PROPAGATION PHENOMENA IN REACTION-ADVECTION-DIFFUSION EQUATIONS

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Abstract

Reaction-advection-diffusion (RAD) equations are a class of non-linear parabolic equations which are used to model a diverse range of biological, physical, and chemical phenomena. Originally introduced in the early twentieth century as a model for population dynamics, they have been used in recent years in diverse contexts including climate change [8], criminal behavior [11], and combustion [72]. These equations are characterized by the combination of three behaviors: spreading, stirring, and growth/decay.

The main focus of mathematical research into RAD equations over the past century has been in characterizing the propagation of solutions. Indeed, these equations are characterized by the invasion of an unstable state by a stable state at a constant rate (for instance, the invasion of empty space by a population until the environmental carrying capacity is reached). In general, this can be characterized by the existence, uniqueness, and stability of traveling wave solutions, or solutions with a fixed profile which move at a constant speed in time. In general, the speed and shape of these traveling waves gives us the speed with which the stable state invades the unstable state.

This thesis assumes the following trajectory, investigating two specific RAD equations: the Fisher-KPP equation, used in population dynamics, and a coupled reactive-Boussinesq system, used to model combustion in a fluid. For the former equation, we prove results regarding the precise spreading rate, and for the latter equation, we prove an existence result for a special solution that generalizes the traveling wave.

In the first part of this thesis, we prove two results quantifying the precise speed of spreading for solutions to the Cauchy problem of the Fisher-KPP equation. The first of these results, concerning localized initial data, provides intuition for a lower order term obtained non-rigorously in [28]. Specifically, we prove a quantitative convergence-to-equilibrium result in a related model, which has been used as a close approximation of the Fisher-KPP equation. The second of these results, concerning non-localized initial data and building on the work of Hamel and Roques [37], quantifies the super-linear in time spreading of the population. Here we compute the highest order

term in the spreading for a broad class of initial data. In addition, we describe precisely the growth to equilibrium.

In the second part of this thesis, we look at a coupled system that models combustion in a fluid, and we prove a qualitative propagation result. Unlike classical models, this relatively new system accounts for the effect of advection induced by the buoyancy force that results from the evolution of the temperature. Essentially, this means that we take into account the phenomenon that "hot air rises." We exhibit a generalized traveling wave solution of this system, called a pulsating front, in two-dimensional periodic domains. To our knowledge, this is the first result regarding the existence of "pulsating fronts" in a coupled system.

Acknowledgments

Though only my name will be attached to this dissertation, its completion would not have been possible without the support and guidance of a number of people. I would like to express my deepest gratitude to them here. Though this list is quite long, it is woefully inadequate, and I apologize to the many people who I have been so fortunate to know and have learned from but who I have been too forgetful to acknowledge here.

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Contents

Chapter 1

Introduction

1.1 A basic reaction-diffusion equation

The logistic equation is an ordinary differential equation that provides one of the most basic models for population dynamics. Let $u(t)$ be a population at time t which grows at rate r up to some carrying capacity K. Then we may model the growth of the population by the ordinary differential equation

$$u_t = ru\left(1 - \frac{u}{K}\right). \tag{1.1.1}$$

This simple model dates back to at least the nineteenth century [69]. It captures the initial Malthusian exponential growth of a population and the limiting behavior when the environment cannot support any more growth.

However, this model does not take into consideration any spatial variation in the environment or any movement of the population. In 1937, Fisher combined this equation with the heat equation, to create the simplest model which accounts for the natural spreading of the population [31]. In this article, he obtained what we now refer to as the Fisher-KPP equation:

$$u_t - Du_{xx} = r\left(1 - \frac{u}{K}\right).$$

Here the constant D models the diffusivity, or motility, of the population. By suitably scaling u, t, and x, we may obtain the equivalent non-dimensional form of the equation that we will henceforth refer to as the Fisher-KPP equation:

$$u_t - u_{xx} = u(1 - u). \tag{1.1.2}$$

The theory that follows from this model, which we will describe in the sequel, has shown to be accurate in matching existing data regarding the spread of a variety of species, such as the muskrat,

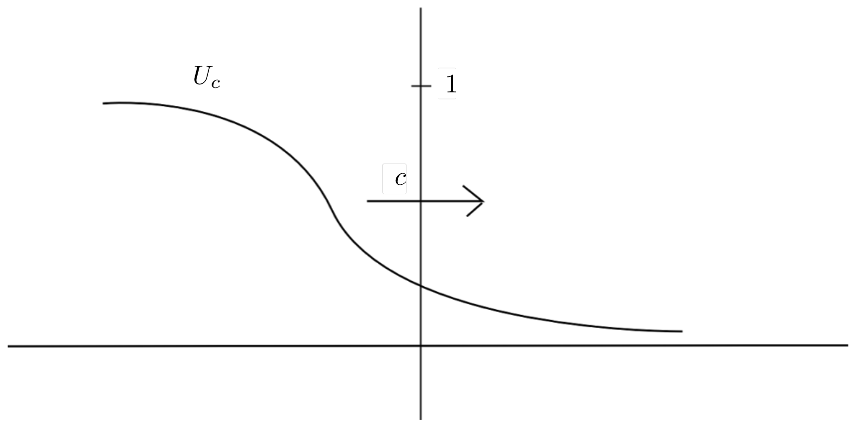


Figure 1.1: A cartoon representation of a traveling wave profile U_c of speed c.

the cabbage butterfly, the sea otter, and rabies [66, Section 3.7].

1.2 Stability of the traveling waves

One of the key features of this equation is the existence of a family of *traveling wave* solutions which, to a large degree, determines the behavior of solutions to the equation (1.1.2) with general initial data. In the one-dimensional case, traveling waves are a pair (c, U) where $U : \mathbb{R} \to \mathbb{R}$ is a fixed profile solving equation (1.1.2) in the moving frame with speed c. In other words, the function $U(x - ct)$ solves equation (1.1.2). The existence of these solutions were first observed by Fisher numerically in his seminal paper [31]. Later that same year, Kolmogorov, Petrovskii, and Piskunov published the first mathematical investigation into solutions of equation (1.1.2), [46]; the attachment of their initials to the name of the equation is out of respect for this work. They showed, rigorously, the existence of traveling wave solutions for all speeds $c \geq 2$, and they proved that if the initial data u_0 is the Heaviside function $\mathbb{1}_{[-\infty,0]}$, then

$$\lim_{t\to\infty} \|u(t,\cdot) - U_{c=2}(\cdot - 2t + s(t))\|_\infty = 0,$$

for some sub-linear function $s(t)$. They proved this by showing that u becomes monotonically "flatter" in time. This result, which can be extended to any initial data that decays faster than e^{-x}, shows that the minimal speed traveling wave solution determines the spreading rate and asymptotic shape of a population.

We note here that this model and the predictions it yields through the minimal traveling wave speed have been checked against experimental data for a variety of species including the muskrat, cabbage butterfly, and the sea otter [66]. In these cases, the predicted spreading speeds given by the minimal speed traveling wave solutions are relatively accurate. In addition, the minimal wave speed has played a role in a debate about the origins of early farming in Europe [2].

The fact that $s(t)$ should be at least logarithmic was, in fact, first noticed by Fisher. However, the precise asymptotics for $s(t)$ were not discovered until the works of Bramson [18, 19] and Uchiyama [68], in which the authors showed that $s(t) \sim \frac{3}{2}\log(t)$. Bramson's proof builds on the work of McKean, who studied the Fisher-KPP equation through a probabilistic interpretation using a branching Brownian motion. Here, a branching Brownian motion is a process where each particle undergoes random motion and splits randomly into two particles at rate one. Each new particle then continues independent branching Brownian motions. A new analytic proof was recently obtained by Hamel, Nolen, Roquejoffre, and Ryzhik [35, 36]. In this paper they looked at solutions to a related equation which can be used to create sub- and super-solutions to equation (1.1.2). The fact that one may pull sub- and super-solutions out of this related equation indicates that it provides an extremely accurate approximation to the tail of solutions to the Fisher-KPP equation.

The work above shows that the "front" of the population is located at $2t - (3/2)\log(t)$, up to a constant shift and some error which vanishes in time. Later, Ebert and van Saarloos argued non-rigorously that the front is located, more precisely, at

$$s(t) = \frac{3}{2}\log(t) + x_0 + \frac{3\sqrt{\pi}}{\sqrt{t}}, \tag{1.2.1}$$

where x_0 is a constant depending on the initial data. While the exact form of the constant x_0 remains a mystery, the $3\sqrt{\pi}/\sqrt{t}$ term, which they obtained through matched asymptotics, is believed to be independent of the initial data. However, there is no rigorous proof of this term in the expansion of the front location.

The first result in this thesis provides some intuition behind the $3\sqrt{\pi}/\sqrt{t}$ term in Ebert and van Saarloos's expansion. To do this, we examine the related equation studied in [35]. In the interest of avoiding technicalities in this section, we describe our result in the language of the equivalent probabilistic model. Fix an arbitrary real number r. Consider a branching Brownian motion where each particle is affected by a drift $-\sqrt{2}t + \frac{r}{\sqrt{2t}}$ and is removed from the population when it hits the origin. Our first result is then the following:

Theorem 1.2.1. *Let $0 < a < b \leq \infty$, and let $N_t^{[a,b]}$ be the number of particles in the process described above which are in $[a, b]$ at time t. Then there is a limit $N_\infty^{[a,b]}$, which is non-zero for some a, b, and there is a constant C, depending only on a, b, and r, such that, if $r = 3\sqrt{\pi}$,*

$$\left|\mathbb{E}\left[N_t^{[a,b]} - N_\infty^{[a,b]}\right]\right| \leq \frac{C}{t\log(t)},$$

and if $r \neq 3\sqrt{\pi}$ then we have that

$$\frac{1}{C\sqrt{t}} \leq \left| \mathbb{E}\left[N_t^{[a,b]} - N_\infty^{[a,b]} \right] \right| \leq \frac{C}{\sqrt{t}}.$$

Essentially, this says that the expected population in a given region stabilizes fastest with the choice of $r = 3\sqrt{\pi}$ in the drift. We note that the difference between the $\sqrt{2}$ factors in the drift in this model and the expansion for the front location in Fisher-KPP, is due only to the scaling of the Brownian motion.

One possible interpretation of the $3\sqrt{\pi}/\sqrt{t}$ term in the Ebert and van Saarloos expansion is that the population to the right of the front location should be unchanging. In our related model, this can occur only with the choice

$$2t - \frac{3}{2}\log t + x_0 - \frac{3\sqrt{\pi}}{\sqrt{t}}.$$

The proof, which is entirely analytic, follows the main techniques of [35] and utilizes a connection between $\mathbb{E}\left[N_t^{[a,b]}\right]$ and the behavior of particles near the origin. The discussion and proof of this result is the subject of Chapter 2. This work appears in [42].

Before we move on to the next result, we briefly comment on the difficulties in understanding the constant shift x_0 in equation (1.2.1). In our proof, the connection between the order $t^{-1/2}$ and the behavior of the system is made possible by shifting to the moving frame, which yields a partial differential equation where the advective term is given by the time derivative of equation (1.2.1). Since x_0 is constant in time, it does not appear in the equation, and thus we may not pick out the correct x_0 by investigating the effect of this advective term. On the other hand, the sub- and super-solutions used in [35] are too imprecise near the front to be helpful to obtain the x_0 directly.

1.3 Accelerating fronts

The above stability results concerning the logarithmic delay hold for any initial data which decays faster than e^{-x} as x tends to infinity. On the other hand, if the initial data decays like $e^{-\lambda x}$, for $\lambda < 1$, then u converges to a U_c where $c = \lambda + 1/\lambda$ [68].

These results are quite classical, and they have been refined and generalized over time (see [12, 20, 21, 35, 36, 64] and references therein). Less well studied is the case when the initial data does not decay exponentially. Using the traveling wave solutions of each speed $c \geq 2$ as a family of sub-solutions, it is easy to see that u will spread super-linearly in time in this case, whence the name "accelerating fronts" derives. With this in mind, one may ask for a characterization of spreading when the initial data u_0 decays sub-exponentially.

To our knowledge the only previous study of this problem is a work by Hamel and Roques published recently where they show that if u_0 decays more slowly than an exponential then u becomes asymptotically flat. One important implication of this finding is that one cannot quantify the spread of the population using one of the traveling wave solutions since u may not converge to a traveling wave due to its asymptotic flatness. Naturally, they instead investigate the movement of the level sets of u. Namely, we define for each $m \in (0,1)$

$$E_m(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R} : u(t,x) = m\}. \tag{1.3.1}$$

Then Hamel and Roques show that

$$E_m(t) \subset \left[u_0^{-1}\left(e^{-(1-\epsilon)t}\right), u_0^{-1}\left(e^{-(1+\epsilon)t}\right)\right]$$

for any ϵ and any t large enough. Assuming further that u_0 decays slower than x^{-2}, they show that one can find constants $c_m < C_m$, depending on m such that

$$E_m(t) \subset \left[u_0^{-1}\left(C_m e^{-t}\right), u_0^{-1}\left(c_m e^{-t}\right)\right].$$

While they obtain some dependence on the upper bound, the result is not sharp. We discuss their results more precisely in Chapter 3.

In this thesis, we prove a more general and more precise result than that of Hamel and Roques. For the sake of clarity, we first state our result in the context of the standard Fisher-KPP equation, (1.1.2), in an imprecise manner. We obtain that if u_0 decays slower than any exponential and satisfies some weak regularity conditions, we have that

$$E_m(t) \sim u_0^{-1}\left(\frac{m}{1-m}e^{-t}\right), \tag{1.3.2}$$

where the notation, "$\sim$," means up to some lower-order error terms which we make more precise in Chapter 3. We point out that the constant $m/(1-m)$ comes directly from the solution of the logistic equation (equation (1.1.1) with $K = 1$). Specifically, if we solve the logistic equation with initial data $\frac{m}{1-m}e^{-t}$, then the population at time t will be m, up to some small error which is exponentially small in t. In fact, we show is that, for any fixed location x, the function $u(t,x)$ grows in time according to the logistic equation up to some small error. In other words, the diffusion is ignorable when the initial data is spread out.

In fact, we extend this result beyond the usual Fisher-KPP nonlinearity to model inhomogeneous environments. We consider general $C^{1,\delta}$ non-linearities $f : \mathbb{R} \times [0,1] \to \mathbb{R}^{\geq 0}$ which are positive on $\mathbb{R} \times (0,1)$ and which have the property that $f(x,u)/u$ is non-decreasing in u for every x. In addition, we suppose that f is L-periodic for some $L > 0$. Here f represents the effect of a heterogeneous environment on the population. In this case, we can expect u to have a complicated shape with level sets which are very wide (cf. Figure 1.2). This contrasts with the situation in a homogeneous environment where each level set consists of a single point. In fact, in the heterogeneous setting,

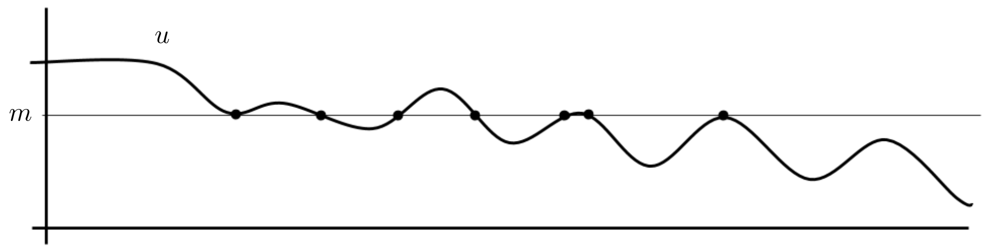


Figure 1.2: A cartoon showing the effect of an inhomogeneous reaction rate. Notice the width of the level set of height m. This width will grow without bound as t tends to infinity.

each level set will become infinitely wide in time. As a result, we look instead at the average-level sets

$$\overline{E}_m(t) \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R} : m = \frac{1}{L} \int_x^{x+L} u(t, y) dy \right\}. \tag{1.3.3}$$

We show here that the location of the average-level sets is determined by the generalized logistic equation, below.

Proposition 1.3.1. *Suppose f is as described above. Then there is a unique (up to translation in time) global-in-time and L-periodic solution to*

$$\begin{cases} \varphi_t = \varphi_{xx} + f(x, \varphi), \\ \lim_{t\to-\infty} \varphi(t, x) = 0, \\ \lim_{t\to\infty} \varphi(t, x) = 1. \end{cases} \tag{1.3.4}$$

Proposition 1.3.1 is new, to our knowledge. However, its proof follows classical arguments used to establish existence and uniqueness of traveling waves solutions to the Fisher-KPP equation. We state it in this thesis for completeness. Before we move on, we point out that $\frac{1}{L}\int \varphi(t, x)dx$ is strictly increasing in time. This is easy to see by integrating the equation for φ.

We now state our results more precisely and in more generality than above, though we leave many details of the full statement for Chapter 3.

Theorem 1.3.2. *Suppose that u_0 decays sufficiently slowly, is sufficiently regular, and is monotonically decreasing. Fix $m \in (0, 1)$ and any $\epsilon > 0$. Let ϕ be the solution to equation* (1.3.4)*, and define $T_{m\pm\epsilon}$ be the unique time such that*

$$\frac{1}{L} \int \varphi(T_{m\pm\epsilon}, x)dx = m \pm \epsilon.$$

Then for all t large enough, depending on m, ϵ, and f, we have that

$$\overline{E}_m(t) \subset \left[u_0^{-1}\left(\frac{1}{L}\int \varphi\left(T_{m+\epsilon} - t, x\right) dx\right), u_0^{-1}\left(\frac{1}{L}\int \varphi\left(T_{m-\epsilon} - t, x\right) dx\right)\right].$$

When f is the standard Fisher-KPP non-linearity, i.e. if $f(x,u) = u(1-u)$, it is quite easy to check that

$$\varphi(x,t) = \frac{e^t}{1+e^t}, \qquad \text{and} \qquad \varphi(T_{m\pm\epsilon} - t) = \frac{m}{1-m}e^{-t} + O(e^{-2t}),$$

which gives the result in equation (1.3.2). In addition, we point out that our work implies that one may find c_m, depending only on m and f, and $\overline{f}_0$, depending only on f, such that

$$\frac{1}{L}\int \varphi(T_m - t, x)dx \sim c_m e^{-\overline{f}_0 t}.$$

Hence, the above becomes

$$\overline{E}_m(t) \sim u_0^{-1}\left(c_m e^{-\overline{f}_0 t}\right).$$

In fact, we prove a more precise result than this, and our result extends to very irregular initial data. However, for the sake of brevity, we relegate this discussion to Chapter 3.

Our strategy here is to build sub- and super-solutions using a product of solutions of the logistic equation and the heat equation. This method for creating sub- and super-solutions is new. The main idea is to exploit a mismatch between the decay of the heat kernel and the decay of the initial conditions in order to show that the factor of the sub- and super-solutions arising from the heat equation can be locally approximated as a constant. With this, we show that the growth/spreading of u is then determined entirely by solutions to a generalized logistic equation. The work presented in Chapter 3 appears in [44].

We note that the phenomenon of super-linear propagation is one that has attracted some attention in recent years. Beyond our own study and that of Hamel and Roques that we mentioned above, this phenomenon has been observed in a number of other settings. Solutions to the Fisher-KPP equation with a fractional Laplacian propagate exponentially [22, 23, 26]. If one replaces the Laplacian with a non-local dispersal operator with a fat tail, the solutions propagate super-linearly as well. In fact, a result similar to Theorem 1.3.2 holds [32, 47, 57]. In a related kinetic reaction-transport equation, which yields the Fisher-KPP equation in the diffusion limit, solutions have been shown to propagate like $t^{3/2}$, [17]. Finally, $t^{3/2}$ propagation has been conjectured in a reaction-diffusion-mutation model with variable motility [16].

1.4 Pulsating fronts in a coupled combustive system

Before we begin, we make a small change in notation. In this section, we use T, instead of u, as the solution to our RAD equation. This reflects the slight change in model; here, we investigate a model for temperature change. We replace the nonlinearity $T(1-T)$ with one to model combustion. In particular, we use a non-negative non-linearity f such that there exists $\theta \in (0,1)$ where $f(T) = 0$ if $T \in [0,\theta] \cup \{1\}$ and $f(T) > 0$ if $T \in (\theta, 1)$ (cf. Figure 1.3). The idea here is that the temperature of the fluid has to reach a certain threshold in order for the fluid to ignite. Any function f satisfying these conditions is called an ignition-type non-linearity [72].

