, it has a tachyon in the spectrum. Let us try to address these issues by introducing fermions.

The worldsheet theory of the bosonic string was described by free bosons in 2d. It is thus natural to start by considering free fermions in 2d. Define the 2d Dirac matrices ρ^a,

$$\rho^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \rho^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{9.1}$$

Satisfying

$$\{\rho^a, \rho^b\} = 2\eta^{ab} . \tag{9.2}$$

As usual define the chirality operator $\bar{\rho} := \rho^0\rho^1 = \sigma^3$. The 2d fermions ψ have two components,

$$\psi^a = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} , \tag{9.3}$$

with chirality ± 1. They are given by Grassmann variables. The action is given by

$$S = -\frac{1}{4\pi}\int d^2\sigma i\bar{\psi}\rho^a\partial_a\psi \tag{9.4}$$

where $\bar{\psi} := \psi^\dagger\rho^0$. Note that $\dagger$ here simply denotes taking the transpose and the complex conjugate. This leads indeed to the massless Dirac equation

$$i\rho^a\partial_a\psi = 0 , \tag{9.5}$$

with suitable boundary conditions that we will discuss later. Note that the components $\psi_\pm$ are both Majorana and Weyl at the same time, *i.e.* they are real, and have definite chirality. The energy-momentum tensor is given by

$$T_{ab} = -\frac{i}{4}\bar{\psi}\rho_a\partial_b\psi - \frac{i}{4}\bar{\psi}\rho_b\partial_a\psi \ . \tag{9.6}$$

Using the equation of motion (9.5), we find that $T^a_a = 0$, *i.e.* the energy-momentum tensor is traceless. It follows that the theory is conformal. Indeed, we can again see this very explicitly by going to lightcone coordinates,

$$S = -\frac{1}{4\pi}\int d^2\sigma i\psi^T(-\mathbf{1}\partial_0 + \sigma_3\partial_1)\psi = \frac{1}{2\pi}\int d^2\sigma i(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+) \tag{9.7}$$

We see that this is indeed invariant under $\sigma^\pm \mapsto \tilde{\sigma}^\pm(\sigma^\pm)$, provided ψ_+ and ψ_- transform as

$$\psi_- \mapsto (\tilde{\sigma}'_-)^{-1/2}\psi_- \qquad \psi_+ \mapsto (\tilde{\sigma}'_+)^{-1/2}\psi_+ \tag{9.8}$$

This means that ψ_- has conformal weight $(0, \frac{1}{2})$ and ψ_+ has conformal weight $(\frac{1}{2}, 0)$. Once we quantize ψ, we will indeed check that it is a primary field of the correct conformal weight. The two non-vanishing components of T are then given by

$$T_{++} = -\frac{i}{2}\psi_+\partial_+\psi_+ \qquad T_{--} = -\frac{i}{2}\psi_-\partial_-\psi_- \ . \tag{9.9}$$

Finally let us compute the Poisson brackets:

$$\{\psi_+(\sigma), \psi_+(\sigma')\}_{PB} = \{\psi_-(\sigma), \psi_-(\sigma')\}_{PB} = 2\pi i\delta(\sigma - \sigma') \tag{9.10}$$

$$\{\psi_-(\sigma), \psi_+(\sigma')\}_{PB} = 0 \tag{9.11}$$

9.2 Neveu-Schwarz and Ramond sectors

Let us now return to the derivation of the equation of motion, and give a more careful derivation, taking into account boundary terms. For ψ_+ we have

$$\begin{aligned}\delta S &= i\int d^2 i\sigma(\delta\psi_+\partial_-\psi_+ + \psi_+\partial_-\delta\psi_+) = i\int d^2\sigma(\delta\psi_+\partial_-\psi_+ - (\partial_-\psi_+)\delta\psi_+) + i\int d^2\sigma\partial_-(\psi_+\delta\psi_+) \\ &= i\int d^2\sigma 2\delta\psi_+\partial_-\psi_+ + i\int_{-\infty}^{\infty} d\tau\,(\psi_+(\tau,0)\delta\psi_+(\tau,0) - \psi_+(\tau,\pi)\delta\psi_+(\tau,\pi))\end{aligned} \tag{9.12}$$

where in the last line we have used that the ψ are Grassman variables and therefore anticommute. The first term gives the Dirac equation $\partial_-\psi_+ = 0$. To make the second term vanish, we need to impose boundary conditions. Naively, we might think that we need to impose periodic boundary conditions $\psi_+(\tau, 0) = \psi_+(\tau, \pi)$ just as for the bosons. Fermionic fields however are intrinsically quantum mechanical objects. By previous arguments, the most we can impose is $\psi_+(\tau, 0) = e^{i\alpha}\psi_+(\tau, \pi)$. For the boundary term to vanish however, we only have the two possibilities $e^{i\alpha} = \pm 1$, *i.e.* we can either impose periodic or anti-periodic boundary conditions. The two cases are called

$$\psi_+(\tau,0) = -\psi_+(\tau,\pi) \quad : \quad \text{Neveu-Schwarz} \tag{9.13}$$
$$\psi_+(\tau,0) = \psi_+(\tau,\pi) \quad : \quad \text{Ramond} \tag{9.14}$$

