A function which models exponential growth or decay can be written in either the form

$$P(t) = P_0 b^t \qquad \text{or} \qquad P(t) = P_0 e^{kt}.$$

In either form, P_0 represents the initial amount.

The form $P(t) = P_0 e^{kt}$ is sometimes called the *continuous exponential* model. The constant k is called the *continuous growth (or decay) rate*. In the form $P(t) = P_0 b^t$, the *growth rate* is $r = b - 1$. The constant b is sometimes called the growth factor. **The growth rate and growth factor are not the same**.

It is a simple matter to change from one model to the other. If we are given $P(t) = P_0 e^{kt}$, and want to write it in the form $P(t) = P_0 b^t$, all that is needed is to note that $P(t) = P_0 e^{kt} = P_0 \left(e^k\right)^t$, so if we let $b = e^k$, we have the desired form. If we want to switch from $P(t) = P_0 b^t$ to $P(t) = P_0 e^{kt}$, it again is just a matter of noting that $b = e^k$, and solving for k in this case. That is, $k = \ln b$.

Template for solving problems: Given a rate of growth or decay r,

- If r is given as a constant rate of change (some fixed quantity per unit), then the equation is linear (i.e. $P = P_0 + rt$)
- If r is given as a percentage increase or decrease, then the equation is exponential
  - if the rate of growth is r, then use the model $P(t) = P_0 b^t$, where $b = 1 + r$
  - if the rate of decay is r, then use the model $P(t) = P_0 b^t$, where $b = 1 - r$
  - if the *continuous* rate of growth is k, then use the model $P(t) = Pe^{kt}$, with k positive
  - if the *continuous* rate of decay is k, then use the model $P(t) = Pe^{kt}$, with k negative

Thus, the form $P(t) = P_0 e^{kt}$ should be used if the rate of growth or decay is stated as a *continuous* rate.

Example: A trash dumpster starts with 5 pounds of garbage. Write a function which represents the amount of garbage in the dumpster after t days given the following rates

a) The amount of garbage increases by 3 lbs per day
   - ∗ Since the rate is stated as a constant 3 pounds per day, the equation is linear. So, the model is $Q(t) = 5 + 3t$, where Q represents the amount of trash, and t is measured in days.

b) The amount of garbage increases by 3% per day
   - ∗ Since the rate is now given as percentage increase, we need to use the exponential model $P(t) = P_0 b^t$. Since the growth rate is $r = .03$, the base of our model should be $b = 1 + .03$. So, if we again let Q represent the amount of trash, we have $Q(t) = 5(1.03)^t$.
     - · Note that the growth *rate* is .03, and the growth *factor* is 1.03

c) The amount of garbage increases continuously by 3% per day
   - ∗ Since the rate is given as a *continuous* percentage increase, we need to use the exponential model $P(t) = P_0 e^{kt}$. We have $k = .03$, so our model is $Q(t) = 5e^{.03t}$

Example: Suppose the population of ants in a colony grows by 4.2% per month.

a) Determine a model which represents the population of the colony after t months. What is the growth rate? What is the *continuous growth rate*?
   - ∗ The model is simply $P(t) = P_0(1.042)^t$.
     The growth rate is $r = .042$ (or 4.2%).
     In order to find the continuous growth rate, we need to convert the model to the form $P(t) = P_0 e^{kt}$. So, we need to solve for k in $1.042 = e^k$. Taking the natural log of both sides, we get $k = \ln(1.042) \approx .04114$. Thus the continuous growth rate is approximately .04114 (or about 4.114%).

b) Suppose instead that the population of ants grows at a continuous rate of 4.2%.
Determine a model which represents the population of the colony after t months. What is the growth rate? What is the continuous growth rate?

* In this case the model is $P(t) = P_0 e^{.042t}$.
To find the growth rate, we convert to the form $P(t) = P_0 b^t$. So, $b = e^{.042} \approx 1.04289$. Thus the growth factor is about 1.04289, and the growth rate is approximately .04289 (or 4.289%). The continuous growth rate is the stated 4.2%.

Example: A container with 1 liter of a liquid is placed in a warm, arid environment. The liquid evaporates at a rate of 2.3% per day. Write a function which represents the amount of liquid (in milliliters) in the container after t days.

– Since the amount of liquid present is decreasing, we have an example of exponential decay.
  * The rate of decay is 2.3%, so the base of our exponential is $b = 1 - .023 = .977$. (If we like, we could also think of the rate of decay, as a rate of growth of -.023, which yields the same result since then $b = 1 + r = 1 + (-.023) = .997$). In any case, our exponential model is $Q(t) = 1000(.977)^t$ (the initial amount is 1000, since our function is supposed to represent the amount in *milliliters*).

EXERCISES

1. Write a model to represent each of the following.
   (a) The amount of a radioactive substance present after t hours if there are initially 500 mg and the half-life is 9 hours.
   (b) The value of a car after t years if the initial price is \$26,000 and the car depreciates at a rate of 11% per year.
   (c) The value of a car after t years if the initial price is \$33,000 and the car depreciates at a rate of \$3700 per year.
   (d) The population of a city in year t if the population in 2005 is 125,000 and the city is growing at a rate of 3.1% per year.

2. Use the fact that after 5715 years a given amount of carbon-14 will have decayed to half the original amount to answer the following.
   (a) Write an equation which represents the amount of carbon-14 present after t years. What does the base of your equation tell you (in practical terms)?
   (b) In 1947, earthenware jars containing what are now known as the Dead Sea Scrolls were found. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the current age of the Dead Sea Scrolls.

3. Suppose that the fruit fly population in a small habitat after t days is given by the logistics equation

$$P(t) = \frac{230}{1 + 56.5e^{-.37t}}$$

   (a) How many fruit flies were initially placed in the habitat?
   (b) What is the carrying capacity of the habitat? [That is, as $t \to \infty$ what happens to $P(t)$?] What feature is this on the graph?
   (c) How many fruit flies are present on day 4?
   (d) When will the population of fruit flies be 180?

4. It is predicted that the population of a certain town in 2030 will be double the population in 2005.
   (a) Determine the annual growth rate.
   (b) Determine the monthly growth rate.
   (c) Determine the continuous growth rate.