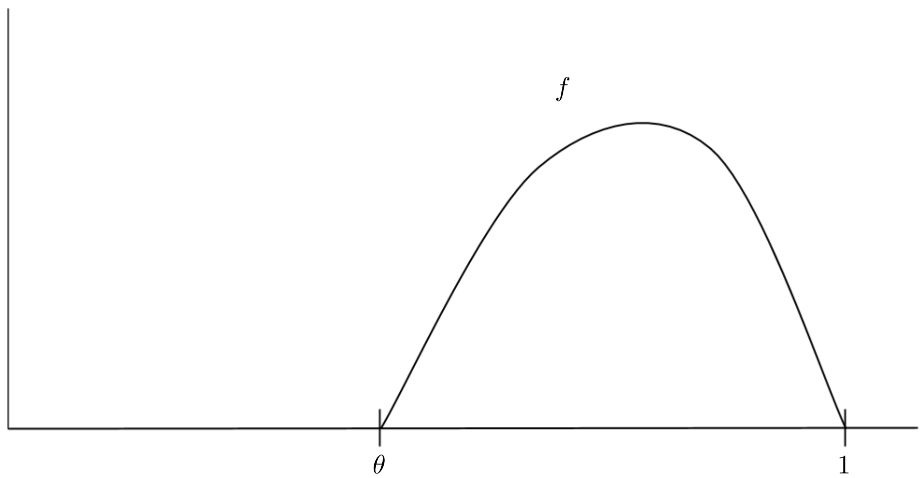


Figure 1.3: A typical ignition-type nonlinearity f.

The results for ignition type non-linearities are quite similar to those discussed above for the Fisher-KPP equation. In particular, there exists a unique $c > 0$ such that there is a traveling wave solution of speed c and such that if T_0 is large enough then T spreads at speed c. This last result holds only under the condition that T_0 be "large enough" in order to prevent the case when $T_0(x) < \theta$ for all x, in which case the reaction term $f(T)$ is uniformly zero for all time and the equation for T simplifies to the well-understood heat equation.

If one considers a typical reaction-diffusion equation on a two-dimensional cylinder $\mathbb{R} \times [a,b]$, it is easy to check that the a direct analogue of the one-dimensional theory holds. However, if one broadens the model to include the effects of advection or of inhomogeneities in the environment, traveling waves cannot exist since the system is no longer translation invariant. Hence, in periodic environments, one must instead use the framework of *pulsating fronts*, first introduced by Shigesada [67]. A pulsating front is a solution which is periodic in the traveling frame. Namely, if u and A are L-periodic in x, if $f(x,\cdot)$ is ignition type for each x, and if f is L-periodic in x, then

$$T_t + u \cdot \nabla T = \nabla \cdot (A \nabla T) + f(x,T)$$

has a unique solution up to translation of the form $T(t,x,z) = T^m(x - ct, x, z)$, where $T^m(s,x,z)$ is L-periodic in x. We discuss this in more detail in Chapter 4. The existence, uniqueness, and stability theory of pulsating fronts is exactly analogous to the theory of traveling waves [9].

This is reassuring, but the physical situation is slightly more complicated than this model lets on. In fact, as the fluid combusts and the temperature changes, the resulting density differences induce advection. This new, induced advection is then time-dependent, but the existence, uniqueness, and stability results discussed above only hold for time-independent advection. One remedy for this is to consider a coupled model, where the advection u satisfies a fluid equation with a gravitational forcing term. There are many choices for fluid equations and boundary conditions, but in this thesis, we consider the system

$$\begin{cases} T_t + u \cdot \nabla T = \Delta T + f(x, T), \\ u_t - \Delta u + \nabla p = T\hat{z}, \\ \nabla \cdot u = 0, \end{cases} \tag{1.4.1}$$

where p is the pressure of the system, an unknown. We assume that T satisfies Neumann boundary conditions and that u satisfies the no-stress boundary conditions, which we will state explicitly in Chapter 4.

There are a number of works investigating this system. However, these results are quite technical and are, in general, perturbative. The major impediment to research into this system is the lack of a comparison principle. As such, we only briefly mention here the few results on which we build. When f does not depend on the variable x, traveling waves have been shown to exist in a number of settings [7, 25, 51, 52]. These results hold only in "flat" cylinders of the form $\mathbb{R} \times \omega$ where $\omega \subset \mathbb{R}^{n-1}$ is a smooth domain. In this setting, one can hope to find traveling waves as the system and the domain are translation invariant. A more in depth discussion of the history of research into this model outside of the existence results mentioned above can be found in Chapter 4.

Our final result in this thesis is an extension of the results mentioned above. We now state an existence result for pulsating fronts in the reactive-Boussinesq system above in "periodic" cylinders.

Theorem 1.4.1. *Let Ω be any domain which is L-periodic in x and bounded in z. Let f be as above. Then there exists T^m and u^m, which are L-periodic in their second variable, such that*

$$T(t, x, z) = T^m(x - ct, x, z), \quad \textit{and} \quad u(t, x, z) = u^m(x - ct, x, z).$$

solves the system equation (1.4.1).

We note that, in Chapter 4, we prove a more general theorem than this, but for the sake of clarity, we state the theorem as such. Theorem 1.4.1 says that there exist pulsating fronts for any periodic domain which is bounded in one direction.

This is the most technical result in this thesis, and requires a single argument involving three sequential approximations of the problem. We follow the general technique of [7]; however, the proof is significantly complicated by the fact that the change of variables to the moving frame yields a degenerate elliptic equation in our setting. This degeneracy causes regularity issues for T. In addition, the coupling of the advection u with the temperature T magnifies these issues. As such, the proof involves a number of original techniques, most notably new estimates in Sobolev spaces and a novel regularization method for the advection. The work presented in this chapter appears in [43].

Chapter 2

The Ebert-van Saarloos constant

2.1 Introduction

In this chapter we consider a probabilistic model, branching Brownian motion, through analytic means. In this process, each particle currently alive moves as an independent Brownian motion and independently splits at a constant rate into two particles, each of which then evolves as the parent.In addition, we are interested in this model where the particles are pushed by a given deterministic drift, $-\dot{X}$ and where particles are killed upon reaching the origin.

This connection to the Fisher-KPP equation and its use as a model for populations undergoing selection have made branching Brownian motion the subject of intense interest in recent years [1, 3, 13,14,20,21,29,35,36,39–41,45,53,54,64]. Early work by McKean, [56], focused on understanding the statistics of the rightmost particle of branching Brownian motion with neither drift nor absorption in order to understand solutions to the Fisher-KPP equation. We describe in more detail the connection between this work and Fisher-KPP at the conclusion of this section. In 1978, Kesten introduced drift and absorption at the origin into the model in [45]. In this work, he showed that $\dot{X} = 2$ is the critical drift, separating systems which die out with probability one from systems with positive survival probability. Recently, more precise results have been obtained regarding the distribution of particles at or near the critical drift and the convergence of the statistics of the rightmost particle; see, for example, [**?**, 1, 13, 14, 39–41, 53]. The preceding is an incomplete bibliography, and the interested reader should investigate the references within the works referenced above.

Our perspective is slightly different. We wish to understand the effect of lower order terms in expansion of the drift on the average number of surviving particles. To be more explicit, let N_t be the number of particles at time t with the position of the ith particle given by Y_t^i, and define

$$v(t, x) = \mathbb{E}^x \left[\sum_{i=1}^{N_t} v_0 \left(Y_t^i \right) \right],$$

for some compactly supported function v_0. Then, for $x \geq X(t)$, v solves the equation

$$\begin{cases} v_t = v_{xx} + v, \\ v(t, X(t)) = 0, \\ v(0,x) = v_0(x), \end{cases} \tag{2.1.1}$$

see, e.g. the "many-to-one" lemma [38, 40]. In particular, if v_0 is the indicator function of $[a, b]$, this is simply the expected number of particles in $[a, b]$. In [35, 36], the authors, using (2.1.1) as an approximation for Fisher-KPP, show that, letting

$$\dot{X} = 2 + \frac{r}{t},$$

the only choice for r which yields non-trivial long-time behavior is $-3/2$. We obtain a correction of order $t^{-3/2}$ which gives us more precise information on the average number of particles by a refinement of their methods and using a connection between the mass of the solution of (2.1.1), v, with the normal derivative at the origin, $v_x(0)$. This allows us to find an expansion for $\dot{X}$, independent of initial data, which yields faster convergence to the limiting mass. We now state this precisely.

Statement of Results

Before we state the main theorem, we give a bit of notation and recast our problem. For ease of exposition, we use the following notation in order to omit tracking multiplicative constants that arise.

Notation 2.1.1. *For two values a and b, which may depend on time and various other data, we write*

$$a \lesssim b,$$

if there is some multiplicative constant, $C > 0$, that is independent of time such that

$$a \leq Cb.$$

Such a constant may depend on initial data or various other constants.

We note that, despite this notation, we occasionally need to introduce an arbitrary constant into our equations. Whenever we do, we denote by C such an arbitrary constant which may change line by line, but which is independent of time.

In addition, in order to avoid the complications inherent in a moving boundary, we shift to a moving frame to obtain the equation

$$\begin{cases} v_t - \dot{X}(t)v_x = v_{xx} + v, & \text{for } x \geq 0 \\ v(t,0) = 0, \\ v(0,x) = v_0(x). \end{cases} \tag{2.1.2}$$

Theorem 2.1.2. *Let v satisfy (2.1.2) with $X(t)$ given by*

$$X(t) = 2(t+1) - \frac{3}{2}\log(t+1) - \frac{\overline{c}}{\sqrt{t+1}}, \tag{2.1.3}$$

and where $v_0(x)$ is a compactly supported function on $[0, \infty]$. Then there exists $\alpha_0 \geq 0$ such that for any $p \in [1, \infty]$ we have

$$\lim_{t\to\infty} \|v(t,x) - \alpha_0 x e^{-x}\|_{L^p([0,\infty])} = 0. \tag{2.1.4}$$

In addition, we have that if $\overline{c} = 3\sqrt{\pi}$, then

$$\left|\alpha_0 - \int_0^\infty v(t,x)dx\right| \lesssim \frac{\log(t)}{t}. \tag{2.1.5}$$

Otherwise, we have that

$$\frac{1}{\sqrt{t}} \lesssim \left|\alpha_0 - \int_0^\infty v(t,x)dx\right| \lesssim \frac{1}{\sqrt{t}}. \tag{2.1.6}$$

Finally, for certain initial data, we have $\alpha_0 > 0$.

The main focus of this chapter is in obtaining the mass stabilization rates in (2.1.5) and (2.1.6). We state the convergence in L^p and the positivity of α_0 in order to reassure the skeptical reader that these rates have meaning. The L^p convergence and the positivity of α_0 arises naturally in our work, and we make note when they become apparent.

Our proof of (2.1.5) in Theorem 2.1.2 involves no direct estimates on the mass of the solution. In systems such as (2.1.2), there is a close relationship between the derivative of the solution at $x = 0$ and its total mass. Hence, we focus on obtaining estimates on the derivative of the solution at the origin. To this end, the main ingredient of the proof of Theorem 2.1.2 is the following lemma.

Lemma 2.1.3. *Let v satisfy (2.1.2) with $X(t)$ and $\overline{c}$ given by (2.1.3) and where the initial data v_0 is compactly supported on $[0, \infty]$. Then there exists a non-negative constant α_0 such that $v_x(t,0)$ converges to α_0. If $\overline{c} = 3\sqrt{\pi}$, we have*

$$|v_x(t,0) - \alpha_0| \lesssim \frac{\log(t)}{t}.$$

On the other hand, if $\bar{c} \neq 3\sqrt{\pi}$*, we have that, for* t *sufficiently large,*

$$\frac{1}{\sqrt{t}} \lesssim |v_x(t,0) - \alpha_0| \lesssim \frac{1}{\sqrt{t}}.$$

Connection With Front Speeds in Fisher-KPP

Much of the renewed interest in understanding the system (2.1.1) and its probabilistic counterpart, branching Brownian motion with drift and absorption, lies in its connection to the Fisher-KPP equation

$$\begin{cases} u_t = u_{xx} + u(1-u), \\ u(0,x) = u_0(x), \end{cases} \tag{2.1.7}$$

where u_0 is some localized smooth function on $\mathbb{R}$ taking values in $[0,1]$. This equation usually arises as a model for population dynamics, see e.g. [30, 31, 58, 72].

The equation (3.0.1) was originally studied in the early twentieth century in [31, 46], and it was observed that traveling wave solutions to the system exist. To be more explicit, there are global in time solutions of the form $u(t,x) = \phi_c(x-ct)$ for profiles ϕ_c and speeds $c \geq 2$. Later, in [4], Aronson and Weinberger showed that solutions with more general initial data spread at the speed of the critical traveling wave and admit the same behavior, in that the steady state $u \equiv 1$ invades the unsteady state $u \equiv 0$. Namely, given a solution to (3.0.1) where u_0 is compactly supported, non-negative, and non-zero, we have

$$\min_{|x| \leq ct} u(x) = 1, \quad \text{for all } c < 2,$$

and

$$\max_{|x| \geq ct} u(x) = 0, \quad \text{for all } c > 2.$$

In this case, as in Kesten's paper [45], the critical speed is 2.

In the celebrated papers [18, 19], Bramson obtained more precise asymptotics of the front location with probabilistic methods. More specifically, he showed that for any $m \in (0,1)$, there exists a shift x_m, depending on the initial conditions and m, such that

$$\{x \in \mathbb{R} : u(t,x) = m\} \subset \left[2t - \frac{3}{2}\log(t) - x_m - o(1), 2t - \frac{3}{2}\log(t) - x_m + o(1)\right].$$

Recently, Roberts simplified the proofs of these results [64]. Ebert and van Saarloos, in [28], obtained, through non-rigorous methods, the next term in the expansion. Namely, using matched asymptotics, they argue that the front speed is given by

$$2t - \frac{3}{2}\log(t) - x_m - \frac{3\sqrt{\pi}}{\sqrt{t}} + O(t^{-1}). \tag{2.1.8}$$

Interestingly, though the constant term has dependence on the initial data, the lower order term is believed to be universal. We point out that the expansion obtained by Ebert and van Saarloos is the same as the expansion we obtain in (2.1.3) through Theorem 2.1.2. In fact, part of the motivation of our work has been to provide some understanding of the $3\sqrt{\pi}$ term, as the Ebert and van Saarloos paper does not provide any interpretation beyond the matched asymptotics in the formal derivation. The most recent works regarding the front location in Fisher-KPP are [35, 36] by Hamel et al. In these papers the authors used PDE methods, which we have borrowed and expanded on here, in order to obtain results similar to those of Bramson at the expense of the precision in the constant term. In addition, we mention a work in preparation, [15], in which the authors, through probabilistic methods, investigate the value of $\overline{c}$ given in (2.1.8) in the Fisher-KPP context.

That our expansion for the critical drift in (2.1.1) is the same as Ebert and van Saarloos's expansion for the front location for (3.0.1) is not a surprise. Solutions to (2.1.1) provide a convenient family of sub and super solutions to (3.0.1), which are exceptionally faithful approximations of the tail of solutions to (3.0.1) provided that $X(t)$ is chosen carefully. As we mentioned above, the authors in [35, 36, 62], use equations such as (2.1.1) in order to obtain precise results about the front position in Fisher-KPP. In addition, we remind the reader that solutions of (2.1.1) and (3.0.1) are connected through their interpretation as statistics of branching Brownian motion [56]. The close relationship between these two probabilistic models has been leveraged to transfer understanding from one system to the other, see e.g. [29, 35, 54] and many of the papers mentioned above.

Outline of the Chapter

The chapter is organized as follows. In Section 2.2, we show how the mass stabilization rates given in Theorem 2.1.2 follow from Lemma 2.1.3. This boils down to leveraging a connection between the total mass of a solution to (2.1.2) and its derivative as $x = 0$ to obtain an ODE governing the total mass. Then we apply the estimates provided by Lemma 2.1.3 to conclude.

In Section 2.3, we switch to self-similar variables in order to reduce the equation to a simple parabolic PDE with a decaying forcing term. This change of variables also used in [35, 36] and is convenient because it changes (2.1.2) into a PDE that has a discrete spectrum which we know explicitly. This allows us to decompose our solution into three parts: a steady state, a slowly decaying function with vanishing normal derivative, and a rapidly decaying remainder. This reduces Lemma 2.1.3 to proving this decomposition. We note that the convergence of v to $\alpha_0 x e^{-x}$ is an easy consequence of this, and, as such, we omit the proof of it.

Finally, in Section 2.4, we give the proof of this decomposition by explicitly solving for the steady state and the slowly decaying function and by obtaining estimates for the quickly decaying remainder. At some point in the analysis, the positivity of α_0 becomes obvious; we share when this happens.

2.2 Deducing population stabilization rates from Lemma 2.1.3

Proof of (2.1.5) *and* (2.1.6)*:* We first cover the case where $\bar{c} = 3\sqrt{\pi}$. Let α_0 be the constant from Lemma 2.1.3. Integrating equation (2.1.2) and defining

$$P(t) = \int_0^\infty v(t,x)dx - \alpha_0,$$

we obtain

$$P'(t) - P = -v_x(t,0) + \alpha_0.$$

To see this, notice that parabolic regularity theory gives us that v is smooth in time and space and that v and v' will decay exponentially as x tends to infinity. Hence, the boundary terms at infinity vanish, and we are justified in pulling the time derivative outside of the integral that defines P.

We point out that $P(t)$ is bounded above uniformly in time. One may see this by, for instance, re-proving the bound (20) from [35] in our setting[1]. We note that this bound follows directly from the analysis in this paper.

Define $E(t) = e^{-t}M(t)$. Notice that $E'(t) = (M' - M)e^{-t}$ and that

$$\lim_{t\to\infty} E(t) = 0$$

since $M(t)$ is bounded above. Hence we have that

$$E'(t) = (\alpha_0 - v_x(t,0))e^{-t}.$$

Integrating both sides of this and applying Lemma 2.1.3, we get

$$\begin{aligned} |M(t)|e^{-t} &= \left|\int_t^\infty E'(s)ds\right| = \left|\int_t^\infty (\alpha_0 - v_x(s,0))e^{-s}ds\right| \\ &\lesssim \int_t^\infty \frac{\log(s)}{s}e^{-s}ds \le \frac{\log(t)}{t}\int_t^\infty e^{-s}ds = \frac{\log(t)}{t}e^{-t}. \end{aligned}$$

Multiplying both sides by the exponential finishes the upper bound for this choice of $\bar{c}$.

If $\bar{c} \neq 3\sqrt{\pi}$, we may apply Lemma 2.1.3 in the same manner to get the desired upper bound in the statement of the proof. In order to obtain the lower bound, let M and E be as above. For t sufficiently large, we may apply Lemma 2.1.3 to obtain that

$$\begin{aligned} |M(t)|e^{-t} &= \left|\int_t^\infty E'(s)ds\right| \gtrsim \int_t^\infty \frac{e^{-s}}{\sqrt{s}}ds \\ &\ge \frac{1}{\sqrt{2t}}\int_t^{2t} e^{-s}ds \gtrsim \frac{e^{-t} - e^{-2t}}{\sqrt{t}}. \end{aligned}$$

[1] Actually, their bound may be used out-of-the-box in our setting by creating a super-solution with well-chosen initial data. Since this is a minor point, we do not go into detail.

Multiplying both sides by e^t and taking t sufficiently large, we obtain the desired lower bound. This finishes the proof. $\square$

2.3 Self-similar variables

In order to understand the solutions to (2.1.2) where $X(t)$ is given by (2.1.3), we change variables a number of times following the development in [35, 36]. First, we remove an exponential to obtain the function $\overline{v} = e^x v$ satisfying

$$\overline{v}_t + \left(\frac{3}{2(t+1)} - \frac{\overline{c}}{2(t+1)^{3/2}}\right)\overline{v}_x = \overline{v}_{xx} + \left(\frac{3}{2(t+1)} - \frac{\overline{c}}{2(t+1)^{3/2}}\right)\overline{v}. \tag{2.3.1}$$

Changing to self-similar variables $\tau = \log(1+t)$ and $y = x(1+t)^{-1/2}$, we obtain

$$w_\tau - \frac{y}{2}w_y - w_{yy} - \frac{3}{2}w = \left(\frac{\overline{c}}{2e^\tau} - \frac{3}{2e^{\tau/2}}\right)w_y - \frac{\overline{c}}{2e^{\tau/2}}w, \tag{2.3.2}$$

where we have defined $w(\tau, y) = \overline{v}(e^\tau - 1, ye^{-\tau/2})$. Let $W(\tau, y) = e^{\tau/2}e^{y^2/8}w(\tau, y)$ and we get

$$W_\tau + MW = \left(\frac{\overline{c}}{2e^\tau} - \frac{3}{2e^{\tau/2}}\right)\left(W_y - \frac{y}{4}W\right) - \frac{\overline{c}}{2e^{\tau/2}}W, \tag{2.3.3}$$

where

$$M = -\partial_y^2 + \left(\frac{y^2}{16} - \frac{3}{4}\right).$$

The initial data and the boundary conditions for W are given by

$$\begin{cases} W(\tau, 0) = 0, \\ W(0, y) = e^{y^2/8}e^y v_0(y). \end{cases}$$

We work mainly with W in the sequel. Notice that the operator on the left hand side of (2.3.3) is the equation of a simple harmonic oscillator with a decaying forcing term. We use the fact that we understand the eigenvalues and eigenfunctions of this operator in the analysis that follows.