The same argument of course also holds for ψ_-. We can choose the periodicity condition independently, so that in total we have four possibilities: NS-NS, NS-R, R-NS, R-R. The Fourier expansion is then

$$\begin{array}{l} \psi_-(\sigma^-) = \sqrt{2}\sum_{r\in\mathbb{Z}+\phi} b_r e^{-2ir\sigma^-} \\ \psi_+(\sigma^+) = \sqrt{2}\sum_{r\in\mathbb{Z}+\phi} \tilde{b}_r e^{-2ir\sigma^+} \end{array} \text{ where } \left\{ \begin{array}{l} \phi = 0 : R \\ \phi = \frac{1}{2} : NS \end{array} \right. \tag{9.15}$$

The Poisson brackets of the modes b and $\tilde{b}$ are given by

$$\{b_r, b_s\}_{PB} = \{\tilde{b}_r, \tilde{b}_s\}_{PB} = i\delta_{r+s,0} \tag{9.16}$$
$$\{b_r, \tilde{b}_s\}_{PB} = 0 \ . \tag{9.17}$$

The modes of T are then given by

$$L_m = -\frac{1}{4\pi}\int_0^\pi d\sigma e^{-2im\sigma} T_{--} \qquad \bar{L}_m = -\frac{1}{4\pi}\int_0^\pi d\sigma e^{2im\sigma} T_{++} \tag{9.18}$$

In terms of the the modes of ψ they are given by

$$L_m = \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}(r+\frac{1}{2}m)b_{-r}b_{m+r} \tag{9.19}$$
$$\bar{L}_m = \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}(r+\frac{1}{2}m)\tilde{b}_{-r}\tilde{b}_{m+r} \tag{9.20}$$

Note that the L_m are bosonic and therefore have integer modes no matter which sector we are in.

9.3 Fermionic string action

The gauge fixed version of the Polyakov action for the bosonic string consisted of D free bosons. As we argued, it was crucial that this had conformal symmetry. Since our free fermions also have conformal symmetry, it makes sense to try to write down the action of a fermionic string which contains both bosons and fermions. Let us assume that it has D free bosons and D free fermions of the type (9.4). The total action is thus

$$S = -\frac{1}{8\pi}\int d^2\sigma \left(\frac{2}{\alpha'}\partial_a X^\mu \partial^a X_\mu + 2i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right)$$
$$= \frac{1}{2\pi}\int d^2\sigma \left(\frac{2}{\alpha'}\partial_+ X \cdot \partial_- X + i(\psi_- \cdot \partial_+ \psi_- + \psi_+ \cdot \partial_- \psi_+)\right) \quad (9.21)$$

The spacetime index μ runs from $0, 1, \dots D-1$. At first, it may seem surprising that the fermsions ψ should carry a vector index μ rather than a spacetime spinor index: naively we would have expected them to correspond to spacetime fermions. The story however is more complicated than that, and we will return to it in the next chapter.

As mentioned above, (9.21) has conformal symmetry. The infintesimal version $\sigma^\pm \mapsto \sigma^\pm + \xi^\pm$ acts as

$$\delta_\xi X^\mu = -(\xi^+ \partial_+ X^\mu + \xi^- \partial_- X^\mu) \quad (9.22)$$
$$\delta_\xi \psi_+^\mu = -(\xi^+ \partial_+ \psi_+^\mu + \xi^- \partial_- \psi_+^\mu + \frac{1}{2}\partial_+ \xi^+ \psi_+^\mu) \quad (9.23)$$
$$\delta_\xi \psi_-^\mu = -(\xi^+ \partial_+ \psi_-^\mu + \xi^- \partial_- \psi_-^\mu + \frac{1}{2}\partial_- \xi^- \psi_-^\mu) \quad (9.24)$$

The additional terms in the transformation of ψ are due to the fact that ψ has conformal weight $\frac{1}{2}$.

(9.22) is a *bosonic* symmetry of the theory, that is, it transforms bosonic fields such as X into bosonic fields, and fermionic fields such as ψ into fermionic fields. (9.21) however also has *fermionic* symmetry, *i.e.* a symmetry that transforms fermions into bosons and vice versa. This symmetry is *supersymmetry*. More precisely, take the Majorana spinor $\epsilon(\tau, \sigma) = (\epsilon^+, \epsilon^-)^T$ satisfying

$$\partial_- \epsilon^+ = 0 \ , \qquad \partial_+ \epsilon^- = 0 \ , \quad (9.25)$$

to be the generator of the transformations

$$\sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu = i(\epsilon^+ \psi_+^\mu + \epsilon^- \psi_-^\mu) \quad (9.26)$$
$$\delta_\epsilon \psi_+^\mu = -\sqrt{\frac{2}{\alpha'}}\epsilon^+ \partial_+ X^\mu \quad (9.27)$$
$$\delta_\epsilon \psi_-^\mu = -\sqrt{\frac{2}{\alpha'}}\epsilon^- \partial_- X^\mu \quad (9.28)$$

Let us check that this is indeed a symmetry of the action. For a start, take $\epsilon^- = 0$.

