Math 223

Disclaimer:
It is not a good idea to rely exclusively on reading through old exam solutions as a way to prepare for the final exam. In particular, this semester's course director may not have written any of the exams available from this page, so the ones he/she gives will almost certainly have a somewhat different flavor.

Topic Warning:
Because the topics taught differ slightly from semester to semester, it is not a good idea to use the old exams to gauge the content of the exams this semester.
1. (30) Solve the following problems. No partial credit.

(a) If $\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}$, and $\vec{v} = a\vec{i} + 2\vec{j} - 4\vec{k}$ (a is a constant), then $\vec{u} \cdot \vec{v}$ is equal to

A. $a - 14$.  
B. $2(a - 7)$  
C. $2a - 10$  
D. $a + 10$  
E. 7

(b) Find a normal vector for the plane $7y = z$.

(c) Find $\frac{\partial f}{\partial y}$, where $f(x, y, z) = y^2e^{xyz}$. Simplify your answer as much as possible, factoring where possible.
(d) If $S$ is the plane $x = 0$ oriented in the positive $x$ direction, then the surface area element $\vec{dA}$ is which one of the following quantities?

A. $\vec{j} \, dx \, dz$
B. $\vec{k} \, dy \, dz$
C. $-\vec{i} \, dy \, dz$
D. $\vec{i} \, dy \, dz$
E. $-\vec{k} \, dy \, dx$

(e) The figure below shows the contour diagram of which function $f(x, y)$?

A. $f(x, y) = 6y - 3x + 6$
B. $f(x, y) = \frac{1-x}{2}$
C. $f(x, y) = e^{-3x-6y+6}$
D. $f(x, y) = -3x - 6y$
E. None of the above.
(f) Let $R$ be the two dimensional region shown in the figure below. What is $\int_R f(x, y)dA$?

A. $\int_0^3 \int_0^{2y/3} f(x, y) \, dy \, dx$

B. $\int_0^3 \int_0^{3x/2} f(x, y) \, dx \, dy$

C. $\int_0^3 \int_0^{2y/3} f(x, y) \, dy \, dx$

D. $\int_0^2 \int_0^{3x/2} f(x, y) \, dy \, dx$

E. $\int_0^3 \int_0^{3x/2} f(x, y) \, dx \, dy$
2. (20) Consider the integral \( \int_{-1}^{0} \int_{0}^{y^2} e^{x/y^2} \, dx \, dy \).

(a) Interchange the order of integration. Show your work, including a sketch of the region of integration.

(b) Evaluate the original integral. Give an exact answer.
3. (30) Let \( \vec{F} = y \hat{i} + 2z \hat{j} + (1 - z) \hat{k} \). Evaluate the following:

(a) \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the straight line from the origin to \((1, 3, 1)\).

(b) \( \int_S \vec{F} \cdot d\vec{A} \) where \( S \) is the rectangle \( 0 \leq x \leq 2, \ 1 \leq y \leq 4, \ z = 0 \), oriented upwards.

(c) \( \int_S \vec{F} \cdot d\vec{A} \) where \( S \) is the sphere of radius 2 centered at the origin, oriented outwards.
4. (20) Let $f(x, y) = \sqrt{1 + 4x + y^2}$, and let $P$ be the point $(1, 2)$.

(a) At $P$, what is the direction of maximal increase for the function $f$? Give your answer as a unit vector.

(b) Find the directional derivative of $f$ at $P$ in the direction of $3\mathbf{i} - 4\mathbf{j}$.
5. (15) Let $H(x, y, z) = x^2 + y^2 + 2z^2$, and let $S$ be the level surface $H(x, y, z) = 4$. Find the coordinates of a point $P$ on the surface $S$ where the tangent plane to $S$ is parallel to the plane $2x + 4z = 0$. 
6. (20) Suppose $S$ is the surface obtained by taking the union of the upper hemisphere of a sphere of radius 2 centered at $(0,0,4)$,

$$S_1 = \{(x,y,z) \text{ such that } x^2 + y^2 + (z-4)^2 = 4 \text{ and } z \geq 4\}$$

and an open cylinder of radius 2 centered around the $z$ axis,

$$S_2 = \{(x,y,z) \text{ such that } 0 \leq z \leq 4 \text{ and } x^2 + y^2 = 4\}.$$ 

The orientation of $S$ is away from the origin.

a.) Sketch the surface $S$.

b.) Evaluate the integral $\int_S (\text{curl}(\vec{F})) \cdot d\vec{A}$, if $\vec{F}$ is the vector field $\vec{F} = y\hat{i} - x\hat{j} + xy\hat{k}$.

Hint: It is strongly recommended to use Stokes’ theorem to simplify the surface integral.
7. (20) Consider the cubic polynomial \( f(x, y) = \frac{5}{2}x^2 - xy + 15x + \frac{1}{75}y^3 - 3y. \)

(a) Find the critical point(s) of \( f(x, y). \)

(b) Use the second derivative test to classify, if possible, the critical point(s) you have found.
8. (15) Let $S$ be the paraboloid $x^2 + y^2 + z = R^2$, $0 < z \leq R^2$, oriented upward, and let $\vec{F} = x\vec{i} + y\vec{j} + z^2\vec{k}$. Find the flux of the vector field $\vec{F}$ through the surface $S$. 
9. (20) Consider the 2-dimensional force field $\vec{F} = (4e^{-2x} + 3y^3)i + 9xy^2j$.

a) Is $\vec{F}$ conservative? If so, find a potential function $f(x, y)$ whose gradient is $\vec{F}$.

b) Find the work done by the force field $\vec{F}$ in moving an object from $P(0, 1)$ to $Q(1, 2)$ along the path $y = 1 + \sin(\pi x / 2)$ from $x = 0$ to $x = 1$. 
10. (10) An asteroid is a cylindrical mass of ice, 100 km tall and with radius 5 km. The density of the asteroid varies linearly along its long dimension, varying from zero at one end to $10 \text{kg/m}^3$ at the other. Set up a triple integral representing the total mass of the asteroid.