In these coordinates, Lemma 2.1.3 is easily reduced to proving the following decomposition as the following gives complete information about $v_x(t, 0)$.

Lemma 2.3.1. *Let W satisfy* (2.3.3) *with smooth, compactly supported initial data. Then there exists α, g, and R such that*

$$W(\tau, y) = \alpha y e^{-y^2/8} + e^{-\tau/2}g(y) + R(\tau, y). \tag{2.3.4}$$

In addition, $|R_y(0)| \lesssim (1+\tau)e^{-\tau}$. Finally, $g_y(0) = 0$ if and only if $\overline{c} = 3\sqrt{\pi}$

We prove this lemma in the sequel by decomposing W into three functions: one part is the steady solution of the equation, one is a slowly decaying function with zero derivative at $y = 0$, and one is a quickly decaying function. First, we show that W is bounded in L^2 and converges to the steady state at the rate $e^{-\tau/2}$. This also shows the existence of α above. Then we prove the existence of g and use the fact that we may solve for it explicitly. Finally, we leverage these facts to prove the existence of R.

2.4 Proof of Lemma 2.3.1

Before we begin the proof, notice that M defines a non-negative definite, symmetric quadratic form on the space

$$X := \{\phi \in L^2 : y\phi \in L^2, \phi_y \in L^2\}, \tag{2.4.1}$$

which we call Q. Namely, for all $\phi \in X$, we define

$$Q(\phi) := \int \phi(M\phi)dy. \tag{2.4.2}$$

Note that Q satisfies the following inequality

$$\int \frac{y}{4}\phi^2 dy \le Q(\phi) + \|\phi\|_2^2. \tag{2.4.3}$$

We use this inequality often in the sequel.

Let $e_0, e_1, \ldots$ denote the eigenfunctions of M and we know by [36] that the first two eigenvalues are 0 and 1. Hence Q is non-negative on X and $Q(\phi) \ge \|\phi\|_2^2$ on $\mathrm{Span}\{e_1, e_2, \ldots\}$. Moreover, we know that

$$e_0(y) = \frac{1}{\sqrt{2\sqrt{\pi}}} y e^{-y^2/8}.$$

First, we show that W converges in L^2 to the steady state at the rate $e^{-\tau/2}$. In addition, this shows the existence of α in Lemma 2.3.1 since e_0 is the steady state of (2.3.3). We remark that the potential positivity of the total mass in Theorem 2.1.2 follows from the work below. We comment further on this following the proof.

Lemma 2.4.1. *Suppose that W satisfies* (2.3.3)*. Then there exists α such that*

$$\|W - \alpha y e^{-y^2/8}\|_2 \lesssim e^{-\tau/2}.$$

Proof. First, we show that W is bounded in L^2. To this end, multiplying (2.3.3) by W and integrating by parts we get

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int |W|^2 dy + Q(W) &= \int \left[\left(\frac{\bar{c}}{2e^{\tau}} - \frac{3}{2e^{\tau/2}}\right) yW^2 - \frac{\bar{c}}{2e^{\tau/2}}W^2\right] dy \\ &\leq 4e^{-\tau/2}\left(\frac{\bar{c}}{2e^{\tau/2}} - \frac{3}{2}\right)\left[Q(W) + \|W\|_2^2\right] + e^{-\tau/2}\frac{\bar{c}}{2}\|W\|_2^2 \\ &\lesssim e^{-\tau/2}\left[Q(W) + \|W\|_2^2\right],\end{aligned}$$

where we used (2.4.3) to obtain the second inequality. We may the re-write this as

$$\frac{1}{2}\frac{d}{d\tau}\|W\|_2^2 + \left(1 - Ce^{-\tau/2}\right)Q(W) \lesssim e^{-\tau/2}\|W\|_2^2.$$

We may choose τ_0 such that, for $\tau \geq \tau_0$, we have $1 - Ce^{-\tau/2} \geq 0$. We note that, by working in the original Euclidean coordinates (t, x), we may easily check that W remains in L^2 for $\tau \in [0, \tau_0]$. Since Q is non-negative, then we have that

$$\frac{d}{d\tau}\|W\|_2^2 \lesssim e^{-\tau/2}\|W\|_2^2.$$

Solving this differential inequality gives us that W is uniformly bounded in L^2.

Now we finish the proof in two steps. First, we look at the projection of W onto e_0. This gives us the steady state. Then we look at the component of W orthogonal to e_0. To be explicit, we decompose W as

$$W(\tau, y) = W_1(\tau)e_0(y) + \tilde{W}(\tau, y), \tag{2.4.4}$$

where $\tilde{W}$ is an element of Span$\{e_1, e_2, \dots\}$.

In order to understand W_1, multiply (2.3.3) by e_0 and integrate by parts to obtain

$$\begin{aligned}(W_1)_\tau &= \int W_\tau(\tau, y)e_0(y)dy + \int MW(\tau, y)e_0(y)dy \\ &= \int \left(\frac{\bar{c}}{2e^{\tau}} - \frac{3}{2e^{\tau/2}}\right)\left(W_y(\tau, y) - \frac{y}{4}W(\tau, y)\right)e_0(y)dy \\ &\quad - \int \frac{\bar{c}}{2e^{\tau/2}}W(\tau, y)e_0(y)dy \\ &\lesssim e^{-\tau/2}\int W(\tau, y)\Big[|(e_0)_y| + ye_0(y)\Big]dy + e^{-\tau/2}|W_1(\tau)| \\ &\lesssim e^{-\tau/2}\|W\|_2^2.\end{aligned}$$

We used here that M is symmetric and that $Me_0 = 0$. We may obtain the same inequality for $-(W_1)_\tau$ by multiplying by $-e_0$ instead. Hence, we obtain

$$|(W_1)_\tau| \lesssim e^{-\tau/2}\|W\|_2 \lesssim e^{-\tau/2}. \tag{2.4.5}$$

Hence there exists α' such that W_1 tends to α' as τ tends to infinity. Moreover, we have that $|\alpha' - W_1| \lesssim e^{-\tau/2}$. Hence we need only show that $\tilde{W}$ decays fast enough in order to finish the proof.

To obtain the decay of $\tilde{W}$, we use (2.4.4) in (2.3.3) to obtain

$$\begin{aligned}\tilde{W}_\tau + M\tilde{W} = e^{-\tau/2}&\left[\left(\frac{\bar{c}}{2e^{\tau/2}} - \frac{3}{2}\right)\left(\tilde{W}_y - \frac{y}{4}\tilde{W}\right) - \frac{\bar{c}}{2}\tilde{W}\right] \\ &+ e^{-\tau/2}\left[\left(\frac{\bar{c}}{2e^{\tau}} - \frac{3}{2e^{\tau/2}}\right)\left(W_1(e_0)_y - \frac{y}{4}W_1e_0\right) - \frac{\bar{c}}{2e^{\tau/2}}W_1e_0\right] - (W_1)_\tau e_0.\end{aligned}$$

Noting that $\tilde{W}$ lives in the span of $e_1, e_2, \ldots$, we have that $Q(\tilde{W}) \geq \|\tilde{W}\|_2^2$. Hence, when we multiply the equation above by $\tilde{W}$, integrate by parts and use our inequality on Q, we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|\tilde{W}\|_2^2 + \left(1 - Ce^{-\tau/2}\right)\|\tilde{W}\|_2^2 \lesssim e^{-\tau/2}\|\tilde{W}\|_2. \tag{2.4.6}$$

Solving this differential inequality yields

$$\|\tilde{W}\|_2 \lesssim e^{-\tau/2},$$

finishing the proof. □

Remark 2.4.2. *By changing coordinates, we see that*

$$W_0(y) = e^{y^2/8}e^y v_0(y),$$

which gives us that

$$\langle W_0, ye^{-y^2/8}\rangle = \int_0^\infty ye^y v_0(y)dy.$$

Hence, if $\int \xi v_0(\xi)d\xi$ *is large enough, then* α_0 *must be positive. Indeed,* (2.4.6) *implies that if* $\tilde{W}$ *is small initially, it stay small, and then* (2.4.5) *implies that if* W_1 *is large initially it remains large. When our equation,* (2.1.2)*, is used to approximate Fisher-KPP, this may be overcome by either choosing a larger initial condition or running the system for sufficiently long in order that this integral is large, depending on whether one is looking for a supersolution or subsolution.*

Now we investigate g and R in (2.3.4). The ansatz implicit in (2.3.4) gives us the following equation

$$
\begin{aligned}
-\frac{e^{-\tau/2}}{2}g+&e^{-\tau/2}Mg + R_\tau + MR \\
&= e^{-\tau/2}\left[\frac{3\alpha}{4}y^2e^{-y^2/8} - \frac{\alpha\bar{c}}{2}ye^{-y^2/8} - \frac{3\alpha}{2}e^{-y^2/8}\right] \\
&\quad + e^{-\tau}\left[\frac{\alpha\bar{c}}{2}e^{-y^2/8} - \frac{\alpha\bar{c}}{4}y^2e^{-y^2/8} - \frac{3}{2}g_y + \frac{3y}{8}g - \frac{\bar{c}}{2}g + \frac{\bar{c}e^{-\tau/2}}{2}\left(g_y - \frac{y}{4}g\right)\right] \\
&\quad + e^{-\tau/2}\left[\left(\frac{\bar{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left(R_y - \frac{y}{4}R\right) - \frac{\bar{c}}{2}R\right],
\end{aligned}
$$

where α is as in Lemma 2.4.1. We separate this into equations for g and R by associating the terms on the right of order $e^{-\tau/2}$ with g and the rest with R. This yields

$$
Mg - \frac{g}{2} = \alpha e^{-y^2/8}\left[\frac{3}{4}y^2 - \frac{\bar{c}}{2}y - \frac{3}{2}\right], \tag{2.4.7}
$$

and

$$
R_\tau + MR = e^{-\tau}f(y,\tau) + e^{-\tau/2}\left[\left(\frac{\bar{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left(R_y - \frac{y}{4}R\right) - \frac{\bar{c}}{2}R\right], \tag{2.4.8}
$$

where f is given by

$$
f(\tau,y) = \left[\frac{\alpha\bar{c}}{2}e^{-y^2/8} - \frac{\alpha\bar{c}}{4}y^2e^{-y^2/8} - \frac{3}{2}g_y + \frac{3y}{8}g - \frac{\bar{c}}{2}g + \frac{\bar{c}e^{-\tau/2}}{2}\left(g_y - \frac{y}{4}g\right)\right]. \tag{2.4.9}
$$

We first show that (2.4.7) is well defined and that g has the property that $g_y(0) = 0$ if and only if $\bar{c} = 3\sqrt{\pi}$. Then we show (2.4.8) is well-defined as well.

Lemma 2.4.3. *There exists a smooth solution $g \in X$ of (2.4.7) which is locally bounded in C^1. In addition, $g_y(0) = 0$ if and only if $\bar{c} = 3\sqrt{\pi}$.*

Proof. First we show, abstractly, that such a solution exists. Then we write down an explicit solution to the equation. This explicit solution allow us to understand $g_y(0)$.

In order to show the existence of a solution $g \in X$, we proceed as in Lemma 2.4.1. Namely, we write

$$
g = g_1e_0 + \tilde{g}, \tag{2.4.10}
$$

where $\tilde{g}$ is in the span of $e_1, e_2, \ldots$. To bound g_1, we multiply (2.4.7) by e_0 and integrate. This gives an explicit formula for g_1 independent of $\tilde{g}$. Namely

$$
g_1 = -2\alpha\int\left[\frac{3}{4}y^2 - \frac{\bar{c}}{2} - \frac{3}{2}\right]e_0(0)e^{-y^2/8}dy. \tag{2.4.11}
$$

Then, fixing g_1 as this value, we simply write down the equation for $\tilde{g}$ given by

$$M\tilde{g} - \frac{\tilde{g}}{2} = \alpha \left[\frac{3}{4}y^2 - \frac{\bar{c}}{2}y - \frac{3}{2}\right] + \frac{g_1}{2}e_0. \tag{2.4.12}$$

On $\mathrm{Span}\{e_1, e_2, \dots\}$ with the norm of X, the operator $M - 1/2$ is coercive, i.e. its spectrum is bounded below by a positive constant. Here we are using that for $\phi \in X \cap \mathrm{Span}\{e_1, e_2, \dots\}$, we have $Q(\phi) \geq \|\phi\|_2^2$. Hence, the Lax-Milgram theorem implies that (2.4.12) is uniquely solvable in $\mathrm{Span}\{e_1, e_2, \dots\}$. This, along with (2.4.11), gives us that (2.4.7) is uniquely solvable in X. The standard elliptic theory, as in [34], then, implies that g is in fact in H^k_{loc} for every k. This, in addition, implies that g is smooth and locally bounded in C^1.

We now solve (2.4.7) explicitly. By changing variables to $z = y^2/4$ and letting $G = e^{z/2}g$, we obtain the equation:

$$-zG_{zz} - \left(\frac{1}{2} - z\right)G_z - G = \alpha\left[3z - \bar{c}\sqrt{z} - \frac{3}{2}\right].$$

Following the work of Ebert and van Saarloos in [28], shows us that the explicit solution is of the form

$$G(z) = \alpha\left[\frac{3}{2} + 2\bar{c}\sqrt{z} - \frac{3}{2}F_2(z) + a_1(1-2z) + a_2H(z)\right]$$

where F_2 and H are given by

$$F_2(z) = \sqrt{\pi}\sum_{n=2}^{\infty}\frac{z^n}{n(n-1)\Gamma(1/2+n)}, \quad \text{and} \quad H(z) = -\frac{\sqrt{z}}{4}\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(-1/2+n)}{\Gamma(3/2+n)}. \tag{2.4.13}$$

As in [28], one may check that

$$\lim_{z\to\infty}\frac{H(z)}{z^{-3/2}e^z} = -\frac{1}{4}, \quad \text{and} \quad \lim_{z\to\infty}\frac{F_2(z)}{z^{-3/2}e^z} = \sqrt{\pi}.$$

Hence H and F_2 are clearly not in L^2 in our original variables, even with the additional $e^{-z/2}$ factor. Thus, we must choose a_2 such that these terms cancel at $z = \infty$. In addition, we must choose a_1 such that $G(0) = 0$. Hence, we obtain

$$\begin{aligned} G(z) &= \alpha\left[\frac{3}{2} + 2\bar{c}\sqrt{z} - \frac{3}{2}G_2(z) + \frac{-3}{2}(1-2z) - 6\sqrt{z}H(z)\right] \\ &= \alpha\left[2\bar{c}\sqrt{z} - \frac{3}{2}G_2(z) + 3z - 6\sqrt{\pi}H(z)\right] \end{aligned}$$

.

By uniqueness, we may return to the original variables to obtain an explicit formula for $g_y(0)$.

Namely, we have that

$$g(y) = \alpha e^{-y^2/8}\left[\bar{c}y - \frac{3}{2}G_2(y^2/4) - \frac{3y^2}{4} - 6\sqrt{\pi}H(y^2/4)\right].$$

With this formula and with the first term in the sequence for H, we can easily see that $g_y(0) = 0$ if and only if $\bar{c} = 3\sqrt{\pi}$, finishing the proof. □

Lemmas 2.4.1 and (2.4.3) tell us that R must decay to zero in L^2 as τ tends to infinity. This is a key fact that we use in the following lemma, which finishes the proof of Lemma 2.3.1.

Lemma 2.4.4. *Let R satisfy* (2.4.8) *with α and g given above. Then we have the following bounds*

$$\|R\|_2 \lesssim \tau e^{-\tau}, \quad \text{and} \quad |R_y(0)| \lesssim \tau e^{-\tau}.$$

Proof. We proceed by decomposing R as we did W and g in the proofs of Lemmas 2.4.1 and 2.4.3. Namely, let

$$R(\tau, y) = R_1(\tau)e_0(y) + \tilde{R}(\tau, y), \tag{2.4.14}$$

where $\tilde{R}$ is orthogonal to e_0. To obtain a bound on R_1, we first note that Lemmas 2.4.1 and 2.4.3 imply that R decays to zero in L^2 as least at the rate $e^{-\tau/2}$. Then we multiply (2.4.8) by e_0 and integrate to obtain

$$(R_1)_\tau = e^{-\tau}\langle f, e_0\rangle - e^{-\tau/2}\langle R, \left(\frac{\bar{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left((e_0)_y + \frac{y}{4}e_0\right) + \frac{\bar{c}}{2}e_0\rangle.$$

Using the formula for f given in (3.1.1), we see that $\langle f, e_0\rangle$ is bounded in X uniformly in time. In addition, we know that $e_0 \in X$ does not depend on time. Hence, applying Cauchy-Schwarz, we obtain

$$|(R_1)_\tau| \lesssim e^{-\tau} + e^{-\tau/2}\|R\|. \tag{2.4.15}$$

Since we know that $\|R\|_2 \lesssim e^{-\tau/2}$, it follows that

$$|(R_1)_\tau| \lesssim e^{-\tau}.$$

This gives us that

$$|R_1(\tau)| = |R_1(\infty) - R_1(\tau)| = \left|\int_\tau^\infty (R_1)_\tau(s)ds\right| \lesssim \int_\tau^\infty e^{-s}ds = e^{-\tau}.$$

This is the desired bound on R_1.

In order to finish the proof of the L^2 bound on R, we need to bound $\tilde{R}$ in L^2. To this end we use the decomposition (2.4.14) along with (2.4.8), to note that $\tilde{R}$ satisfies

$$\begin{aligned}\tilde{R}_\tau + M\tilde{R} = e^{-\tau} f(y,\tau) + e^{-\tau/2}\left[\left(\frac{\overline{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left(\tilde{R}_y - \frac{y}{4}\tilde{R}\right) - \frac{\overline{c}}{2}\tilde{R}\right] \\ + R_1 e^{-\tau/2}\left[\left(\frac{\overline{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left((e_0)_y - \frac{y}{4}e_0\right) - \frac{\overline{c}}{2}e_0\right] - (R_1)_\tau e_0.\end{aligned}$$

Multiplying this by $\tilde{R}$ and integrating by parts yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\|\tilde{R}\|_2^2 + Q(\tilde{R}) \\ &\qquad \lesssim e^{-\tau}\|\tilde{R}\|_2 + e^{-\tau/2}\left(\int (y+1)\tilde{R}^2 dy\right) + e^{-\tau/2}R_1\|\tilde{R}\|_2 + |(R_1)_\tau|\|\tilde{R}\|_2.\end{aligned}$$

Again using the inequality (2.4.3) along with the inequality on R_1 and $(R_1)_\tau$ that we just obtained, we note that

$$\frac{d}{d\tau}\|\tilde{R}\|_2^2 + (2 - Ce^{-\tau/2})\|\tilde{R}\|_2^2 \lesssim e^{-\tau}\|\tilde{R}\|_2, \tag{2.4.16}$$

where C is some universal constant.

We will now solve this differential inequality. We do this in two steps. First, we obtain a rough bound on the decay of $\tilde{R}$, and then we will leverage this to get the correct decay. To start, we may set τ_0 to be the time when $2 - Ce^{-\tau/2} \geq 7/4$ for all $\tau \geq \tau_0$. Then we see that, for $\tau \geq \tau_0$, we have

$$\frac{d}{d\tau}\|\tilde{R}\|_2^2 + \frac{7}{4}\|\tilde{R}\|_2^2 \lesssim e^{-\tau}\|\tilde{R}\|_2.$$

This gives us that

$$\frac{d}{d\tau}\left(e^{\frac{7\tau}{4}}\|\tilde{R}\|_2^2\right) \lesssim e^{-\tau/8}\left(e^{\frac{7\tau}{4}}\|\tilde{R}\|_2^2\right)^{1/2}.$$

Integrating this in τ implies that $\|\tilde{R}\|_2 \lesssim e^{-7\tau/8}$.