We get

$$2\pi\delta_\epsilon\mathcal{L} = \sqrt{\frac{2}{\alpha'}}i\left(\partial_+(\epsilon^+\psi_+^\mu)\partial_-X^\mu + \partial_+X_\mu\partial_-(\epsilon^+\psi_+^\mu)) - \epsilon^+\partial_+X_\mu\partial_-\psi_+^\mu - \psi_+^\mu\partial_-(\epsilon^+\partial_+X^\mu)\right)$$
$$= \sqrt{\frac{2}{\alpha'}}i\left(\partial_+(\epsilon^+\psi_+^\mu)\partial_-X^\mu + (\partial_-\epsilon^+)\partial_+X\cdot\psi_+ + \epsilon^+\psi_+\cdot\partial_+\partial_-X + (\partial_-\epsilon^+)\psi_+\cdot\partial_+X\right)$$
$$= \sqrt{\frac{2}{\alpha'}}i\left(\partial_+(\epsilon^+\psi_+\cdot\partial_-X) + 2(\partial_-\epsilon^+)\psi_+\cdot\partial_+X\right) \ . \quad (9.29)$$

If we choose $\partial_-\epsilon^+ = 0$ as indicated above, we are left with only the first term, which is a total derivative. The action is therefore indeed invariant under (9.26). The same computation of course also works for ϵ^-. Moreover, the coefficient of $(\partial_-\epsilon^+)$ we can directly read off the Noether currents J_+ and J_-,

$$J_+(\sigma^+) = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_+^\mu\partial_+X_\mu \quad (9.30)$$
$$J_-(\sigma^-) = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_-^\mu\partial_-X_\mu \quad (9.31)$$

Note that since (9.26) is a fermionic symmetry, $J_\pm$ is fermionic.

Let us now discuss the equations of motions. From the Dirac equation $\rho^a\partial_a\psi = 0$ it follows that

$$\partial_-\psi_+^\mu = 0 \qquad \partial_+\psi_-^\mu = 0 \ . \quad (9.32)$$

Just like ∂_+X^μ and ∂_-X^μ, ψ_-^μ and ψ_+^μ are purely left- and right-moving, respectively. In particular this shows that (9.26) makes sense, since left-moving fields are mapped to left-moving fields and similar for right-moving fields. Similarly T and J only have left- and right-moving components, namely

$$T_{++}(\sigma^+) = -\frac{1}{\alpha'}\partial_+X^\mu\partial_+X_\mu - \frac{i}{2}\psi_+^\mu\partial_+\psi_{+\mu} \quad (9.33)$$
$$T_{--}(\sigma^-) = -\frac{1}{\alpha'}\partial_-X^\mu\partial_-X_\mu - \frac{i}{2}\psi_-^\mu\partial_-\psi_{-\mu} \quad (9.34)$$

for the energy-momentum tensor and

$$J_+(\sigma^+) = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_+^\mu\partial_+X_\mu \quad (9.35)$$
$$J_-(\sigma^-) = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_-^\mu\partial_-X_\mu \quad (9.36)$$

for the supercurrent J. Again, we can express their modes in terms of the oscillator modes. It is useful to split the contribution to L_m into the contributions coming from the bosons and fermions,

$$L_m = -\frac{1}{4\pi}\int_0^\pi d\sigma e^{-2im\sigma}T_{--} = L_m^{(\alpha)} + L_m^{(b)} \tag{9.37}$$

$$G_r = -\frac{1}{\pi\sqrt{2}}\int_0^\pi d\sigma e^{-2ir\sigma}J_- \tag{9.38}$$

and similarly for the right movers. Note that r is either half-integer or integer, depending on whether the fermions are in the NS or the R sector. We then find

$$L_m^{(\alpha)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_{m-n}\cdot\alpha_n \tag{9.39}$$

$$L_m^{(b)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}(r+\frac{1}{2}m)b_{-r}\cdot b_{m+r} \tag{9.40}$$

$$G_r = \sum_{m\in\mathbb{Z}}\alpha_{-m}\cdot b_{r+m} \tag{9.41}$$

9.4 Quantization

Let us now quantize the fermionic string. The Poisson brackets are given by:

$$\{\psi_+^\mu(\sigma),\psi_+^\nu(\sigma')\}_{PB} = \{\psi_-^\mu(\sigma),\psi_-^\nu(\sigma')\}_{PB} = 2\pi i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{9.42}$$

$$\{\psi_-^\mu(\sigma),\psi_+^\nu(\sigma')\}_{PB} = 0 \tag{9.43}$$

The bosonic part of course works exactly the same way as before. As usual we will replace ψ by an operator. Because of the fermionic statistics, we will replace the Poisson bracket by an anti-commutator

$$\{\cdot,\cdot\}_{PB} \to \{\cdot,\cdot\}\ . \tag{9.44}$$

The anti-commutator of the fermion modes thus becomes

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}\ . \tag{9.45}$$

Moreover the fermionic modes and the bosonic modes commute. The Hilbert space is then the standard fermionic Fock space. Namely, we define creation operators as the operators with negative modes, $b_r, r<0$, and annililation operators as operators with positive modes, $b_r, r>0$. The vacuum $|0\rangle$ is then annihilated by all annihilators,

$$b_r^\mu|0\rangle = \tilde{b}_r^\mu|0\rangle = 0 \qquad \forall\mu, r>0\ . \tag{9.46}$$

The Fock space is then obtained by acting with creation operators on $|0\rangle$. Note howevert that because of

$$(b_{-r}^\mu)^2 = 0\ , \tag{9.47}$$

unlike in the bosonic case, the occupation number of a given mode is always either 0 or 1. Just as in the bosonic case, this space is not positive definite, due to the the states coming from the $\tilde{b}^0$ and b^0.

In the NS sector, this takes care of all the modes. In the Ramond sector however we also have zero modes b_0^ν, which will play a crucial role. We postpone discussing these modes until we discuss the spectrum of the fermionic string.

Normal ordering for the fermionic modes is defined as

$$: b_r^\mu b_s^\nu := \begin{cases} b_r^\mu b_s^\nu & : \quad s \geq r \\ -b_s^\nu b_r^\mu & : \quad s < r \end{cases} . \tag{9.48}$$

We can now write down the quantized version of the $N = 1$ superconformal algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8} m(m^2 - 2\phi)\delta_{m+n,0} && (9.49) \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} && (9.50) \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{\phi}{2}\right)\delta_{r+s,0} && (9.51) \end{aligned}$$

where again $\phi = 0$ for R and $\phi = \frac{1}{2}$ for NS. (9.49)–(9.51) is called the $N = 1$ *superconformal algebra*. It is the $N = 1$ supersymmetric extension of the Virasoro algebra. We see that each fermion adds $c = \frac{1}{2}$ to the central charge. In the Ramond sector, the central term of the Virasoro algebra may look slightly unusual. We could bring it to the standard form by redefining the zero mode $L_0 \mapsto L_0 + \frac{D}{16}$, but we find it more convenient to leave it like that.

9.5 Light cone quantization

We again have a vector space that is not positive definite. In the bosonic case, we identified the worldsheet conformal symmetry (9.22) as a gauge symmetry. For the fermionic string, we also want to identifiy the worldsheet supersymmetry (9.26) as a gauge symmetry. From what we have said so far, it is not obvious why we would do this. The deeper reason is that properly we would start with the supersymmetry analogue of the (un-gauge-fixed) Polyakov action (2.12). We did not go this way because the supersymmetric Polyakov action contain the gravitino χ, which is somewhat cumbersome to work with. The upshot is that we can again fix the metric h_{ab} to η_{ab}, and we can also fix the gravitino χ to obtain the gauged fixed action (9.21). This however means that we also need to impose the equations of motion for h and χ as constraint equations. The equation of motion for h is again

$$T_{ab} = 0 , \tag{9.52}$$

whereas the equation of motion for χ is setting (**??**) to zero,

$$J_a = 0 \ . \tag{9.53}$$

The residual gauge symmetry of the gauge fixed Polyakov action is thus the full $N = 1$ superconformal symmetry.

We want to eliminate the negative norm states by fixing the residual worldsheet gauge. By the same arguments as for the bosonic string, we can choose worldsheet coordinates τ, σ so that

$$X^+ = x^+ + 2\alpha' p^+ \tau \ . \tag{9.54}$$

We can then use superconformal symmetry to also eliminate the + direction of the fermion,

$$\psi^+_\pm = 0 \ . \tag{9.55}$$

(Note that here $\psi^+ = (\psi^0 + \psi^-)/\sqrt{2}$.) We can now impose the constraint equation

$$T_{--}(\sigma^-) + 4a = 0 \ , \qquad T_{++}(\sigma^+) + 4\tilde{a} = 0 \ , \tag{9.56}$$

where we already allowed for normal ordering constants a and $\tilde{a}$. We have defined the normal ordering constants such that (9.56) is equivalent to

$$L_m - a = 0 \ , \qquad \bar{L}_m - \tilde{a} = 0 \ . \tag{9.57}$$

We do this by using (9.56) to express X^- in terms of the other modes,

$$\partial_\pm X^- = -\frac{1}{2p^+} T^\perp_{\pm\pm} = \frac{1}{2p^+} \sum_{i=2}^{D-1} \left(\frac{1}{\alpha'} \partial_\pm X^i \partial_\pm X^i + \frac{i}{2} \psi^i_\pm \partial_\pm \psi^i_\pm\right) \ . \tag{9.58}$$