Call $\psi(\tau) = e^{\tau}\|\tilde{R}\|_2$ and we have that

$$\psi_\tau - Ce^{-\tau/2}\psi \lesssim 1.$$

From above we have that $\psi(\tau) \lesssim e^{\tau/8}$. The combination of these implies that

$$\psi_\tau \lesssim 1,$$

which in turn implies that $\psi(\tau) \lesssim 1 + \tau$. This gives us the desired inequality

$$\|\tilde{R}\|_2 \lesssim (\tau + 1)e^{-\tau},$$

which finishes the proof of the initial bound of R.

To finish the proof we need to bound $R_y(0)$. This, however, is a simple consequence of our bounds on the L^2 norm of R and the right hand side of (2.4.8). Indeed, with these, the standard parabolic regularity theory, which may be found, for example, in [48, 49], give us the desired bound on $R_y(0)$, finishing the proof. $\square$

Chapter 3

Super-linear propagation in the Fisher-KPP equation

In this article, we study the long time behavior of solutions to the Fisher-KPP equation

$$\begin{cases} u_t = u_{xx} + f(x,u), & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.0.1}$$

where f is an L-periodic function in x, for some $L > 0$, which vanishes when u is 0 and 1 and where u_0 is a function taking values between 0 and 1 that decays slowly as x tends to infinity. This equation comes from adding a diffusive term to the logistic equation, and it was originally introduced, with $f(x,u) = u(1-u)$, as model for population dynamics in the first half of the twentieth century [31,46]. In this interpretation, u stands for the local population density normalized by the carrying capacity, while f accounts for the effect of an inhomogeneous environment on the growth of this population up to the carrying capacity.

Fisher and Kolmogorov, Petrovskii, and Piskunov showed that if the initial data is localized then the population would spread at a constant rate in the form of a traveling wave. This has been made more precise and succinct over time [18, 19, 33, 35, 36, 50, 64, 68]. In these works, the highest order term in the spreading rate is obtained by studying the linearized problem while the full non-linearity is only felt in the form of a lower-order logarithmic correction. When u_0 is not localized, but decays exponentially, the behavior is similar. Here, the population converges to a traveling wave, the speed of which is given again by the linearized problem [68].

Recently, the case when u_0 decays slower than any exponential and f is homogeneous was investigated by Hamel and Roques [37]. They show that u becomes asymptotically flat in time implying that a traveling wave cannot give the limiting behavior of u. As a result, they investigate the location of the level sets of u and show that, for each $m \in (0,1)$, the level sets satisfy

$$\{x \in \mathbb{R} : u(t,x) = m\} \subset \left[u_0^{-1}\left(C_m e^{-f'(0)t}\right), u_0^{-1}\left(c_m e^{-f'(0)t}\right)\right],$$

where $c_m < C_m$ are constants depending on m, which they cannot obtain explicitly. These results hold for only a small class of initial data; for more general data, they obtain significantly less precise results. The main results of this chapter are to utilize a new approach in order to improve the precision of their result in the homogeneous setting for a broader class of initial data. Beyond this, we obtain these results in the previously unstudied inhomogeneous setting with the same precision. Finally, we show that in the inhomogeneous case, u grows at each point according to the solution to a periodic partial differential equation that is an analogue to the logistic equation when the non-linearity depends on the spatial variable. This provides an analogue to the asymptotic flatness result of Hamel and Roques in the inhomogeneous setting.

3.1 Main results

We now state our results more precisely. First we assume that there is a $\delta > 0$ such that $f(x,r)$ is a positive, $C^{1,\delta}$ function with respect to r and that it satisfies

$$\begin{cases} f(x,r) \le f_r(x,0)r \text{ for all } r \in [0,1], \\ \frac{f(x,r)}{r} \text{ is decreasing in } r, \text{ and} \\ f_r(x,0) > 0 \text{ and} \\ f_r(x,1) < 0 \end{cases} \tag{3.1.1}$$

We note that some of the assumptions on the signs of f and its derivatives can be relaxed, but we use them here to simplify the arguments.

We assume that u_0 is a positive function that decays to zero super-exponentially. By this we mean that, for any $\epsilon \in (0,\infty)$

$$\lim_{x\to\infty} u_0(x) = 0, \quad \text{and} \quad \lim_{x\to\infty} u_0(x)e^{\epsilon x} = \infty. \tag{3.1.2}$$

As pointed out in [37], examples of simple functions which satisfy equation (3.1.2) are

- $u_0(x) \sim \alpha e^{-\beta x/\log(x)}$,
- $u_0(x) \sim \alpha e^{-\beta x^\gamma}$, $\gamma \in (0,1)$,
- $u_0(x) \sim \alpha x^{-\beta}$, and
- $u_0(x) \sim \alpha \log(x)^{-\beta}$.

Here, by $\sim$ we mean that $u_0(x)$ divided by the right hand side has limit equal to one as x tends to infinity. Though our technique can handle any functions of this type, the presentation is much simpler if we restrict to initial data, u_0, which are eventually monotonic and which decay faster than $e^{-\beta\sqrt{x}}$ for any β. Namely, we assume that there is a point $x_0 > 0$ such that

$$u_0'(x) < 0 \text{ for all } x \geq x_0, \quad \text{and } \lim_{x\to\infty} \frac{u_0'(x)}{u_0(x)/\log(u_0(x))} = 0. \tag{3.1.3}$$

We will comment on the effect of this assumption later. In addition, since we are studying the effect of a slowly decaying tail at ∞, we additionally assume that

$$\liminf_{x\to-\infty} u_0(x) > 0 \tag{3.1.4}$$

in order to prevent pathological behavior at $-\infty$. Again, we note that the positivity assumption on u_0 can be relaxed, but we use it here to simplify the presentation.

Classical results address the case when $u_0 \sim e^{-\lambda x}$ [18,19,31,46,50,68]. If $\lambda > 1$ then u converges to a traveling wave with speed 2 in the correct moving frame, while if $\lambda < 1$, u converges to a traveling wave with speed $\lambda + 1/\lambda$ in the moving frame. In short, this means that the level sets of u_0 move with a constant speed, which increases to infinity as λ decreases to zero, and that any two level sets travel at the same speed and remain a bounded distance from one another for all time. Hamel and Roques point out in [37] that, using these as a family of sub-solutions, we can see immediately that the level sets of any solution with initial data like equation (3.1.2) will spread to the right super linearly in time. We now calculate the speed of these level sets explicitly.

3.1.1 The homogeneous case

Our first result concerns the spreading of the level sets in the homogeneous setting.

Theorem 3.1.1. *Suppose that u solves equation* (3.0.1) *with f depending only on u and not on the ambient space variable, x. Suppose further that f satisfies equation* (3.1.1) *and if u_0 satisfies equations* (3.1.2) *and* (3.1.4)*. In addition, let u_0 satisfy equation* (3.1.3)*. Let φ be the unique (up to translation) global in time solution to*

$$\varphi_t = f(\varphi).$$

Then, for each $m \in (0,1)$, define $T^m \in \mathbb{R}$ to be the unique time that $\varphi(T^m) = m$. Then, for any $r > 0$, we have

$$\limsup_{T\to\infty} \sup_{x \geq u_0^{-1}(\varphi(T^m - T)) - rT} u(T,x) \leq m. \tag{3.1.5}$$

and

$$\liminf_{T\to\infty} \inf_{x \leq u_0^{-1}(\varphi(T^m - T)) + rT} u(T,x) \geq m. \tag{3.1.6}$$

In order to make this result clearer, we first consider a specific example. If f is the standard

Fisher-KPP non-linearity, i.e. $f(u) = u(1-u)$, then

$$\varphi(T) = \frac{e^T}{1+e^T}, \qquad \text{implying that} \qquad \varphi(T^m - T) = \frac{m}{1-m}e^{-T} + O(e^{-2T}),$$

where $O(e^{-2T})$ is bounded by a constant multiple of e^{-2T} for all T. For a general non-linearity, f, one can easily see (by linearization) that $\varphi(T^m - T) = c_m e^{-f'(0)T}$, where c_m is a constant depending only on f and m. In addition, defining the level set of u as

$$E_m(t) \stackrel{\text{def}}{=} \{x : u(t,x) = m\},$$

if we look at, e.g., initial data such that $u_0(x) = x^{-\alpha}$ for large enough x, Theorem 3.1.1 implies that

$$E_m(t) \sim \left(\frac{1-m}{m}e^t\right)^{1/\alpha}. \tag{3.1.7}$$

Notice that the non-linearity affects the constant $\frac{1-m}{m}$, as the linearized equation would yield only $1/m$. The fact that the full non-linearity affects the highest order behavior is in stark contrast to the case when u_0 decays exponentially, where the linearized equation completely determines the highest order behavior.

An easy corollary of Theorem 3.1.1 is the following.

Corollary 3.1.2. *For any $\epsilon > 0$, under the assumptions of Theorem 3.1.1, there exists $T_0 > 0$ depending on all parameters such that if $T > T_0$ then*

$$E_m(T) \subset \left[u_0^{-1}\left(\varphi(T_{m+\epsilon} - T)\right), u_0^{-1}\left(\varphi(T_{m-\epsilon} - T)\right)\right].$$

Again, it is instructive to look at the example that we considered above. Corollary 3.1.2 implies that, for any ϵ and any t large enough, we have

$$E_m(t) \subset \left[\left(\frac{m}{1-m} - \epsilon\right)e^{t/\alpha}, \left(\frac{m}{1-m} + \epsilon\right)e^{t/\alpha}\right].$$

Remark 3.1.3. *We comment briefly on the assumption on the decay of the first derivative of u_0, equation* (3.1.3)*. In practice, this means that u_0 decays slower than $e^{-\sqrt{x}}$. We could remove this restriction, assuming only that u_0 satisfies equation* (3.1.2)*. In this case, we would obtain a linear in time error term in equations* (3.1.5) *and* (3.1.6) *and in Corollary 3.1.2. In other words, there is a constant $r > 0$ such that, for any $m \in (0,1)$ and $\epsilon > 0$, we have*

$$E_m(T) \subset \left[u_0^{-1}\left(\varphi(T_{m+\epsilon} - T)\right) - rT, u_0^{-1}\left(\varphi(T_{m-\epsilon} - T)\right) + rT\right],$$

for T large enough. This result is an improvement over known results. We omit proving this here, *in the interest of presenting a single, clear argument and result.*

3.1.2 The inhomogeneous case

When the non-linearity, f, depends periodically on x, the level sets fail to capture the spreading behavior. Intuitively this is because the function u_0 is very flat at infinity, so it will grow as if it were a solution to the periodic, inhomogeneous Fisher-KPP equation. In particular, u will oscillate in space enough to cause the level sets to become very wide.

To circumvent this, we look instead at the locations of the intervals of a given mean oscillation. In other words, we seek to describe the location of the average-level sets

$$\overline{E}_m(t) \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R} : \fint_x^{x+L} u(t,y)dy = m \right\},$$

where we define, for any finite measure space X and any $L^1(X)$ function $\gamma(x)$,

$$\fint_X \gamma(x)dx \stackrel{\text{def}}{=} \frac{1}{|X|}\int_X \gamma(x)dx.$$

In this regard, we find similar results to Theorem 3.1.1 in the inhomogeneous setting.

As we alluded to above, instead of the logistic equation, the behavior of u for very large x is captured by the global in time solution to

$$\begin{cases} \varphi_t = \varphi_{xx} + f(x,\varphi), & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{T}, \\ \fint \varphi(0,x)dx = \frac{1}{2}, \end{cases} \tag{3.1.8}$$

where $\mathbb{T} = [0,L]$ with the ends identified. We briefly note that, although the techniques to show the well-posedness of equation (3.1.8) are similar to those used in the theory of well-posedness for the Fisher-KPP equation, see e.g. [12], we are unable to find a treatment of it in the literature. As such, we dedicate Section 3.5 to proving the existence and uniqueness of this problem.

We now formulate our results precisely.

Theorem 3.1.4. *Suppose that u solves equation* (3.0.1) *and that f is an L-periodic function satisfying the conditions in equation* (3.1.1). *Suppose u_0 satisfies the conditions in equations* (3.1.2) *to* (3.1.4). *For each $m \in (0,1)$ and each $T > 0$, we may solve the terminal value problem*

$$\begin{cases} \varphi_t^{T,m} = \varphi_{xx}^{T,m} + f(x,\varphi^{T,m}), & \text{on } [0,T]\times[0,L], \\ \frac{1}{L}\int_0^L \varphi^{T,m}(T,x)dx = m, \\ \varphi^{T,m}(0) \equiv B(m,T), \end{cases} \tag{3.1.9}$$

with periodic boundary conditions and where $B(m,T)$ is an unknown. For any $r > 0$, we have that

$$\limsup_{T\to\infty} \sup_{x \geq u_0^{-1}(B(m,T))-rT} \fint_x^{x+L} u(T,y)dy \leq m, \tag{3.1.10}$$

and

$$\liminf_{T\to\infty} \inf_{x \leq u_0^{-1}(B(m,T))+rT} \fint_x^{x+L} u(T,y)dy \geq m. \tag{3.1.11}$$

This result gives an analogous result to the Theorem 3.1.1. Namely, that to find an interval $[nL, nL+L]$ such that $u(t,x)$ take the value m on average, one has to choose n such that $u_0(x)$ roughly looks like $B(m,T)$ for $x \in [nL, nL+L]$. It is simple to check that the system equation (3.1.9) is well defined and is continuous in m and T. It is also easy to check that $B(m,T)$ is increasing in m and decreasing in T.

In Theorem 3.1.4, the $B(m,T)$ term is a bit mysterious. In the homogeneous case, we characterize the analogous term in terms of the global-in-time solution to the logistic equation. In fact, we may do the same here.

Proposition 3.1.5. *Let the conditions of Theorem 3.1.4 be satisfied. Let ψ_0 and f_0 be the unique positive solutions to the eigenvalue problem*

$$\begin{cases} (\psi_0)_{xx} + f_u(x,0)\psi_0 = f_0\psi_0, & x \in \mathbb{T}, \\ \int \psi_0^2 dx = 1. \end{cases}$$

Let T^m be the unique time that $\fint \varphi(T^m)dx = m$. Then

$$\lim_{T\to\infty} \frac{B(m,T)\int \psi_0(x)dx}{\int \varphi(T^m - T, x)\psi_0(x)dx} = 1.$$

Again we may obtain an easy corollary regarding the movement of the average-level sets $\overline{E}_m(t)$.

Corollary 3.1.6. *Let the hypotheses of Proposition 3.1.5 be satisfied. For any $\epsilon > 0$, and any $m \in (0,1)$ there is a time T_0, depending on all parameters, such that if $T \geq T_0$, then*

$$\overline{E}_m(t) \subset \left[u_0^{-1}\left(\varphi(T_{m+\epsilon} - T)\right), u_0^{-1}\left(\varphi(T_{m-\epsilon} - T)\right)\right].$$

As a part of the proof, we will show that there is a positive constant $\alpha > 0$ such that , for very large negative times, $\varphi(t,x) \sim \alpha e^{f_0 t}\psi_0(x)$. Hence Proposition 3.1.5 and Corollary 3.1.6 imply that there is a constant c_m, depending only on m and f, such that

$$\overline{E}_m(t) \sim u_0^{-1}\left(c_m e^{-f_0 t}\right)$$

holds for large times t. The constant c_m is, in principle, computable given a specific non-linearity f

by solving the global in time periodic problem.

Finally, we prove a result which is analogous to the asymptotic flatness result of [37]. This final result shows that u grows exactly as in the global in time solution of the periodic problem as long as time is suitably shifted.

Theorem 3.1.7. *Let the conditions of Theorem 3.1.4 be satisfied. Let ψ_0 be as in Proposition 3.1.5. Define*

$$S_n = \left(\int \psi_0(x)dx\right)^2 u_0(nL). \tag{3.1.12}$$

Then

$$\lim_{T\to\infty} \max_{n\in\mathbb{Z}} \|u(T,\cdot) - \varphi(T^{S_n} + T,\cdot)\|_{L^\infty([nL,nL+L])} = 0.$$

3.1.3 Relationship to previous work

Slowly decaying initial data

As we have mentioned above, the most closely related work is that of Hamel and Roques [37]. In this novel article, the authors present three main results regarding the solutions to the homogeneous Fisher-KPP equation (i.e. (3.0.1) with f depending only on u):

- (Theorem 1.1) if the initial data, u_0, satisfies a mild condition on the second derivative of u_0 along with the slower-than-exponential decay as in equation (3.1.2), they show that for any fixed ϵ
$$E_m(t) \subset u_0^{-1}\left(\left[e^{-(f'(0)+\epsilon)t}, e^{-(f'(0)-\epsilon)t}\right]\right)$$
holds for t large enough;
- (Theorem 1.4) if u_0 satisfies a very strong condition on the decay of the second derivative, they show that there exists constants C and C_m such that
$$E_m(t) \subset u_0^{-1}\left(\left[Cme^{-f'(0)t}, C_m e^{-f'(0)t}\right]\right);$$
- (Theorem 1.5) if $(u_0)_x/u_0$ is in L^p for some $p < \infty$, then $\|u_x(t)/u(t)\|_\infty \to 0$.

In addition, they assume some eventual monotonicity and convexity assumptions on the initial data. We note that the condition on the decay of u_0 that they assume in Theorem 1.4 restricts their result to functions which decay at least as slowly as x^{-2} for the classic Fisher-KPP nonlinearity, i.e. $f(u) = u(1-u)$.

Hamel and Roques prove Theorem 1.1 and Theorem 1.4 by carefully building sub- and super-solutions using the function, $u_0(x)e^{f'(0)t}$, and using the fact that we may bound f below by a second order Taylor approximation at zero. We note that the difference in precision between the constant in their upper bound C_m in theorem 1.4 and our upper bound, $m/(1-m)$, in equation (3.1.7), can be attributed to a more careful analysis of the effect of the non-linearity f carried out by utilizing a new method to create sub- and super-solutions. In addition, we point out that our approach handles a much broader class of initial data and that it may be adapted to handle the case when u_0 is not differentiable (see Remark 3.1.3).

Accelerating fronts in other models

Beyond the work of Hamel and Roques, there has been recent interest in super-linear in time propagation phenomena, or accelerating fronts. In a series of articles [22,23,26], Cabré, Coulon, and Roquejoffre investigated the phenomenon of exponential spreading of level sets of solutions to the Fisher-KPP with the fractional Laplacian. We note that the techniques in this article can be adapted to yield analogous results when u_0 decays slower than the critical cases, which they investigated. Accelerating fronts have also been observed in biological models with a non-local dispersal operator with fat tails [32, 47, 57] and in a kinetic reaction-transport equation [17]. Finally, they have been conjectured to occur in a reaction-diffusion-mutation model with variable motility [16].

3.1.4 Plan of the chapter

The results of Theorem 3.1.1 are a special case of Theorem 3.1.4, and, hence, we only prove Theorem 3.1.4. We do so in Section 3.2 by creating sub- and super-solutions using the heat equation and equation (3.1.9), with carefully chosen initial data for both. Then we prove Proposition 3.1.5 in Section 3.3 through an eigenfunction decomposition. This characterizes the location of the level sets in terms of the global in time problem equation (3.1.8). In Section 3.4, we prove Theorem 3.1.7, showing that at large times u locally looks like the global in time solution to equation (3.1.8). Finally, we show that equation (3.1.8) admits a unique global in time solution in Section 3.5.

Notation

In order to make the discussion clearer, we adopt the following notation. The constant, C, denotes an arbitrary constant independent of time which may change line-by-line. In addition, we occasionally use $o(1)$ to mean a constant that tends to zero along with some quantity which we will specify. We will make clear, in each usage, the dependencies of C and $o(1)$.

3.2 Spreading of the average-level sets

In this section, we prove Theorem 3.1.4 by creating sub- and super-solutions by decomposing u into a product of purely diffusive and purely reactive terms. We utilize the Feynman-Kac formula along with our condition on the oscillations, equation (3.1.3), in order to control the diffusive term.

In addition, we state a lemma which will allow us to translate the simple condition on the decay of u_0, equation (3.1.3), to a more useful condition on the oscillations of u_0 very far to the right.

Lemma 3.2.1. *Suppose that $u_0 \in C^1$ is strictly decreasing and satisfies*

$$\lim_{x\to\infty} \frac{(u_0)_x(x)}{u_0(x)/\log(u_0(x))} = 0.$$

Let λ_t be some quantity for which there exist constants $c_1, c_2 > 0$ such that $\lambda_t \le c_1 e^{-c_2 t}$. Then, for any choice of c_3, we have that

$$\lim_{t\to\infty} \frac{u_0\left(u_0^{-1}(\lambda_t) \pm c_3 t\right)}{\lambda_t} = 1.$$

The limit may be taken uniformly for any c_1 which is bounded above.

We will use this with the choice $\lambda_t = B(m,t)$. To this end, we need the following lemma that guarantees the exponential decay of $B(m,t)$ in time, which will be useful in its own right.

Lemma 3.2.2. *Let $B(m,T)$ be defined as in Theorem 3.1.4. Let f_0 be the largest eigenvalue of $\Delta + f_u(x,0)$. Then $f_0 > 0$ and there exists positive constants C, independent of m, and C_m, depending on m, such that*

$$\frac{m}{C} e^{-f_0 T} \le B(m,T) \le C_m e^{-f_0 T}. \tag{3.2.1}$$

Moreover, there exists $\delta_0 > 0$ such that if $m \le \delta_0$, then

$$B(m,T) \le C m e^{-f_0 T}. \tag{3.2.2}$$

We prove these lemmas in Section 3.2.3.

3.2.1 An upper bound

To begin, we define

$$x_m(T) = u_0^{-1}(B(m,T)), \tag{3.2.3}$$

and we define a super-solution as $\overline{v} = \varphi^{T,m}\overline{w}$ where $\varphi^{T,m}$ satisfies equation (3.1.9) and $\overline{w}$ satisfies

$$\begin{cases} \overline{w}_t = \overline{w}_{xx} + 2\left(\frac{\varphi_x^{T,m}}{\varphi^{T,m}}\right)\overline{w}_x, \\ \overline{w}(0,x) = \max\left\{\frac{u_0(x)}{B(m,T)}, 1\right\}. \end{cases}$$

For an illustration of this, see Figure 3.1. Indeed, we notice that $\overline{v}(0,x) \ge u_0(x)$ for every x and

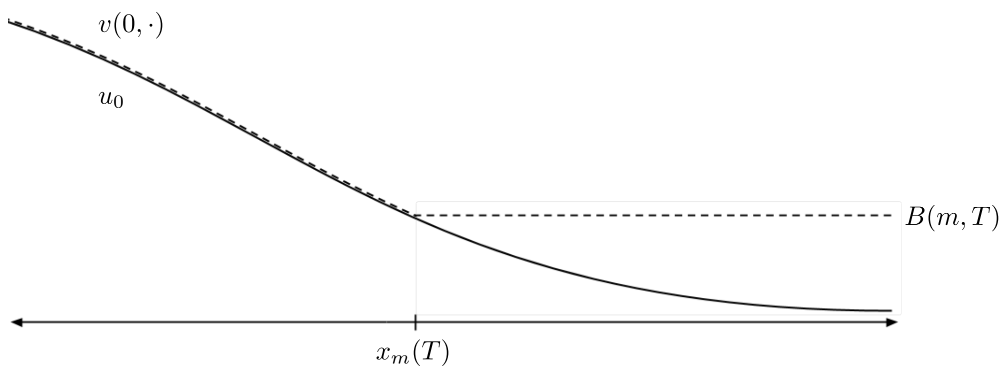


Figure 3.1: A cartoon of the initial data u_0 (the solid line) and $v(0,\cdot)$ (the dashed line).

that

$$\begin{aligned}
\overline{v}_t - \overline{v}_{xx} - f(x,\overline{v}) &= \varphi_t^{T,m}\overline{w} + \varphi^{T,m}\overline{w}_t - \varphi_{xx}^{T,m}\overline{w} - 2\varphi_x^{T,m}\overline{w}_x - \varphi^{T,m}\overline{w}_{xx} - f(x,\varphi^{T,m}\overline{w}) \\
&= \left[\varphi_{xx}^{T,m} + f(x,\varphi^{T,m})\right]\overline{w} + \varphi^{T,m}\left[\overline{w}_{xx} + 2\frac{\varphi_x^{T,m}}{\varphi^{T,m}}\overline{w}_x\right] - \varphi_{xx}^{T,m}\overline{w} - 2\varphi_x^{T,m}\overline{w}_x \\
&\quad - \varphi^{T,m}\overline{w}_{xx} - f(x,\varphi^{T,m}\overline{w}) \\
&= \overline{w}\varphi^{T,m}\left[\frac{f(x,\varphi^{T,m})}{\varphi^{T,m}} - \frac{f(x,\varphi^{T,m}\overline{w})}{\overline{w}\varphi^{T,m}}\right] \geq 0.
\end{aligned} \tag{3.2.4}$$

The inequality in the last line follows from the fact that $\overline{w} \geq 1$, by the maximum principle, and from the fact that $f(x,s)/s$ is decreasing in s by equation (3.1.1).

As a result of the above decomposition, we need only show that $\overline{w}$ does not deviate much from 1 near $x_m(T)$. To this end, we will use the Feynman-Kac representation of $\overline{w}$, see e.g. [5]. First, it follows from the standard parabolic regularity theory that

$$\|\varphi_x^{T,m}/\varphi^{T,m}\|_\infty \leq C_0, \tag{3.2.5}$$

where C_0 is a constant independent of t, T, and m, see e.g. [48,49]. We define a random process X_t as the solution to

$$\begin{cases} dX_t = 2\frac{\varphi_x^{T,m}(T-t,X_t)}{\varphi^{T,m}(T-t,X_t)}\, dt + \sqrt{2}\, dB_t, \\ X_0 = x, \end{cases} \tag{3.2.6}$$

where B_t is a standard Brownian motion. We may then represent w as

$$\overline{w}(t,x) = \mathbb{E}^x\left[\overline{w}(0,X_t)\right].$$

Fix $r_1 > 0$, to be determined later, and any r_2. For $x \geq x_m(T) - r_2 T$, we estimate this at time T as

$$\overline{w}(T,x) = \mathbb{E}^x\left[\overline{w}(0,X_T)\mathbb{1}_{\{X_T \leq x - r_1 T\}}\right] + \mathbb{E}^x\left[(\overline{w}(0,X_T)-1)\mathbb{1}_{\{X_t \geq x - r_1 T\}}\right] + \mathbb{E}^x\left[\mathbb{1}_{\{X_t \geq x - r_1 t\}}\right]. \tag{3.2.7}$$

First, we may use Lemma 3.2.2 to find a constant C with which we may bound $\|\overline{w}(0)\|_\infty \leq Ce^{f_0 T}$. Then, if $r > 2C_0 + 2\sqrt{f_0}$, with C_0 as in equation (3.2.5), the first term in equation (3.2.7) may be bounded as

$$\begin{aligned}\mathbb{E}^x\left[\overline{w}(0,X_T)\mathbb{1}_{\{X_T \leq x - rT\}}\right] &\leq Ce^{f_0 T}\mathbb{P}\left\{X_T \leq x - rT\right\} \leq Ce^{f_0 T}\mathbb{P}\left\{\sqrt{2}B_T - 2C_0 T \leq -rT\right\}\\ &\leq Ce^{f_0 T}\mathbb{P}\left\{\sqrt{2}B_T \leq -2\sqrt{f_0}T\right\} \leq \frac{C}{\sqrt{T}}.\end{aligned}$$

The last inequality follows from a standard estimate on the tail of a Gaussian. Thus, the first term tends to zero as T tends to infinity. The third term in equation (3.2.7) can be shown to converge to 1 in the exact same manner as above. Notice that these limits did not depend on m.

Hence, we need only show that the second term converges to zero. To this end we bound the second term as

$$\begin{aligned}\mathbb{E}^x\left[(\overline{w}(0,X_T)-1)\mathbb{1}_{\{X_t \geq x - r_1 T\}}\right] &\leq \mathbb{E}^x\left[\frac{u_0(x - r_1 T) - u_0(x_m(T))}{B(m,T)}\right]\\ &\leq \mathbb{E}^x\left[\frac{u_0(x_m(T) - (r_1 + r_2)T) - u_0(x_m(T))}{B(m,T)}\right].\end{aligned}$$

Using the definition of $x_m(T)$, equation (3.2.3), this tends to zero as a consequence of Lemma 3.2.1 and Lemma 3.2.2. We notice that this limit can be taken independent of m if m is bounded away from 1 as a consequence of the tracking the dependence on m through Lemma 3.2.1 and Lemma 3.2.2.

Hence we obtain that if $x \geq x_m(T) - r_2 T$, then $\overline{w}(T,x) \leq 1 + o(1)$, where $o(1)$ tends to zero as T tends to infinity. This yields

$$u(T,y) \leq \varphi^{T,m}(T,y)\overline{w}(T,y) \leq \varphi^{T,m}(T,y)[1 + o(1)]. \tag{3.2.8}$$

By construction, $⨍ \varphi^{T,m}(T,y)dy = m$. Hence, averaging over $[x, x+L]$ gives us the upper bound equation (3.1.10).

We note that, to take the limits above, we used only estimates on the Gaussian and Lemma 3.2.1 and Lemma 3.2.2. Hence, these limits can be taken independent of m as long as m is uniformly bounded away from 0 and 1.

3.2.2 A lower bound

The argument here is similar to the argument in Section 3.2.1, above. As before we fix $T > 0$ and $m \in (0,1)$, define $x_m(T)$ as in equation (3.2.3), and we $\underline{v}(t,x) = \varphi^{T,m}(t,x)\underline{w}(t,x)$ where $\varphi^{T,m}(t,x)$

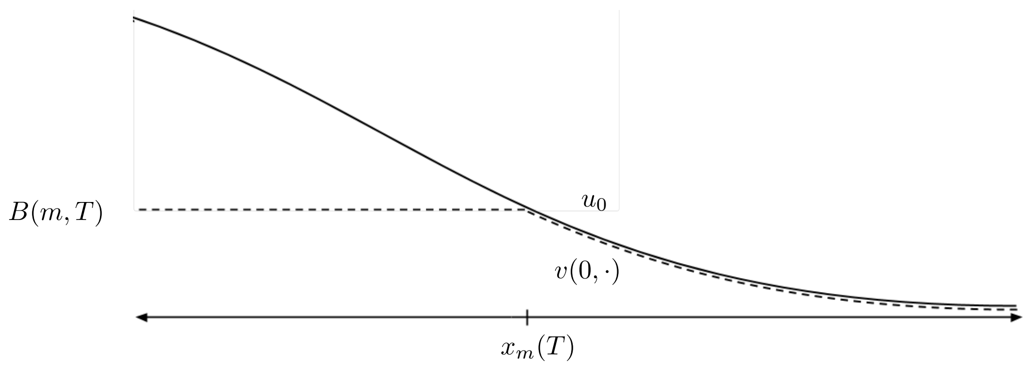


Figure 3.2: A cartoon of the initial data u_0 (the solid line) and $v(0,\cdot)$ (the dashed line).

is defined in equation (3.1.9). Here, we instead define $\underline{w}$ as the solution to

$$\begin{cases} \underline{w}_t = \underline{w}_{xx} + 2\frac{\varphi_x^{T,m}}{\varphi^{T,m}}\underline{w}_x, \\ \underline{w}(0,x) = \min\left\{\frac{u_0(x)}{B(m,T)}, 1\right\}. \end{cases}$$

For an illustration of this, see Figure 3.2. Notice that $\underline{v}(0,x) \leq u_0(x)$ for all x. The computation above, equation (3.2.4), shows that

$$\underline{v}_t - \underline{v}_{xx} - f(x,\underline{v}) = \underline{w}\varphi^{T,m}\left[\frac{f(x,\varphi^{T,m})}{\varphi^{T,m}} - \frac{f(x,\varphi^{T,m}\underline{w})}{\underline{w}\varphi^{T,m}}\right] \leq 0,$$

where the last inequality follows since $\underline{w} \leq 1$, as a consequence of the maximum principle, and since $f(x,s)/s$ is decreasing for $s \in [0,1]$.

Again, defining X_t by equation (3.2.6), we obtain that

$$\underline{w}(t,x) = \mathbb{E}^x\left[\underline{w}(0,X_t)\right].$$

Fix $r_1 > 0$, to be determined below, and any r_2. For any $x \leq x_m(T) + r_2T$, we decompose $w(T,x)$ as

$$\underline{w}(T,x) = \mathbb{E}^x\left[\underline{w}(0,X_T)\mathbb{1}_{\{X_T \geq x+r_1T\}}\right] + \mathbb{E}^x\left[(\underline{w}(0,X_T)-1)\mathbb{1}_{\{X_t \leq x+r_1T\}}\right] + \mathbb{E}^x\left[\mathbb{1}_{\{X_t \leq x+r_1T\}}\right].$$

For $x \leq x_m(T) + r_2T$, we may again choose r_1 large enough that the first term on the right hand side tends to zero as T tends to infinity and that the last term on the right tends to one as T tends to infinity. In addition, Lemma 3.2.1 and Lemma 3.2.2 imply that the second term on the right tends to zero as T tends to infinity. Hence, we obtain, for any $x \leq x_m(T) + r_2T$,

$$u(T,x) \geq \varphi^{T,m}(T,x)\underline{w}(T,x) \geq \varphi^{T,m}(T,x)(1-o(1)). \tag{3.2.9}$$

Using that, by construction, $\fint \varphi^{T,m}(T,y)dy = m$, integrating over $[x-L,x]$ yields the lower bound in equation (3.1.11) for any $x \le x_m(T) + r_2 T$. Again, we note that the limit in equation (3.2.9) may be taken uniformly for m bounded away from 1.

3.2.3 Controlling the oscillations

In this section we prove Lemma 3.2.1 and Lemma 3.2.2. We first prove Lemma 3.2.1, by using an ODE argument to compare u_0 to a sequence of equations which decay at the critical order but with increasingly slower rates.

Proof of Lemma 3.2.1. We will prove this lemma for $u_0^{-1}(\lambda_t) - c_3 t$, though the proof is exactly analogous in the case of $u_0^{-1}(\lambda_t) + c_3 t$. Fix $\epsilon > 0$, and define $h_\epsilon(x) = e^{-\sqrt{\epsilon x}}$. Notice that

$$h' = -\frac{\epsilon}{2}\frac{h}{\log(h)}. \tag{3.2.10}$$

Define

$$x_0 = u_0^{-1}(\lambda_t) \quad \text{and} \quad y_0 = h^{-1}(\lambda_t).$$

We may then shift h to obtain $g(x) = h(x - x_0 + y_0)$. Notice that $g(x_0) = u_0(x_0)$. The condition on the decay of λ_t and the fact that u_0 decays slower than any exponential implies that $u_0^{-1}(\lambda_t) - c_3 t$ tends to infinity. Hence we may choose t large enough, uniformly in c_1 if c_1 is bounded above, such that if $x \ge x_0 - c_3 t$, then

$$0 \ge u_0'(x) \ge -\frac{\epsilon}{2}\left|\frac{u_0(x)}{\log(u_0(x))}\right|. \tag{3.2.11}$$

We note that u_0 is a supersolution to equation (3.2.10) as a consequence of equation (3.2.11). Hence $u_0(x) \le g(x)$ for all $x_0 - c_3 t \le x \le x_0$. Using this, along with the fact that $u_0(x_0) = g(x_0)$, we obtain

$$0 \le u_0(x_0 - c_3 t) - u_0(x_0) \le u_0(x_0 - c_3 t) - g(x_0) \le g(x_0 - c_3 t) - g(x_0). \tag{3.2.12}$$

To conclude, we estimate the last term, $g(x_0 - c_3 t) - g(x_0)$. Returning to our function h and manipulating the equation, we obtain

$$\begin{aligned} g(x_0 - c_3 t) - g(x_0) &= h(y_0 - c_3 t) - h(y_0) \\ &= \exp\{-\sqrt{\epsilon(y_0 - c_3 t)}\} - \exp\{-\sqrt{\epsilon y_0}\} \\ &= \exp\{-\sqrt{\epsilon y_0}\}\left[\exp\left\{\sqrt{\epsilon y_0}\left(1 - \sqrt{1 - (c_3 t / y_0)}\right)\right\} - 1\right]. \end{aligned}$$

It is a simple consequence of Taylor's theorem that there is a constant C such that

$$1 - \sqrt{1 - (c_3 t / y_0)} \le C(c_3 t / y_0).$$

Applying this in the equation above, and using that, by construction, $e^{\sqrt{\epsilon y_0}} = \lambda_t$, we obtain

$$g(x_0 - c_1 t) - g(x_0) \leq \lambda_t \left[\exp\left\{ \frac{C\sqrt{\epsilon} c_3 t}{\sqrt{y_0}} \right\} - 1 \right].$$

Then using that $\lambda_t \leq c_1 e^{-c_2 t}$, we may solve for y_0 to obtain

$$y_0 \geq \frac{1}{\epsilon^2} \left(c_2 t - \log(c_1) \right)^2.$$

Returning to equation (3.2.12), and using that $u_0(x_0) = \lambda_t$, by construction, we obtain

$$0 \leq u_0(x_0 - c_3 t) - \lambda_t \leq g(x_0 - c_3 t) - g(x_0) \leq \lambda_t \left[\exp\left\{ \epsilon \frac{C c_3 t}{c_2 t - \log(c_1)} \right\} - 1 \right].$$

Adding λ_t to to all sides of each inequality and then dividing by λ_t yields

$$1 \leq \frac{u_0(x_0 - c_3 t)}{\lambda_t} \leq \exp\left\{ C' \epsilon \right\},$$

where C' is a universal constant which bounds $Cc_3 t/(c_2 t - \log(c_1))$. Again, C' can be chosen uniformly in c_1 as long as c_1 is bounded above. The result then follows by using the fact that we can choose ϵ as small as we would like by taking t larger. $\square$

We will now prove Lemma 3.2.2 in order to make Lemma 3.2.1 applicable. The idea of the proof is to use the linearized equation to control the growth of $B(m, T)$.

Proof of Lemma 3.2.2. First, we verify that f_0 is positive. To see this, we simply use the Rayleigh quotient characterization of f_0 as

$$f_0 = \max_{\psi \in H^1(\mathbb{T})} \frac{\int \left(f_u(x, 0)\psi(x)^2 - |\nabla \psi(x)|^2 \right) dx}{\int_{\mathbb{T}} \psi(x)^2 dx},$$

where $\mathbb{T} = [0, L]$ with the ends identified. One can verify the positivity of f_0 by testing the equation with the function $\psi \equiv 1$ and using the positivity of $f_u(x, 0)$.

Fix $T > 0$, and let ψ_0 be the unique eigenfunction associated with f_0 with $\|\psi_0\|_2 = 1$. It follows from the Krein-Rutman theorem that this function exists and is positive [27]. Notice that $\overline{\varphi}(t, x) = \alpha \psi_0(x) e^{f_0 t}$ satisfies

$$\overline{\varphi}_t = \Delta \overline{\varphi} + f_u(x, 0)\overline{\varphi} \geq \Delta \overline{\varphi} + f(x, \overline{\varphi}).$$

Hence, if $\alpha \psi_0(x) \geq B(m, T)$, we have that $\overline{\varphi}(t, x) \geq \varphi^{T,m}(t, x)$ for all x and all t, where $\varphi^{T,m}$ is defined by equation (3.1.9). Choosing

$$\alpha = \frac{B(m,T)}{\min_x \psi_0(x)},$$

this implies that

$$B(m,T)\frac{\|\psi_0\|_\infty}{\min_x \psi_0(x)}e^{f_0 T} = \alpha\|\psi_0\|_\infty e^{f_0 T} \geq \fint \overline{\varphi}(T,x)dx \geq \fint \varphi^{T,m}(T,x)dx = m.$$

Rearranging this inequality yields the lower bound on $B(m,T)$.