Similarly, imposing $J_\pm = 0$ gives

$$\psi^-_\pm = -\sqrt{\frac{2}{\alpha'}} \frac{1}{p^+} J^\perp_\pm \ . \tag{9.59}$$

Note that there is no ordering ambiguity here, so that we do not need to introduce an ordering constant. In terms of the modes, we get

$$\alpha^-_n = \sqrt{\frac{2}{\alpha'}} \frac{1}{p^+} (L^\perp_n - a\delta_{0,n}) \ , \qquad b^-_r = \sqrt{\frac{2}{\alpha'}} \frac{1}{p^+} G^\perp_r \ . \tag{9.60}$$

Just as in the bosonic case, we have thus used the residual gauge symmetry to eliminate all modes associated with the 0 and 1 directions. Our Hilbert space is thus manifestly positive definite. However, we have broken manifest spacetime Poincar'e invariance, and therefore need to check for anomalies. To this end we again have to compute the commutator of the Lorentz generators

$$[J^{i-}, J^{j-}] \ . \tag{9.61}$$

The computation is tedious, but straightforward along the same lines as in the bosonic case. The total result is

$$[J^{i-}, J^{j-}] = -(\ell_s p^+)^2 \sum_{m=1}^{\infty} (\Delta_m - m)(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m) \ . \tag{9.62}$$

where

$$\Delta_m = m\left(\frac{D-2}{8}\right) + \frac{1}{m}\left(2\left(a + \left(\frac{1}{2} - \phi\right)\frac{D-2}{8}\right) - \frac{D-2}{8}\right) \ . \tag{9.63}$$

Note that the condition on the normal ordering constant depends on R versus NS. This is due to the fact that in (9.49) we had shifted the zero mode in the R sector relative to the canonical definition. The condition for the Poincaré anomaly to vanish is in the NS sector

$$D = 10 \ , \qquad a_{NS} = \frac{1}{2} \ , \tag{9.64}$$

and in the R sector

$$D = 10 \ , \qquad a_R = 0 \ . \tag{9.65}$$

Fermionic string theory thus has to live in 10 dimensions, and we have $a = \phi$.

9.6 Ramond Ground States

Let us now discuss the spectrum of the fermionic string in detail. This is very similar to what we did in section 3.5. One of the motivations for introducing the fermionic string was to find fermionic particles. Naively, one might have expected that such particles come from fermionic excitation modes b^μ and $\tilde{b}^\mu$. Note however that these states transform in the fundamental of the spacetime Lorentz group $SO(1, D-1)$. This means that we can only get integer spacetime spin. For spacetime fermions, we would want to have half-integer spin however. Even though the ψ^μ are worldsheet fermions, their modes do not directly lead to spacetime fermions.

This is where the fermionic zero modes b^μ_0 in the Ramond sector become important. They lead to degenerate ground states in the Ramond sector. Let $|\alpha\rangle$ denote a ground state, *i.e.* a highest weight state such $b^\mu_r|\alpha\rangle = 0 \forall r > 0$. From the commutation relations it then immediately follows that $b^\mu_0|\alpha\rangle$ is again a ground state. The ground states thus form a representation of the zero modes $b^\mu_0, \mu = 0, \ldots, D-1$. The zero modes themselves form a Clifford algebra

$$\{b^\mu_0, b^\nu_0\} = \eta^{\mu\nu} \ . \tag{9.66}$$

On the space of ground states $|\alpha\rangle$ they can be represented as Dirac matrices γ^μ in D dimensions by defining

$$b_0^\mu|\alpha\rangle = \frac{1}{\sqrt{2}}(\gamma^\mu)^\alpha_\beta|\beta\rangle \ , \tag{9.67}$$

which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The Ramond sector zero modes thus play the role of the *spacetime* Dirac matrices! For $D = 10$ even, we can thus construct the Ramond ground states in the following way. Define the operators

$$B^0_\pm \ := \ \frac{i}{\sqrt{2}}(b_0^0 \pm b_0^1) \tag{9.68}$$

$$B^j_\pm \ := \ \frac{1}{\sqrt{2}}(b_0^{2j} \pm i b_0^{2j+1}) \qquad j = 1, \ldots, 4 \ . \tag{9.69}$$

These operators have the commutation relations of ordinary fermionic creation- and annihilation operators,

$$\{B^i_+, B^j_-\} = \delta^{ij} \ . \tag{9.70}$$

We can thus define a highest weight state $|0\rangle_R$ such that

$$B^j_+|0\rangle_R = 0 \ , \tag{9.71}$$

and obtain a represenation of the Clifford algebra (9.66) by acting with the creation operators. Since the creation operators are fermionic, the resulting representation has dimension $2^5 = 32$, and is spanned by states

$$|\mathbf{s}\rangle = |s_0, s_1, s_2, s_3, s_4\rangle \qquad s_j = \pm\frac{1}{2} \ . \tag{9.72}$$

In this notation we have $|0\rangle_R = |\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$, and the creation operators B^j_- lower $s_j = \frac{1}{2}$ to $s_j = -\frac{1}{2}$.

Light-cone quantization eliminates the modes b^0_r and b^1_r, and in particular also $B^0_\pm$. We are thus left with four pairs $B^j_\pm$, whose representation is given by

$$|\mathbf{s}\rangle = |s_1, s_2, s_3, s_4\rangle \qquad s_j = \pm\frac{1}{2} \ , \tag{9.73}$$

which has dimension $2^4 = 16$. This is a representation of $SO(8)$, the little group of massless particles. More precisely, the generators S^{ij} of $SO(8)$ for the representation (9.73) are given by

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{i}{2}[b_0^i, b_0^j] \ . \tag{9.74}$$

It is straightforward to check that the S^{ij} indeed satisfy the commutation relations of $SO(8)$. From (9.74) however we immediately see that the representation (9.73) is reducible: S^{ij} is a bilinear in the B^i, so that it changes the total sum $\sum_i s_i$ by an even number. We can thus decompose it into two irreducible representations,

$$\mathbf{16} = \mathbf{8}_c + \mathbf{8}_s \ . \tag{9.75}$$

As is customary we denote representations by their dimensions, adding a subscript if there is more than one representation of a given dimension. The irreducible representations are spanned by

$$\mathbf{8}_s = \langle |s_1, s_2, s_3, s_4\rangle : \sum s_i \in 2\mathbb{Z}\rangle \tag{9.76}$$

$$\mathbf{8}_c = \langle |s_1, s_2, s_3, s_4\rangle : \sum s_i \in 2\mathbb{Z}+1\rangle \tag{9.77}$$

We will denote by $|\alpha\rangle$ the states in $\mathbf{8}_s$ and by $|\dot{\alpha}\rangle$ the states in $\mathbf{8}_c$. Note that $\mathbf{8}_s$ and $\mathbf{8}_c$ differ by their chirality: Defining as usual the chirality operator as

$$\Gamma = \gamma^2\gamma^3 \cdots \gamma^9 \ , \tag{9.78}$$

it is clear that they have chirality +1 and -1. Note that there is of course a third 8-dimensional $SO(8)$ representation, the fundamental or vector representation, which we will denote by $\mathbf{8}_v$.

9.7 Spectrum

Let us now discuss the spectrum of the fermionic string.

Let us first introduce the level operators

$$N = N^\alpha + N^b \ , \qquad N^\alpha = \sum_{0<m\in\mathbb{Z}} \sum_{i=2}^{24} \alpha^i_{-m}\alpha^i_m \qquad N^b = \sum_{0<r\in\mathbb{Z}+\phi}^{\infty} \sum_{i=2}^{24} b^i_{-r} b^i_r \tag{9.79}$$

and analogous for the right movers $\tilde{N}$. The condition $L_0^\perp = \tilde{L}_0^\perp$ leads to the level matching condition

$$N = \tilde{N} \ . \tag{9.80}$$

To obtain the mass spectrum, we can again use the constraint equation

$$L_0 + \bar{L}_0 - a - \tilde{a} = 0 \ . \tag{9.81}$$

Note that because we set $\alpha^+_{n\neq 0} = 0$ and $\psi^+_\pm = 0$, we have

$$L_0 = -\alpha_0^+\alpha_0^- + L_0^\perp = \frac{1}{2}\alpha_0 \cdot \alpha_0 + N = \frac{1}{4}\alpha' p^2 + N \tag{9.82}$$

and similarly for $\bar{L}_0$. From (9.81) we thus get for the mass spectrum

$$M^2 = -P^2 = \frac{2}{\alpha'}(N + \tilde{N} - a - \tilde{a}) \ . \tag{9.83}$$

Note that now a and $\tilde{a}$ depend on the sector that we are in. In the Ramond sector, we get another interesting equation from the constraint $G_0 = 0$. Acting on a Ramond ground state $|\alpha\rangle$, we find

$$0 = G_0|\alpha\rangle = \alpha_{0\mu} b_0^\mu |\alpha\rangle = \frac{\sqrt{\alpha'}}{2} p_\mu (\gamma^\mu)^\alpha_\beta |\beta\rangle \ , \tag{9.84}$$

which is exactly the massless Dirac equation in $D = 10$. This agrees with the interpretation of $|\alpha\rangle$ as a spacetime fermion.

We can now write down the spectrum using (9.83). The result for the first few levels is given in table 9.1. At first sight, the spectrum looks promising. We now have both bosons, coming from the NS-NS and the R-R sectors, and fermions, coming from the R-NS and NS-R sector. There are however serious issues. For a start, the tachyon is still in the spectrum. The fermionic string has other problems too. It turns out that it is not modular invariant, so that it is impossible to define consistent higher loop amplitudes. Moreover, it is not spacetime supersymmetric. To see this, note that supersymmetry maps bosons to fermions of the same mass. A boson and fermion related by supersymmetry are called *superpartners*. The tachyon is a boson of mass squared $-2/\alpha'$, and there is no fermion that could be its superpartner. The fermionic string is therefore not spacetime supersymmetric.