The upper bound is similar, but requires slightly more work. We first obtain a bound for small m by using a modulation technique found in [35,36]. Notice that our conditions on f guarantee the existence of M such that

$$f(x,s) \geq f_u(x,0)s - Ms^{1+\delta}, \tag{3.2.13}$$

holds for all $s \in [0,1]$. Define $\underline{\varphi}(t,x) = \alpha(t)\psi_0(x)e^{f_0 t}$. We find conditions on α to make $\underline{\varphi}$ a sub-solution of φ. To this end, we compute

$$\begin{aligned}
\underline{\varphi}_t - \underline{\varphi}_{xx} - f(x,\underline{\varphi}) &= \frac{\alpha'}{\alpha}\underline{\varphi} + f_0\underline{\varphi} - \alpha(\psi_0)_{xx}e^{f_0 t} - f(x,\underline{\varphi}) \\
&= \frac{\alpha'}{\alpha}\underline{\varphi} + f_0\underline{\varphi} - \alpha[f_0\psi_0 - f_u(x,0)\psi_0]e^{f_0 t} - f(x,\underline{\varphi}) \\
&= \frac{\alpha'}{\alpha}\underline{\varphi} + f_0\underline{\varphi} - f_0\underline{\varphi} + f_u(x,0)\underline{\varphi} - f(x,\underline{\varphi}) \\
&\leq \frac{\alpha'}{\alpha}\underline{\varphi} + M\underline{\varphi}^{1+\delta}
\end{aligned}$$

Hence $\underline{\varphi}$ is a sub-solution if we choose

$$\frac{\alpha'}{\alpha^{1+\delta}} = -Me^{f_0\delta t}\|\psi_0\|_\infty^\delta \quad \text{and} \quad \alpha(0) = \frac{B(m,T)}{\|\psi_0\|_\infty}.$$

In other words, we may let

$$\alpha(t) = \frac{B(m,T)}{\|\psi_0\|_\infty}\left[\frac{f_0\delta}{B(m,T)^\delta M[e^{f_0\delta t}-1] + f_0\delta}\right]^{1/\delta}$$

Hence at time $t = T$, we may average $\underline{\varphi}$ and use that $\underline{\varphi}$ sits below $\varphi^{T,m}$ to obtain

$$\frac{B(m,T)}{\|\psi_0\|_\infty}\left[\frac{f_0\delta}{B(m,T)^\delta M[e^{f_0\delta T}-1] + f_0\delta}\right]^{1/\delta} e^{f_0 T} = \fint \underline{\varphi}(T,x)dx \leq \fint \varphi^{T,m}(T,x)dx = m.$$

Re-arranging this we see that

$$f_0\delta B(m,T)^\delta e^{f_0\delta T} \leq m^\delta \|\psi_0\|_\infty^\delta \left[B(m,T)^\delta e^{f_0\delta T} M + f_0\delta\right].$$

If $m^\delta M\|\psi_0\|_\infty^\delta \leq f_0\delta/2$, then we may rearrange the above to obtain

$$B(m,T)^\delta \leq 2m^\delta \|\psi_0\|_\infty^\delta e^{-f_0\delta T}, \tag{3.2.14}$$

finishing the claim for small m.

The argument is somewhat more complicated for general m. We prove it in two steps. First, we show that there is R_1, which is uniformly bounded if m is uniformly bounded away from 0 and 1, such that $B(m/2, T-R_1) \geq B(m,T)$. To see this, we take $C_{f,m}$ to be a positive constant, arising from parabolic regularity and the smoothness and positivity of f, such that $f(x,\varphi^{T,m}(t,x)) \geq C_{f,m}$ if the average of φ is between $m/2$ and m. Then we choose $R_1 = m/(2C_{f,m})$ and we will show by contradiction that this is the correct choice of R_1. If not, and instead $B(m/2, T-R_1) < B(m,T)$ holds, then the maximum principle gives that

$$\frac{m}{2} = \fint \varphi^{T-R_1,m/2}(T-R_1,x)dx < \fint \varphi^{T,m}(T-R_1,x)dx. \tag{3.2.15}$$

Using our choice of $C_{f,m}$ above, we may integrate the equation for $\varphi^{T,m}$ in space for any $t \in [T-R_1, T]$ to obtain

$$\partial_t \fint \varphi^{T,m}(t,x)dx = \fint f(x,\varphi^{T,m})dx \geq C_{f,m}.$$

Integrating this in time from $T-R_1$ to T, using the bound in equation (3.2.15), and using our choice of R_1 yields

$$m = \fint \varphi^{T,m}(T,x)dx \geq C_{f,m}R_1 + \fint \varphi^{T,m}(T-R_1,x)dx > \frac{m}{2} + \frac{m}{2},$$

which is clearly a contradiction.

Having proven this fact, we now bootstrap the result for small m to obtain the statement for general m. For any m, we may find N such that $m/2^N$ is small enough to apply the work above and obtain equation (3.2.14). Indeed, we choose N large enough that

$$\left(\frac{m}{2^N}\right)^\delta M\|\psi_0\|_\infty^\delta < f_0\delta/2,$$

and as a result obtain that, for any $\tilde{T} > 0$,

$$B\left(\frac{m}{2^N}, \tilde{T}\right) \leq C\left(\frac{m}{2^N}\right) e^{-f_0\tilde{T}}. \tag{3.2.16}$$

In addition to this, we may iterate the procedure above to find $R_1, R_2, \ldots, R_N$ such that

$$B\left(\frac{m}{2^i}, T-(R_1+\cdots+R_i)\right) \leq B\left(\frac{m}{2^{i+1}}, T-(R_1+\cdots+R_{i+1})\right)$$

holds for any i, and hence,

$$B(m,T) \leq B\left(\frac{m}{2^N}, T-(R_1+\cdots R_N)\right). \tag{3.2.17}$$

Combining equations (3.2.16) and (3.2.17) and defining $C_m = Cm2^{-N}e^{-f_0(R_1+\cdots R_N)}$, we obtain

$$B(m,T) \leq C\frac{m}{2^N}e^{-f_0(T-R_1-R_2-\cdots-R_N)} = C_m e^{-f_0 T},$$

where we have defined $C_m = Cm2^{-N}e^{f_0(R_1+\cdots+R_N)}$. This concludes the proof.

□

3.3 Characterization of the speed

In this section, we will show how to characterize $B(m,T)$ in terms of the the global-in-time solution φ of equation (3.1.8). In the homogeneous case, there is nothing to prove since φ and $\varphi^{T,m}$ are equal up to a translation in time, by the uniqueness of solutions of ordinary differential equations. In the inhomogeneous setting, we need to deal with the fact that φ is not necessarily flat, while $\varphi^{T,m}(0,x) \equiv B(m,T)$. The idea of the proof is to run the system for time $T/2$ and use spectral estimates to show that the $\varphi(T^m - T/2 + t)$ and $\varphi^{T,m}(T/2+t)$ will be close. From then on, they must remain close.

Before we begin, we will need one fact about the global in time solution. We delay the proof of this lemma until Section 3.5 as it will be crucial in the proof of the well-posedness of equation (3.1.8).

Lemma 3.3.1. *Let φ be any solution of equation* (3.1.8)*. Then there exist positive constants α and ω such that*

$$\lim_{t\to-\infty}\frac{\varphi(t,x)}{\psi_0(x)e^{f_0 t}} = \alpha, \quad \text{and} \quad \lim_{t\to\infty}\frac{1-\varphi(t,x)}{\psi_1(x)e^{-f_1 t}} = \omega$$

where ψ_0 and f_0, defined in Proposition 3.1.5, are the normalized eigenfunction and principle eigenvalue of $\Delta + f_u(x,0)$, and where ψ_1 and f_1 are the normalized eigenfunction and principle eigenvalue of $\Delta - f_u(x,1)$.

Now, our first step is to prove the following lemma.

Lemma 3.3.2. *Suppose that γ_0 is a $C^\infty(\mathbb{T})$ function such that $\|\gamma_0\|_\infty \leq Ce^{-f_0 T}$ for some $C > 0$.*

$$\begin{cases} \gamma_t = \gamma_{xx} + f(x,\gamma), \\ \gamma(0,x) = \gamma_0(x). \end{cases}$$

Fix $\epsilon > 0$. Then there is $T_0 > 0$, which can be chosen uniformly if C is bounded above, such that if $T \geq T_0$, we have that

$$\left\| \gamma(T/2) - \psi_0 \left(\int \gamma_0 \psi_0 \right) e^{f_0 T/2} \right\|_2 \leq \epsilon e^{-f_0 T/2}.$$

Proof. Define

$$\Gamma(t,x) = \gamma(t,x) e^{f_0(T-t)},$$

and we can easily see that Γ satisfies

$$\Gamma_t - \Gamma_{xx} + f_0 \Gamma = f(x,\gamma) e^{f_0(T-t)} = \left(\frac{f(x,\gamma)}{\gamma} \right) \Gamma.$$

Letting $\Gamma_1(t) = \int \Gamma(t,x)\psi_0(x)dx$, we notice that

$$\dot{\Gamma}_1 = e^{f_0(T-t)} \int \left(f(x,\gamma) - f_u(x,0)\gamma \right) \psi_0(x) dx.$$

First, we notice that $\dot{\Gamma}_1 \leq 0$. In addition, since ψ_0 multiplied by a large constant is a super-solution of Γ, it follows that Γ is uniformly bounded. We utilize this, along with equation (3.2.13), to obtain

$$\begin{aligned} 0 \geq \dot{\Gamma}_1 &\geq -M e^{f_0(T-t)} \int \gamma(x)^{1+\delta} \psi_0(x) dx \geq -MC' e^{f_0(T-t)} e^{\delta f_0 (t-T)} \int \gamma(x)\psi_0(x)dx \\ &\geq -MC' e^{\delta f_0(t-T)} \Gamma_1. \end{aligned}$$

Dividing by Γ_1, integrating in time, and exponentiating yields

$$1 \geq \frac{\Gamma_1(T/2)}{\Gamma_1(0)} \geq \exp\left\{ -e^{-\frac{\delta f_0}{2}T} C'' \left[1 - e^{-\frac{\delta f_0}{2}T} \right] \right\}.$$

Defining $\gamma_1(t) = \int \gamma(t,x)\psi_0(x)dx$, we may rewrite this inequality as

$$|\gamma_1(T/2) - e^{f_0 T/2}\gamma_1(0)| \leq o(1) e^{-f_0 T/2}. \tag{3.3.1}$$

Now we let $\Gamma_2(t,x) = \Gamma(t,x) - \Gamma_1(t)\psi_0(x)$, and we bound Γ_2. Notice that Γ_2 satisfies the equation

$$\begin{aligned} (\Gamma_2)_t - &(\Gamma_2)_{xx} - (f_u(x,0) - f_0)\Gamma_2 \\ &= \left[f(x,\gamma)e^{f_0(T-t)} - f_u(x,0)\Gamma \right] - \psi_0(x) e^{f_0(T-t)} \int (f(y,\gamma) - f_u(y,0)\gamma(y))\psi_0(y)dy \end{aligned} \tag{3.3.2}$$

Because Γ_2 is orthogonal to the the eigenspace for eigenvalue f_0 of $\Delta + f_u(x,0)$, then there is α such that

$$\int \left[(\Gamma_2)_{xx} + (f_u(x,0) - f_0)\Gamma_2 \right] \Gamma_2 dx \leq -\alpha \|\Gamma_2\|_2^2.$$

Hence, multiplying equation (3.3.2) by Γ_2 and, again, using that Γ_2 and ψ_0 are perpendicular yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Gamma_2\|_2^2 + \alpha\|\Gamma_2\|_2^2 &\leq e^{f_0(T-t)}\int\left[f(x,\gamma) - f_u(x,0)\gamma\right]\Gamma_2 dx \\ &\leq Me^{f_0(T-t)}\int \gamma^{1+\delta}|\Gamma_2|dx = Me^{\delta f_0(t-T)}\int \Gamma^{1+\delta}\Gamma_2 dx\end{aligned}$$

As we noted above, $\|\Gamma\|_\infty$ is uniformly bounded in time. Hence, we may write this as

$$\frac{d}{dt}\|\Gamma_2\|_2^2 + 2\alpha\|\Gamma_2\|_2^2 \leq Ce^{\delta f_0(t-T)}\|\Gamma_2\|_2,$$

where C is a new constant independent of t and T. Solving this differential inequality and using that $\|\Gamma_2(0)\| \leq C$ for some constant C, yields constants $C > 0$ and $\theta > 0$ such that $\|\Gamma_2(T/2)\|_2 \leq Ce^{-\theta T}$.

The combination of this inequality with equation (3.3.1) yields that

$$\|\gamma(T/2) - e^{f_0T/2}\gamma_1(0)\psi_0\|_2 \leq C|\gamma_1(T/2) - e^{f_0T/2}\gamma_1(0)| + e^{-f_0T/2}\|\Gamma_2\|_2 \leq o(1)e^{-f_0T/2},$$

finishing the proof. □

We now show how to use this lemma to conclude Proposition 3.1.5. Fix $m \in (0,1)$ and define the starting time $S(m,T)$ as the time such that

$$\int \varphi(S(m,T),x)\psi_0(x)dx = B(m,T)\int\psi_0. \tag{3.3.3}$$

We first show that $⨍\varphi(S(m,T)+T,x)dx \to m$ as T tends to infinity. Then, defining T^m as the unique time such that

$$⨍\varphi(T^m,x)dx = m,$$

we will leverage this to show that

$$\lim_{T\to\infty}\frac{\int\varphi(T^m - T,x)\psi_0(x)dx}{B(m,T)\int\psi_0} = 1, \tag{3.3.4}$$

finishing the proof.

By parabolic regularity, Lemma 3.3.2, and Lemma 3.2.2, we have that

$$\|\varphi(S(m,T)+T/2) - \varphi^{T,m}(T/2)\|_2 \leq o(1)e^{-f_0T/2},$$

where $o(1)$ tends to zero as T tends to infinity (uniformly for m bounded away from 1). By parabolic regularity this may be strengthened to

$$\|\varphi(S(m,T)+T/2) - \varphi^{T,m}(T/2)\|_\infty \leq o(1)e^{-f_0T/2}.$$

Define $\eta(t,x) = \varphi(S(m,T) + T/2 + t, x) - \varphi^{T,m}(T/2 + t, x)$ and notice that η solves

$$\eta_t - \eta_{xx} = c(t,x)\eta,$$

where $c(t,x) = (f(x,\varphi^1) - f(x,\varphi^2))/(\varphi^1 - \varphi^2) \leq f_u(x,0)$. Hence, $o(1)\psi_0 e^{f_0(t-T/2)}$ is a super-solution to the equation above with initial data larger than $\eta(0,x)$. This implies that

$$|\eta(T/2,x)| \leq o(1)\psi(x) \leq o(1),$$

holds for every x. Using the definition of η, this implies that

$$\|\varphi(S(m,T) + T, \cdot) - \varphi^{T,m}(T,\cdot)\|_\infty \leq o(1).$$

By our definition of $\varphi^{T,m}$, this implies that

$$\lim_{T\to\infty} \fint \varphi(S(m,T) + T, x)dx = m,$$

holds, where the limit may be taken uniformly in m if m is bounded away from 1.

Now we will show equation (3.3.4). Suppose this does not hold. Let $\theta > 1$, without loss of generality, be such that there is a sequence $T_1, T_2, T_3, \ldots$ such that

$$\lim_{n\to\infty} \frac{\int \varphi(T^m - T_n, x)\psi_0(x)dx}{B(m,T_n)\int \psi_0} > \theta.$$

We note that the proof is similar for $\theta < 1$ and the limit above is less than θ. Using Lemma 3.3.1 and the definition of $S(m,T)$, we can choose N_0 be such that if $n \geq N_0$ then

$$\frac{e^{T^m - T_n}}{e^{S(m,T_n)}} > \theta,$$

and hence,

$$T^m - T_n - S(m,T_n) > \log(\theta) > 0.$$

By parabolic regularity, along with the smoothness and positivity of f, we may find ϵ depending only on m and θ such that $f(x,\varphi(t,x)) > \epsilon$ for all $t \in [T_n + S(m,T_n), T^m]$. This holds because φ is bounded away from 0 and 1 on this interval. Integrating the equation for φ on this time interval yields

$$\varphi(T^m) - \varphi(T_n + S(m,T_n)) = \int_{T_n+S(m,T_n)}^{T^m} f(x,\varphi)dx \geq \epsilon \log(\theta).$$

The average of the first term is m, and the average of the second term converges to m for large n. This implies that the left hand side tends to zero, which is a contradiction. This finishes the proof.

3.4 Convergence to the global in time problem

In this section, we prove Theorem 3.1.7. We will do this in two steps. First we will prove the closeness of u and $\varphi^{T,m}$. Then we will show that $\varphi^{T,m}$ and φ are close when suitably translated in time. To this end, we state two lemmas.

Lemma 3.4.1. *Define* $\varphi^{T,m}$ *as in equation* (3.1.9)*, and define* $I_{m,T}$ *to be the interval* $[nL, nL+L)$ *containing* $u_0^{-1}(B(m,T))$*, we have that*

$$\lim_{T\to\infty} \max_{m\in(0,1)} \left\| u(T,\cdot) - \varphi^{T,m}(T,\cdot) \right\|_{L^\infty(I_{m,T})} = 0. \tag{3.4.1}$$

Lemma 3.4.2. *Define* $S(m,T)$ *as in equation* (3.3.3)*. Then,*

$$\lim_{T\to\infty} \max_{t\in[T/2,\infty]} \|\varphi(S(m,T)+t,\cdot) - \varphi^{T,m}(t,\cdot)\|_\infty = 0.$$

This limit can be taken uniformly for m *which is bounded away from* 1*.*

We will prove equation (3.4.1) at the end of this section. The proof of Lemma 3.4.2 follows from our choice of $S(m,T)$ and from work in Lemma 3.3.2. Hence, we omit it. We now show how to conclude Theorem 3.1.7 using these two lemmas.

Proof of Theorem 3.1.7. First, denote $I_{m,T}$ as $[nL, nL+L]$. Define T^{S_n} as the unique time such that

$$⨍ \varphi(T^{S_n}, x)dx = S_n,$$

where S_n is as in equation (3.1.12). We show that if T (or equivalently n) is large enough, then T^{S_n} and $S(m,T)$ are close, and that we can obtain this in a uniform way for m bounded away from 1. To this end, notice that, by the definition of $I_{m,T}$,

$$u_0(nL) \geq B(m,T) \geq u_0(nL+L).$$

In addition, by Lemma 3.2.1 and Lemma 3.2.2, we have that

$$\frac{u_0(nL)}{B(m,T)} = 1 + o(1), \tag{3.4.2}$$

where $o(1)$ tends to zero as T tends to infinity in a uniform way if m is bounded away from 1. Using

Lemma 3.3.1, we then compute that

$$1+o(1)=\frac{u_0(nL)}{B(m,T)}=\left(\frac{\fint \varphi(T^{S_n},x)dx}{\left(\int \psi_0(x)dx\right)^2}\right)\left(\frac{\int \psi_0 dx}{\int \varphi(S(m,T),x)\psi_0(x)dx}\right)$$
$$=\left(\frac{\fint \alpha(\psi_0(x)+o(1))e^{f_0 T^{S_n}}dx}{\left(\int \psi_0(x)dx\right)^2}\right)\left(\frac{\int \psi_0 dx}{\int \alpha(\psi_0(x)+o(1))e^{f_0 S(m,T)}\psi_0(x)dx}\right)=\frac{e^{f_0 T^{S_n}}}{e^{f_0 S(m,T)}}(1+o(1)).$$

Again, the constant $o(1)$ tends to zero uniformly as long as m is bounded away from 1. This yields the claim that $T^{S_n}-S(m,T)$ tends to zero as T tends to infinity.

Fix $\epsilon>0$. Using Lemma 3.4.1, choose T_1 large enough that if $T\geq T_1$, then

$$\max_{m\in(0,1)}\|u(T,\cdot)-\varphi^{T,m}(T,\cdot)\|_{L^\infty(I_{m,T})}\leq \epsilon/2. \tag{3.4.3}$$

In addition, choose $m_1<1$, depending only on f and ϵ, such that if $m\geq m_1$, then

$$\varphi^{T,m}(T,x)\geq 1-\epsilon/2,$$

for all $x\in\mathbb{T}$ and any T.

We will consider three cases. First, if n is small enough that $\varphi(T^{S_n},x)\geq B(m_1,T)$, the maximum principle implies that

$$\varphi(T^{S_n}+T,x)\geq \varphi^{T,m_1}(T,x)\geq 1-\epsilon/2.$$

This, combined with equation (3.4.3), implies that

$$\|\varphi(T^{S_n}+T,x)-u(T,\cdot)\|_{L^\infty([nL,nL+L])}\leq\epsilon,$$

for $T\geq T_1$.