9.8 Superstrings

For these reasons, we want to modify the fermionic string to turn it into a *superstring*. It turns out that a relatively small modification is enough to achieve that: the Gliozzi-Scherk-Olive (GSO) projection. To this end, let us first introduce the notion of fermion parity. The fermion parity operator $(-1)^F$ measures the fermion parity of a state. It anticommutes with all fermions

$$\{(-1)^F, b_r^\mu\} = 0 \ , \tag{9.85}$$

and commutes with all bosons. To completely define the action of $(-1)^F$, in the NS sector we fix

$$(-1)^F|0\rangle = -|0\rangle \ , \tag{9.86}$$

and in the Ramond sector we fix

$$(-1)^F|\alpha\rangle = |\alpha\rangle \qquad (-1)^F|\dot{\alpha}\rangle = -|\dot{\alpha}\rangle \ . \tag{9.87}$$

Similarly we define a left-moving fermion parity operator $(-1)^{\bar{F}}$. The fermion parity of the states can then be found in table 9.1. The GSO projection is then to restrict to states which have certain fermion parity.

$\alpha' M^2$:	states and $SO(8)$ reps	$(-1)^{\bar{F}}$	$(-1)^F$	
NS-NS				
-2	$\|0\rangle_L \otimes \|0\rangle_R$ $\mathbf{1}$ $\qquad$ $\mathbf{1}$	-1	-1	$\mathbf{1}$
0	$\tilde{b}^i_{-1/2}\|0\rangle_L \otimes b^j_{-1/2}\|0\rangle_R$ $\mathbf{8}_v$ $\qquad$ $\mathbf{8}_v$	$+1$	$+1$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$
R-R				
0	$\|\alpha\rangle \otimes \|\beta\rangle$ $\mathbf{8}_s$ $\qquad$ $\mathbf{8}_s$	$+1$	$+1$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_s$
0	$\|\dot{\alpha}\rangle \otimes \|\dot{\beta}\rangle$ $\mathbf{8}_c$ $\qquad$ $\mathbf{8}_c$	-1	-1	$\mathbf{1}+\mathbf{28}+\mathbf{35}_c$
0	$\|\dot{\alpha}\rangle \otimes \|\beta\rangle$ $\mathbf{8}_c$ $\qquad$ $\mathbf{8}_s$	-1	$+1$	$\mathbf{8}_v+\mathbf{56}_v$
0	$\|\alpha\rangle \otimes \|\dot{\beta}\rangle$ $\mathbf{8}_s$ $\qquad$ $\mathbf{8}_c$	$+1$	-1	$\mathbf{8}_v+\mathbf{56}_v$
R-NS				
0	$\|\alpha\rangle \otimes b^j_{-1/2}\|0\rangle_R$ $\mathbf{8}_s$ $\qquad$ $\mathbf{8}_v$	$+1$	$+1$	$\mathbf{8}_c+\mathbf{56}_c$
0	$\|\dot{\alpha}\rangle \otimes b^j_{-1/2}\|0\rangle_R$ $\mathbf{8}_c$ $\qquad$ $\mathbf{8}_v$	-1	$+1$	$\mathbf{8}_s+\mathbf{56}_s$
NS-R				
0	$\tilde{b}^i_{-1/2}\|0\rangle_L \otimes \|\beta\rangle$ $\mathbf{8}_v$ $\qquad$ $\mathbf{8}_s$	$+1$	$+1$	$\mathbf{8}_c+\mathbf{56}_c$
0	$\tilde{b}^i_{-1/2}\|0\rangle_L \otimes \|\dot{\beta}\rangle$ $\mathbf{8}_v$ $\qquad$ $\mathbf{8}_c$	$+1$	-1	$\mathbf{8}_s+\mathbf{56}_s$

Table 9.1: Spectrum of the fermionic string

9.8.1 Type IIA superstring

To eliminate the tachyon, in the NS sector we need to project to states which have positive fermion parity,

$$(-1)^F = +1 \qquad (-1)^{\bar{F}} = +1 \ . \tag{9.88}$$

In the Ramond sector we can choose for instance

$$(-1)^F = +1 \qquad (-1)^{\bar{F}} = -1 \ . \tag{9.89}$$

Note that only the relative sign between $(-1)^F$ and $(-1)^{\bar{F}}$ matters. Exchanging the sign in (9.93) gives the same particle content, simply changing the (overall) definition of chiral and anti-chiral.

What is the spectrum of type IIA superstring theory? We can easily read it off from table 9.1 by only picking out states with the correct fermion parity. First of all, note that the GSO projection eliminated the tachyon. With the choice above, the bosonic spectrum is

$$[\mathbf{1} + \mathbf{28} + \mathbf{35}_v]_{NS-NS} + [\mathbf{8}_v + \mathbf{56}_v]_{R-R} \tag{9.90}$$

The NS-NS particles are basically the same that we already encountered for the closed bosonic string. They come from the decomposition of the tensor product of two fundamentals of $SO(8)$ $\mathbf{8}_v \otimes \mathbf{8}_v$ into the symmetric traceless, anti-symmetric and trace parts. $\mathbf{1}$ is a massless scalar particle, the *dilaton*. The symmetric traceless part $\mathbf{35}_v$ is of course again the graviton. Superstring theory is thus also a theory of gravity.

$\mathbf{28}$ on the other hand is the antisymmetric part of $\mathbf{8}_v \otimes \mathbf{8}_v$. We can thus interpret it as a NS-NS 2-form. Similarly, we want to interpret the R-R fields as forms. Clearly $\mathbf{8}_v$ is simply a R-R 1-form. $\mathbf{56}_v$ on the other hand is the completely antisymmetric part of $\mathbf{8}_v \otimes \mathbf{8}_v \otimes \mathbf{8}_v$, and thus corresponds to a R-R 3-form.

The fermions are

$$[\mathbf{8}_s + \mathbf{56}_s]_{R-NS} + [\mathbf{8}_c + \mathbf{56}_c]_{NS-R} \tag{9.91}$$

We see that type IIA is not a chiral theory: all chiral representations appear in pairs. The $\mathbf{56}$ are two gravitini, *i.e.* the superpartners of the graviton. The $\mathbf{8}$ are dilatini, *i.e.* superpartners of the dilaton.

9.8.2 Type IIB superstring

First note that even though the number of fields of a given mass is the same as in type IIA, the spectrum is different as far as their spin is concerned. In the NS sector we again choose

$$(-1)^F = +1 \qquad (-1)^{\bar{F}} = +1 \ . \tag{9.92}$$

In the Ramond sector we choose the same sign

$$(-1)^F = +1 \qquad (-1)^{\bar{F}} = +1 \ . \tag{9.93}$$

$$[\mathbf{1} + \mathbf{28} + \mathbf{35}_v]_{NS-NS} + [\mathbf{1} + \mathbf{28} + \mathbf{35}_s]_{R-R} \tag{9.94}$$

The NS-NS particles are exactly the same as in type IIB. In particular, there is also a graviton in the spectrum. The R-R fields can again be interpreted as forms again: **1** is a 0-form, and **28** is a R-R 2-form. Finally $\mathbf{35}_s$ is a self-dual 4-form: Indeed a 4-form has $\binom{8}{4} = 70$ components, half of which are eliminated by the self-duality condition.

The fermions are

$$[\mathbf{8}_c + \mathbf{56}_c]_{R-NS} + [\mathbf{8}_c + \mathbf{56}_c]_{NS-R} \tag{9.95}$$

This gives again two gravitini and two dilatini. Note however that they are in different representations of $SO(8)$ than for type IIA. In particular, type IIB is a chiral theory: there are representations with a specific chirality whose anti-chiral partners do not appear.

9.8.3 Spacetime supersymmetry

Both type IIA and type IIB have gravitini in their spectrum, that is the massless particles in the R-NS and NS-R sector. They are called gravitini because they are the superpartners of the graviton. Let us denote the gravitino field corresponding to $\mathbf{8}_s + \mathbf{56}_s$ by $S^{\mu\alpha}$. $S^{\mu\alpha}$ is in fact the Noether supercurrent corresponding to a spacetime supersymmetry. In particular we can obtain the corresponding Noether charge Q^α by the usual surface integral

$$Q^\alpha = \int_\Sigma dn_\mu S^{\mu\alpha} \ . \tag{9.96}$$

From this we see that each supercharge is a spinor with 8 real components, that is a Majorana-Weyl (real, with definite chirality) spinor of the little group $SO(8)$. (Alternatively it is a Majorana-Weyl spinor of $SO(1,9)$, whose 16 components are halved once we impose the Dirac equation $\gamma \cdot \partial S = 0$.)

From the spectrum we see that there are always two gravitini, one coming from the R-NS sector, the other one coming from NS-R. Both theories thus have two supercharges Q_1 and Q_2. We therefore say that they have $N = 2$ (spacetime) supersymmetry. This is where the name 'type II' comes from.[1] In type IIA, the two supercharges have opposite chirality,

$$Q_1^\alpha \ , \qquad Q_2^{\dot\alpha} \ , \tag{9.97}$$

whereas in type IIB they have the same chirality,

$$Q_1^\alpha \ , \qquad Q_2^\alpha \ . \tag{9.98}$$

Note that we are being very sketchy here. To show that the theory actually has spacetime supersymmetry with the supercharges listed above, one would have to work much harder. In particular everything we have said so far would imply that we also have the same supercharges without the GSO projection, which is not the case. The GSO projection ensures that the vertex operators corresponding to the supercharges are actually local operators in the theory.

[1] There is also a type I superstring, which only has $N = 1$ supersymmetry. It involves open strings, which we did not discuss this theory here.

Bibliography