The second case is when n is such that $\fint\varphi(T^{S_n},x)dx\leq B(m_1,T)\left(\int\psi_0\right)^2$. This implies that $m\leq m_1$. Indeed, using the definition of S_n, we obtain that

$$B(m_1,T)\left(\int\psi_0\right)^2\geq\fint\varphi(T^{S_n},x)dx=S_n=u_0(nL)\left(\int\psi_0\right)^2\geq B(m,T)\left(\int\psi_0\right)^2.$$

Thus, $B(m,T)\leq B(m_1,T)$, implying that $m\leq m_1$. Hence we may find T_2, depending only on m_1, f, and ϵ, such that if $T\geq T_2$ then T^{S_n} is close enough to $S(m,T)$ that

$$\|\varphi(T^{S_n}+T,\cdot)-\varphi(S(m,T)+T,\cdot)\|_{L^\infty}\leq\epsilon/8.$$

In addition, increasing T_2 if necessary, we may apply Lemma 3.4.2 to get that if $T\geq T_2$ then

$$\|\varphi(S(m,T)+T,\cdot)-\varphi^{T,m}(T,\cdot)\|_\infty\leq\epsilon/16.$$

The combination of these two inequalities with equation (3.4.3) implies that if $T \geq T_1 + T_2$, then

$$\|\varphi(T^{S_n} + T, \cdot) - u(T, \cdot)\|_{L^\infty([nL, nL+L])} \leq \epsilon.$$

Finally, we handle the case when neither of these is true. That both of these conditions do not hold, along with parabolic regularity, implies that we may find a positive constant $\theta < 1$, which depends only on m_1 and f, such that

$$\theta \fint \varphi(T^{S_n}, x)dx \leq B(m_1, T) \left(\int \psi_0 \right)^2.$$

Using Lemma 3.2.2 we may find $t_0 > 0$, depending only on f and m_1, such that

$$\fint \varphi(T^{S_n}, x)dx \leq B(m_1, T - t_0) \left(\int \psi_0 \right)^2.$$

Following the work above, we then have that $B(m, T) \leq B(m_1, T - t_0)$. Hence we obtain

$$(1 - m) = \fint (1 - \varphi^{T,m}(T, x))dx \geq \fint (1 - \varphi^{T-t_0, m_1}(T, x))dx \geq Ce^{f_1 t_0}.$$

The constant C depends only on m_1 and f, and the last inequality comes from considering a constant multiple of $\psi_1 e^{f_1 t}$ as a sub-solution to $1 - \varphi^{T-t_0}$ on the interval $[T - t_0, T]$. This shows that m is bounded away from 1 independent of T. Hence we may argue exactly as in the last paragraph, finishing the proof. □

We will now show how to conclude Lemma 3.4.1. The main idea is largely the same: when m is large enough, both u and $\varphi^{T,m}$ are near 1, and when m is bounded from 1, we may take the limit in T uniformly in m.

Proof of Lemma 3.4.1. Fix $\epsilon > 0$, and we will choose T, independent of m, such that

$$\|u(T) - \varphi^{T,m}(T)\|_{L^\infty(I_{m,T})} \leq \epsilon.$$

To this end, we utilize the work from Section 3.2.

First, we may use parabolic regularity to find $m_1 < 1$ depending only on f and ϵ such that $1 - \varphi^{T,m_1}(T, x) \leq \epsilon/2$. We may now consider, separately, the cases when $m \leq m_1$ and when $m \geq m_1$.

If $m \geq m_1$, using a sub-solution, $v = \varphi^{T,m_1}\underline{w}$, as in Section 3.2.2, we may choose T, depending only on ϵ and f, such that

$$\underline{w}(T, x) \geq (1 - \epsilon)/(1 - \epsilon/2),$$

for any $x \leq \max I_{m_1,T}$. Hence, we have that

$$1 \geq u(T,x) \geq v(T,x) \geq \varphi^{T,m_1}(T,x)w_\ell \geq (1-\epsilon/2)\frac{1-\epsilon}{1-\epsilon/2} = 1-\epsilon.$$

for any $x \leq \max I_{m_1,T}$. By the maximum principle, we can see that $\varphi^{T,m} \geq \varphi^{T,m_1}$ if $m \geq m_1$. Hence we have that, if $m \geq m_1$

$$\|u(T) - \varphi^{T,m}\|_{L^\infty(I_{m,T})} \leq \epsilon.$$

The case when $m \leq m_1$ follows from our work above. Indeed, since m is bounded away from 1, we may choose T such that $\overline{w}$ and $\underline{w}$ from Section 3.2.1 and Section 3.2.2, respectively, satisfy

$$1-\epsilon/2 \leq \underline{w}(T,x), \quad \text{and} \quad \overline{w}(T,x) \leq 1+\epsilon/2,$$

for any $x \in I_{m,T}$. Hence we have that, for any $x \in I_{m,T}$,

$$\left(1-\frac{\epsilon}{2}\right)\varphi^{T,m}(T,x) \leq \underline{w}(T,x)\varphi^{T,m}(T,x) \leq u(T,x) \leq \overline{w}(T,x)\varphi^{T,m}(T,x) \leq \left(1+\frac{\epsilon}{2}\right)\varphi^{T,m}(T,x),$$

which finishes the proof. □

3.5 Well-posedness of the global in time solution

In this section, we establish the existence and uniqueness (up to translation) of the solution to equation (3.1.8). In other words, we prove the following proposition.

Proposition 3.5.1. *There exists a unique global in time solution of equation* (3.1.8).

We will prove this in a series of three lemmas. First, we will exhibit the existence of a solution. Afterwards, we show that any solution to equation (3.1.8) must look grow and decay exponentially as t tends to $-\infty$ and ∞, respectively. Finally, we will use this qualitative property and the sliding method to establish uniqueness.

Lemma 3.5.2. *There exists a solution of equation* (3.1.8).

Proof. We establish existence by considering the limit of family of initial value problems. Define φ^n to be the solution to the parabolic equation on $[T_n, \infty] \times \mathbb{T}$

$$\begin{cases} \varphi^n_t = \varphi^n_{xx} + f(x,\varphi^n), \\ \fint \varphi^n(0,x)dx = 1/2, \\ \varphi^n(T_n,x) \equiv 1/n, \end{cases}$$

where $T_n \leq 0$ is an unknown. It is easy to check that $(n \min \psi_0)^{-1}\psi_0(x)e^{f_0(t-T_n)}$ is a super-solution, where ψ_0 and f_0 are the principle eigenfunction and eigenvalue of $\Delta + f_u(x,0)$. Thus, we obtain

$$\frac{1}{2} = \fint \varphi^n(0,x)dx \leq \frac{\|\psi_0\|_\infty}{n \min \psi_0} e^{-f_0 T_n}.$$

This implies that $T_n \leq C\log(n)$, for some constant C. Hence we may use parabolic regularity, along with the compactness of Hölder spaces to obtain convergence along a subsequence to a global in time function φ which satisfies

$$\begin{cases} \varphi_t = \varphi_{xx} + f(x,\varphi), \\ \fint \varphi(0,x)dx = 1/2, \end{cases}$$

finishing the proof. □

Now we establish that the asymptotics of any solution φ to equation (3.1.8) look like the solutions to the linearized problems around 0 and 1. This amounts to proving Lemma 3.3.1, as stated in Section 3.3.

Proof of Lemma 3.3.1. We will prove the claim for t tending to $-\infty$. The result for t tending to infinity follows by looking at $\tilde{\varphi}(t,x) = 1 - \varphi(-t,x)$ and arguing similarly. Our strategy is to first show that φ is bounded above and below by a multiple of $e^{f_0 t}$. Then we will appeal to parabolic regularity to obtain the result.

To this end, we first assume, by contradiction, that $e^{-f_0 t}\varphi(t,x)$ tends to zero along some sequence t_n, tending to $-\infty$, for some choice of x_n. By the parabolic Harnack inequality, it must converge to zero uniformly in space. Hence, we may assume that, for any ϵ, there exists N such that if $n \geq N$ then $\|e^{-f_0 t_n}\varphi(t_n)\|_\infty \leq \epsilon$. In this case, we may apply Lemma 3.2.2 to find t_n large enough that $\varphi^{-t_n,1/4}(t - t_n, x)$ is a super-solution to φ on $[-t_n, 0] \times \mathbb{T}$. Using this inequality at time $t = 0$, we have that

$$\frac{1}{4} = \fint \varphi^{-t_n,1/4}(-t_n,x)dx \geq \fint \varphi(0,x)dx = \frac{1}{2}.$$

This is clearly a contradiction, finishing the lower bound.

Now we will show that there is some C such that $e^{-f_0 t}\varphi(t,x) \leq C$ for $t \leq 0$. To this end, suppose that there is some sequence t_n tending to $-\infty$ such that $e^{-f_0 t_n}\varphi(t_n, x_n) \to \infty$ for some sequence x_n. Again, by the parabolic Harnack inequality, this implies that $\min e^{-f_0 t_n}\varphi(t_n) \to \infty$. Hence, we may apply Lemma 3.2.2 and take n large enough that $\varphi^{-t_n,3/4}(t - t_n, x)$ is a sub-solution of φ on $[-t_n, 0] \times \mathbb{T}$. Using this inequality at time $t = 0$ yields

$$\frac{3}{4} = \fint \varphi^{-t_n,3/4}(-t_n,x)dx \leq \fint \varphi(0,x)dx = \frac{1}{2}.$$

This is a contradiction, giving us the desired upper bound.

Now, letting $\varphi(t,x) = e^{-f_0 t}\varphi(t,x)$, we will show that φ converges to $\alpha\psi_0$. To this end, we may write $\Phi_1 = \int \Phi(t,x)\psi_0(x)dx$ and $\Phi_2 = \Phi - \psi_0\Phi_1$. Then as in our work in Lemma 3.3.2, we may see that there are positive constants C and θ, independent of time such that

$$\begin{aligned}
&0 \geq \dot{\Phi}_1 \geq -Ce^{\delta f_0 t}\Phi_1,\\
&\frac{d}{dt}\|\Phi_2\|_2^2 + 2\theta\|\Phi_2\|_2^2 \leq Ce^{\delta f_0 t}\|\Phi_2\|_2.
\end{aligned}$$

Solving these differential inequalities and using that Φ is bounded for all time gives us that, as t tends to $-\infty$, $\Phi_1(t) \to \alpha$, for some α, and that $\Phi_2(t) \to 0$. Since Φ is bounded away from zero, we get that $\alpha > 0$. This gives us the desired result. ☐

Finally, we use this qualitative information, along with the sliding method, to prove uniqueness of any solution to equation (3.1.8).

Lemma 3.5.3. *Any solution to equation* (3.1.8) *is unique.*

Proof. Suppose that φ^1 and φ^2 are both solutions to equation (3.1.8) with $⨍ \varphi^i(0,x)dx = 1/2$. Using Lemma 3.3.1, we may choose h large enough such that

$$\varphi^1(t,x) \leq \varphi^2(t+h,x),$$

holds for all t and all x. Then we define

$$h_0 = \inf\{h : \varphi^1(t,x) < \varphi^2(t+h,x)\}. \tag{3.5.1}$$

Since φ^1 and φ^2 have the same spatial average at $t = 0$, then $h_0 \geq 0$. We wish to show that $h_0 = 0$. To this end, let us assume that $h_0 > 0$. There are two cases. Using Lemma 3.2.2, we can find α_1 and α_2 such that $\varphi^1(t,x) \to \alpha_1\psi_0$ as t tends to $-\infty$ that $\varphi^2(t,x) \to \alpha_2\psi_0$ as t tends to $-\infty$. The first case is that $\alpha_1 = e^{f_0 h_0}\alpha_2$. In other words, φ^1 and φ^2 touch at time $t = -\infty$. Define $\eta(t,x) = \varphi^2(t+h_0,x) - \varphi^1(t,x)$ and η satisfies

$$\eta_t = \eta_{xx} + \frac{f(x,\varphi^2(t+h_0,x)) - f(x,\varphi^1(t,x))}{\varphi^2(t+h_0,x) - \varphi^1(t+h,x)}\eta.$$

Fix $T < 0$ to be determined later. Since $\varphi^i \sim \alpha_i e^{f_0 T}$ and since $\alpha_1 = e^{f_0 h_0}\alpha_2$, we notice that $\eta(T,x) \leq o(1)e^{f_0 T}$ where $o(1)$ tends to zero as T tends to $-\infty$. From this it is easy to see that $o(1)\psi_0 e^{f_0 t}$ is a super-solution to equation for η starting from any time T. This implies that $\eta(0,x) \leq o(1)$. Since this holds independent of our choice of T, we get that that $\eta(0,x) = 0$.

On the other hand, φ^2 is bounded away from 0 and 1 on the time interval $[0,h_0]$. Hence, there exists $\theta > 0$ such that $f(x,\varphi^2) \geq \theta$ on the time interval $[0,h_0]$. Integrating the equation for φ^2 in time and space, along with this fact and the fact that $\eta(0,x) = 0$, yields

$$\begin{aligned} 0 &= \fint \eta(0,x)dx = \fint \varphi^2(h_0,x)dx - \fint \varphi^1(0,x)dx = \fint \varphi^2(h_0,x)dx - m \\ &= \fint \varphi^2(h_0,x)dx - \fint \varphi^2(0,x)dx = \int_0^{h_0} \fint f(x,\varphi^2(s,x))dxds \geq \int_0^{h_0} \theta ds = h_0\theta > 0. \end{aligned}$$

The second equality on the second line comes from integrating the equation for φ^2 in time and space. The above inequality is clearly a contradiction.

Hence, it must be that, instead, $\alpha_1 < e^{f_0 h_0}\alpha_2$. Choose $\epsilon > 0$ small enough that $\alpha_1 e^\epsilon < e^{h_0 - 2\epsilon}\alpha_2$. Take t_0 to be a very large negative number such that if $t \leq t_0$ then

$$\varphi^1(t,x) \leq \alpha_1 e^\epsilon \psi_0(x) e^{f_0 t}$$

and

$$\varphi^2(t + h_0) \geq \alpha_2 e^{f_0(t+h_0-\epsilon)}\psi_0(x).$$

Notice then that $\varphi^2(t + h_0 - \epsilon, x) > \varphi^1(t,x)$ for all $t < t_0$. In addition, the comparison principle assures us that $\varphi^2(t + h_0 - \epsilon, x) > \varphi^1(t,x)$ for all $t \geq t_0$ as well. This contradicts our choice of h_0 in equation (3.5.1).

Hence we may conclude that h_0 must be zero. This implies that $\varphi^1(t,x) \leq \varphi^2(t,x)$ for all t and all x. The proof may be applied just as easily to show that $\varphi^2 \leq \varphi^1$ as well. Hence we conclude that $\varphi^1 \equiv \varphi^2$, concluding the proof. $\square$

Chapter 4

Pulsating fronts in a 2D reactive-Boussinesq system

4.1 Introduction

In this chapter, we establish the existence of pulsating traveling front solutions to the reactive Boussinesq system with the no stress boundary conditions. The reactant, or temperature, T, and the fluid velocity, u, satisfy a system composed of coupling the reaction-advection-diffusion equation for T and the linearized Navier-Stokes equation for u via the Boussinesq approximation as below

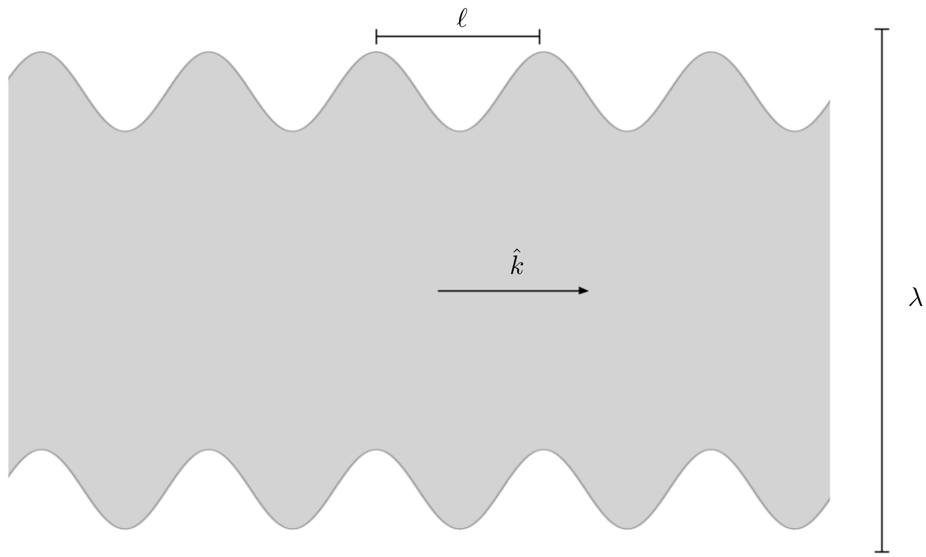
$$\begin{aligned} T_t + u \cdot \nabla T &= \Delta T + f(x,z,T) \\ u_t - \Delta u + \nabla p &= T\hat{z} \\ \nabla \cdot u &= 0. \end{aligned} \tag{4.1.1}$$

Here, the reaction term, f, is smooth and ignition type, and $\hat{z}$ is the vertical unit vector in $\mathbb{R}^2$. That is, there exists a positive number $\theta_0 \in (0,1)$ such that

$$\begin{aligned} f(x,z,r) &= 0 && \text{for all } r \leq \theta_0 \\ f(x,z,r) &> 0 && \text{for all } r \in (\theta_0, 1) \\ f(x,z,r) &\leq 0 && \text{for all } r \geq 1. \end{aligned} \tag{4.1.2}$$

We consider equation (4.1.1) in a smooth, periodic cylinder $\Omega \subset \mathbb{R}^2$. Specifically, there is a unit vector $\hat{k}$ and positive real numbers ℓ and λ such that

$$\Omega + \ell\hat{k} = \Omega \quad \text{and} \quad \Omega \subset \{(x,z) \in \mathbb{R}^2 : |(x,z) \cdot \hat{k}^{\perp}| \leq \lambda\}.$$



An example of a domain Ω

For an example of such a domain, see the figure below. In addition, we require that f is ℓ-periodic in the direction of $\hat{k}$. In addition, we assume that there exists a positive constant C and two constants $r_1, r_2 \in (\theta_0, 1)$ such that $f(x, z, r) \geq C$ for all $(x, z, r) \in \Omega \times (r_1, r_2)$. For ease of notation, when no confusion will arise, we will refer to $f(x, z, T)$ simply as $f(T)$. The temperature is normalized so that $0 \leq T \leq 1$.

In order to simplify the notation, it is convenient to rotate this domain to make it horizontal. In other words, we change variables so that $\Omega \subset \mathbb{R} \times [-\lambda, \lambda]$ and so that Ω is ℓ-periodic in x. Then equation (4.1.1) becomes

$$\begin{aligned} T_t + u \cdot \nabla T &= \Delta T + f(x, z, T) \\ u_t - \Delta u + \nabla p &= T\hat{e} \\ \nabla \cdot u &= 0, \end{aligned} \tag{4.1.3}$$

where u is the fluid velocity measured relative to the new coordinate system. Here $\hat{e}$ is simply $\hat{z}$ rotated by the angle between $\hat{k}$ and $\hat{x}$. Notice that f is now ℓ-periodic in x. We will consider this problem with the Neumann boundary conditions for T and the no stress boundary conditions for u.

Namely,

$$\begin{aligned} \frac{\partial T}{\partial \eta} &= 0 \text{ on } \partial\Omega \\ u \cdot \eta &= 0 \text{ on } \partial\Omega \\ \omega &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.1.4}$$

where $\omega = \partial_z u_1 - \partial_x u_2$, $u = (u_1, u_2)$, and η is the outward pointing unit normal vector of Ω.

The study of front propagation in reaction-diffusion equations dates back to the pioneering works of Kolmogorov, Petrovskii and Piskunov [46] and Fisher [31]. More recently, a lot of studies considered reaction-advection-diffusion equations in a prescribed flow. For an overview of many of these results, we point the reader to [72] and references found within. These results were obtained under the assumption that the fluid flow, u, was prescribed, and hence, is unaffected by the change in temperature. On the other hand, the Boussinesq approximation as in equation (4.1.1) accounts for the density difference by a buoyancy force in the equation for the fluid flow. This is the term in the right hand side of the second line of equation (4.1.1) and equation (4.1.3). In [55], Malham and Xin studied the regularity problem for this system, with the full Navier-Stokes equation governing the advection. Over the next decade, traveling waves were shown to exist in systems similar to that in equation (4.1.1) when Ω is a flat, non-vertical cylinder. First, Berestycki, Constantin, and Ryzhik showed existence of traveling waves in two dimensions when the fluid is governed by the full Navier-Stokes equation with no stress boundary conditions [7]. Later, this was extended to include the no slip boundary conditions in two and three dimensional slanted cylinders in [25, 52]. Finally, traveling waves were also shown to exist in n-dimensional cylinders with no slip boundary conditions with Stokes' equation governing the fluid flow in [51]. To our knowledge, the case of more general periodic domains, unbounded in one direction, has not been studied.

In the case of flat cylinders, one has to worry that the traveling waves are not those constructed in the previous theory. Namely, if the traveling waves are planar and thus depend only on x, one can show that $u \equiv 0$ and thus, that the solutions are the same as those developed in the reaction-diffusion equation long ago. It turns out that the existence of planar fronts depends on the alignment of gravity and the domain. To be more specific, if $\hat{e}$ in equation (4.1.3) is not horizontal, then non-planar fronts exist. On the other hand, if $\hat{e}$ is horizontal, then the existence of non-planar fronts depends on the Rayleigh number [6]. In this chapter, non-planar fronts will not be an issue. When the boundary is not flat, as in this chapter, non-trivial planar fronts will not satisfy the boundary conditions for T.

Pulsating fronts

When the setting of the problem involves either an inhomogeneous medium or a non-flat infinite cylinder, traveling waves can not exist. However, a generalization of the traveling wave solution, the pulsating front, was defined first in [67] based on observations the authors made in numerical simulations. A few years later, J. Xin independently discovered this structure in [70, 71] and gave the first rigorous proof of its existence, though in $\mathbb{R}^n$ and not infinite cylinders. For further results on the existence of pulsating fronts in various settings see [9]. In that paper, the authors prove the existence of pulsating fronts in many settings under the condition that the advection is periodic and prescribed. In this thesis, we define a pulsating front to be a solution to equation (4.1.3) such that there is some positive constant $c > 0$, called the front speed, such that

$$\begin{aligned} T(t + \frac{\ell}{c}, x, z) &= T(t, x - \ell, z) \\ u(t + \frac{\ell}{c}, x, z) &= u(t, x - \ell, z), \end{aligned} \tag{4.1.5}$$

and there is some constant $\theta_- \in (0, 1]$ such that

$$\begin{aligned} \lim_{x\to\infty} T(t, x, z) &= 0 \\ \lim_{x\to-\infty} T(t, x, z) &= \theta_- \\ \lim_{x\to\infty} u(t, x, z) &= 0 \\ \lim_{x\to-\infty} u(t, x, z) &= 0. \end{aligned} \tag{4.1.6}$$

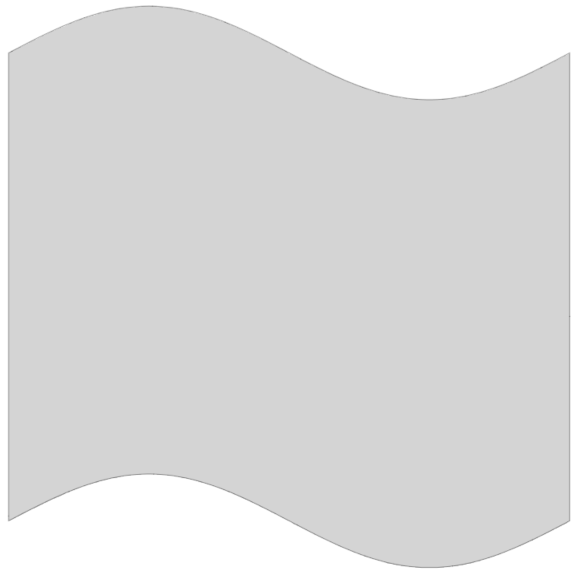
The moving frame

Because of the conditions in equation (4.1.5), it is natural to look for functions in the moving frame. That is, we wish to find $T^m(s, x, z)$ and a real number c such that T^m is periodic in x and satisfies $T(t, x, z) = T^m(x - ct, x, z)$. We make the same change of variables for fluid velocity, u to obtain u^m. We define the moving frame as the set

$$\mathbb{R} \times \Omega_p = \{(s, x, z) \in \mathbb{R}^3 : (x, z) \in \Omega, 0 \le x \le \ell\}.$$

The set Ω_p is the period cell of Ω. Let $P = \{(x, z) \in \Omega_p : x = 0, \ell\}$ be the periodic boundary of Ω_p. Let B be the complement, $B = \partial\Omega_p \setminus P$. This change of variables leads to the following system of equations

$$\begin{aligned} -cT^m_s + u^m \cdot \tilde{\nabla} T^m &= LT^m + f(T^m) \\ -cu^m_s - Lu^m + \tilde{\nabla} p &= T^m \hat{e} \\ \tilde{\nabla} \cdot u^m &= 0, \end{aligned}$$



The period cell Ω_p

where

$$\begin{aligned} L &= \tilde{\nabla} \cdot \tilde{\nabla}, \\ \tilde{\nabla} &= (\partial_x + \partial_s, \partial_z). \end{aligned} \tag{4.1.7}$$

The functions satisfy the following boundary conditions

$$\begin{aligned} \eta \cdot \tilde{\nabla} T^m &= 0 \quad \text{on } \mathbb{R} \times B \\ \lim_{s \to -\infty} T^m(s, x, z) &= \theta_- \quad \text{uniformly on } \Omega_p \\ \lim_{s \to \infty} T^m(s, x, z) &= 0 \quad \text{uniformly on } \Omega_p \\ u^m \cdot \eta &= 0 \quad \text{on } \mathbb{R} \times B \\ \omega^m &= 0 \quad \text{on } \mathbb{R} \times B, \end{aligned}$$

and where all the function satisfy periodic boundary conditions on P.

We will use the following stream function formulation for u. For each t, we let $u = \nabla^{\perp} \Psi =$

$(\Psi_z, -\Psi_x)$ for Ψ which solves the following system of equations:

$$\begin{aligned} \Delta\Psi &= \omega && \text{on } \Omega \\ \Psi &= 0 && \text{on } \partial\Omega \\ \lim_{x\to\infty} \Psi(t,x,z) &= 0 && \text{uniformly in } z \\ \lim_{x\to-\infty} \Psi(t,x,z) &= 0 && \text{uniformly in } z \end{aligned} \tag{4.1.8}$$

In the moving frame we represent Ψ with Ψ^m, which solves the following equations:

$$\begin{aligned} L\Psi^m &= \omega^m \\ \Psi^m &= 0 && \text{on } \mathbb{R}\times B \\ \lim_{s\to\infty} \Psi^m(s,x,z) &= 0 && \text{uniformly on } \Omega_p \\ \lim_{s\to-\infty} \Psi^m(s,x,z) &= 0 && \text{uniformly on } \Omega_p. \end{aligned} \tag{4.1.9}$$

We impose periodic boundary conditions on P.

If we have a solution (c, T^m, u^m) to the above system of equations, then $(c, \tilde{T}, \tilde{u})$ is also a solution, where $\tilde{T}^m(s,x,z) = T^m(s+s_0,x,z)$ and $\tilde{u}^m(s,x,z) = u^m(s+s_0,x,z)$. Hence, we also impose the extra condition that

$$\max_{s\geq 0,(x,z)\in\Omega_p} T^m(s,x,z) = \theta_0. \tag{4.1.10}$$

The main result

Our main result is the following theorem.

Theorem 4.1.1. *Let the nonlinearity, f, be ignition type as in equation* (4.1.2)*. Then there exists a pulsating front solution (c,T,u) to the system equation* (4.1.3) *- equation* (4.1.4)*; that is, T and u satisfy the conditions equation* (4.1.5) *and equation* (4.1.6)*. The solutions satisfy the following: $c > 0$, $T \in C^{1+\alpha,2+\alpha}$, $u \in C^{1+\alpha,2+\alpha}$, $f(T) \not\equiv 0$. Moreover, there is a constant $C_{\Omega,p} > 0$, which depends only on Ω and p such that if f satisfies*

$$f(T) \leq C_{\Omega,p}(T-\theta_0)_+^p, \tag{4.1.11}$$

with $p > 2$, then the left limit is one. In other words

$$\lim_{x\to-\infty} T(t,x,z) = 1.$$

The assumption equation (4.1.11) is made for purely technical reasons and is similar to that in [7, 25, 51, 52].

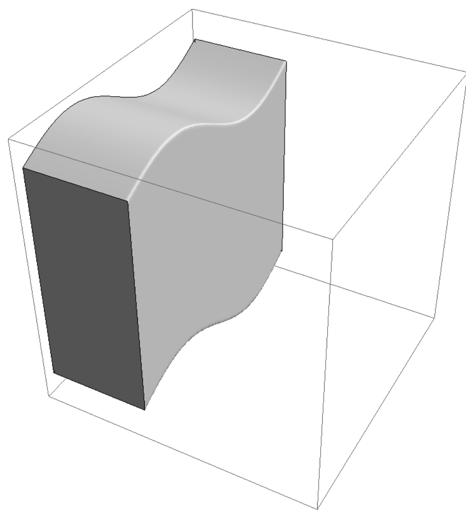
The general idea of the proof is to marry the approaches from [7] and [9,71]. However, significant difficulties arise from having an unknown fluid flow, unlike in [9, 71], and from having a non-elliptic operator after changing variables to the moving from, unlike in [7]. The proof proceeds as follows. First, we consider a finite domain, $[-a, a] \times \Omega_p$, and examine a regularized version of the problem. The operator L is not elliptic so we add a second order term with an ϵ weight. In addition, we smooth the vorticity by convolution in the equation relating it to the fluid velocity. Smoothing the vorticity is novel and does not appear in the analysis of previous works. It provides the necessary regularity for u in order to apply the main result from [10] when taking the limit $\epsilon \to 0$.

In Section 4.2, we obtain a priori estimates on the finite domain so that we can apply a fixed point theorem to find a solution to this related problem. The main difference between our work in this section and the analogous work in [7] is that we use a regularized version of the stream function formulation for the fluid velocity. This allows us to get the requisite estimates to pass to the infinite cylinder. These estimates are contained in Proposition 4.2.1.

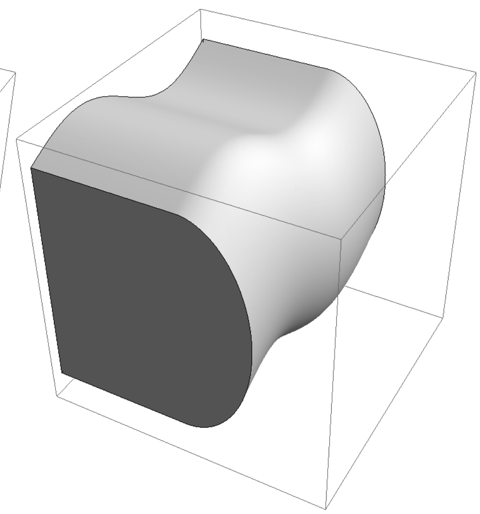
In Section 4.3, we will use these bounds to take the limit $a \to \infty$. In order to take the limit $\epsilon \to 0$, we will obtain new estimates independent of ϵ. The main difficulties in this section are the lower bound on the front speed in Proposition 4.3.15 and the upper bound on the fluid velocity. The proof of the lower bound, while similar in spirit to the lower bound in [9], is more difficult because of the lack of monotonicity results used in that paper. The upper bound on the fluid velocity involves new estimates on a family of degenerate elliptic equations.

Finally, in Section 4.4, we will use the bounds from Section 4.3 along with various parabolic and elliptic regularity results to obtain new estimates on our functions. This will allow us to conclude the proof of Theorem 4.1.1 by de-convolving the vorticity and thereby obtaining a solution to equation (4.1.3).

Before we begin the next section, we will comment on difficulties arising from generalizing this result. Naturally, one might consider the same setup in three or more spatial dimensions. However, the proof that follows requires the use of a variant of the Sobolev embedding theorem for parabolic Sobolev spaces, and the estimates that we have obtained in this chapter are not strong enough to allow us to apply this result in higher dimensions. Another possible avenue for generalizing the results of this chapter would be to consider the more physical no slip boundary conditions. It is harder to obtain estimates for u with the no slip boundary conditions because the equation for the vorticity no longer satisfies Dirichlet boundary conditions. For a possible strategy to obtain bounds with no slip boundary conditions, see the development in [25]. Finally, one could consider reaction terms f which are time periodic. Unfortunately, the moving frame in this setting adds an extra dimension which will again cause problems when applying the Sobolev embedding theorem. For results in the case of prescribed advection, we point the reader to [59, 60, 65] and references therein.



The right end of the domain R_a.



The right end of the domain D_a.

4.2 The problem in a finite domain

We begin by considering this problem on the domain $R_a = [-a, a] \times \Omega_p$ and with the elliptic operator $L_\epsilon = L + \epsilon \partial_s^2$ where L is the linear operator defined in equation (4.1.7) and where $0 < \epsilon < 1/2$. This regularization method was first used in [71]. Also, following the development in [25], in order to avoid regularity issues, we let D_a be a smooth domain such that $R_{a+2\ell} \subsetneq D_a \subsetneq R_{a+3\ell}$. We define the "ends" of D_a as

$$E = \{(s, x, z) \in \partial D_a : (x, z) \notin P, s \geq a + 2\ell \text{ or } s \leq -a - 2\ell\}.$$

We also introduce some new notation. Let φ be a non-negative function in $C_c^\infty(\mathbb{R}^3)$ with $\|\varphi\|_{L^1(\mathbb{R}^3)} = 1$. Define $\varphi_\delta(s, x, z) = \delta^{-3}\varphi(s/\delta, x/\delta, z/\delta)$. If g is a function on D_a which satisfies periodic boundary conditions on P and which vanishes on the rest of the boundary, we can extend it to all of $\mathbb{R}^3$ such that the H^1 and C^1 norms increase by a factor of at most 2. Then we define

$$\tilde{g}_\delta(s, x, z) = (f * \varphi_\delta)(s, x, z) = \frac{1}{\delta^3} \int g(s - s', x - x', z - z')\varphi(s', x', z')ds'dx'dz'.$$

We will use this to smooth the vorticity in the first few sections of this chapter. This introduces δ-dependence which propagates to every function. Hence, decorating functions with a δ subscript becomes ambiguous since one cannot tell if the subscript indicates that the function has been convolved or simply solves a PDE with a coefficient which depends on δ. The tilde notation is meant to clear up this ambiguity.

Since convolutions are continuous as maps from Sobolev spaces to themselves and Hölder spaces to themselves, any bounds we obtain for g will carry bounds on $\tilde{g}_\delta$.

We consider a regularized problem, which will, in the limit as δ and ϵ tend to zero and as a tends to infinity, converge to the solution of the pulsating front problem. For ease of notation, when no confusion will arise, we'll suppress the dependence of the functions on a, δ, and ϵ. Moreover, in this section, we will wish to prove existence by making use of the Leray-Shauder degree theory. Hence, we will add a $\tau \in [0, 1]$ into our equations. This gives us the problem below:

$$\begin{aligned} -cT^m_s + u^m \cdot \tilde{\nabla} T^m &= L_\epsilon T^m + \tau f(T^m) \text{ on } R_a \\ -c\omega^m_s - L_\epsilon \omega^m &= \tau \hat{e} \cdot \tilde{\nabla}^\perp T^m \text{ on } D_a \\ L_\epsilon \Psi^m &= \tilde{\omega}^m \text{ on } D_a \\ u^m &= (\Psi^m_z, -\Psi^m_s - \Psi^m_x), \end{aligned} \tag{4.2.1}$$

with the following boundary conditions:

$$\begin{aligned} \eta \cdot \tilde{\nabla} T^m &= 0 \quad \text{ on } [-a, a] \times B \\ T^m(-a, x, z) &= 1 \\ T^m(a, x, z) &= 0 \\ \Psi^m &= 0 \quad \text{ on } ([-a - 2\ell, a + 2\ell] \times B) \cup E \\ \omega^m &= 0 \quad \text{ on } ([-a - 2\ell, a + 2\ell] \times B) \cup E. \end{aligned} \tag{4.2.2}$$

In addition, all functions satisfy periodic boundary conditions on boundary portion $[-a, a] \times P$. In order to make equation (4.2.1) well-defined, we extend the function T^m from R_a to D_a by reflection as in [25]. Explicitly, we write

$$T^m(s, x, z) = \begin{cases} -T^m(-2a - s, x, z) + 2T^m(-a, x, z) & \text{if } s < -a \\ -T^m(2a - s) + 2T^m(a, x, z) & \text{if } s > a. \end{cases}$$

Notice that the extension does not increase the $C^{1,\alpha}$ norm of T^m, up to a multiplicative factor.

We wish to show, using the Leray-Schauder fixed point theorem, that we can solve the system equation (4.2.1) - equation (4.2.2) with solutions which are uniformly bounded in a. Namely, we will find T^m, ω^m, and Ψ^m which are bounded uniformly in a in $C^{2,\alpha}$ (but not necessarily in ϵ or δ) and which solve the first, second, and third equations in equation (4.2.1), respectively.

In order to find a solution we will prove the following a priori bounds.

Proposition 4.2.1. *There exists a real number a_0 and a positive constant $C = C(\epsilon, \delta)$ such that if*

T^m is a solution to problem equation (4.2.1) - equation (4.2.2) and if $a \geq a_0$, then

$$|c| + \|T^m\|_{C^{2,\alpha}} + \|\Psi^m\|_{C^{3,\alpha}} + \|\omega^m\|_{C^{2,\alpha}} + \|\nabla T^m\|_{L^2} + \|\Psi^m\|_{H^3} + \|\omega^m\|_{H^2} < C.$$

Here all bounds hold on R_a.

In order to do this we will get a series of bounds which will eventually close. We will start by proving a relationship between the front speed, c, and the flow velocity, u^m. Then we will get a series of L^2 and H^1 bounds. First we will show a bound of ω^m in terms of T^m. Then we will get a bound of u^m in terms of ω^m. Finally, we will get a bound of T^m in terms of c. Putting these all together we end up with an inequality where the same norm of T^m shows up on both sides, though with a larger exponent on the left than on the right, effectively finishing the proposition above.

Virtually all the bounds in this section will depend on ϵ and δ unless stated otherwise. On the other hand, the bounds will be independent of a, allowing us to take local limits as a tends to infinity. In the following sections, we will obtain new bounds allowing us to take limits as ϵ and then δ tend to zero.

A bound on the front speed by fluid velocity

We start by proving a bound on c. This proof is essentially the same as that of [7].

Lemma 4.2.2. *There exists $a_0 > 0$ and $C_0 > 0$ so that if (c, T^m, u^m) solves equation (4.2.1)-equation (4.2.2) and equation (4.1.10) then for all $a \geq a_0$ and all $\epsilon \in (0, 1/4)$ we have*

$$-C_0(1 + \|u^m\|_{L^\infty}) \leq c \leq C_0(1 + \|u^m\|_{L^\infty}).$$

Proof. Let ψ_e be the principal eigenfunction (normalized to have L^∞ norm 1) of the operator

$$-\Delta\psi_e + 2\partial_x\psi_e = \mu_e\psi_e, \quad \psi_e > 0$$

defined in Ω_p, with the boundary conditions

$$-\eta_1\psi_e + \eta \cdot \nabla\psi_e = 0 \text{ on } B,$$

and periodic boundary conditions in x (that is, at $x = 0$ and $x = \ell$). As in [9], Proposition 5.7, such an eigenfunction exists and is positive on $\overline{\Omega_p}$. Moreover, we can see by multiplying the equation by ψ_e and integrating that the principal eigenvalue μ_e is positive.

Define the function

$$\gamma_A(s, x, z) = Ae^{-(s+a)}\psi_e(x, z).$$

Notice that γ_A satisfies the same boundary conditions as T^m on $[-a, a] \times B$ and $[-a, a] \times P$ but not at $s = -a$ or $s = a$. Then

$$\begin{aligned}
-c(\gamma_A)_s + u^m \cdot \tilde{\nabla}\gamma_A - L_\epsilon \gamma_A &= Ae^{-(s+a)}[c\psi_e - u_1^m \psi_e + u^m \cdot \nabla\psi_e - (1+\epsilon)\psi_e - \Delta\psi_e + 2(\psi_e)_x] \\
&= Ae^{-(s+a)}[c\psi_e - u_1^m \psi_e + u^m \cdot \nabla\psi_e - (1+\epsilon)\psi_e + \mu_e\psi_e] \\
&\geq Ae^{-(s+a)}[(c - \|u_1^m\|_{L^\infty} - (1+\epsilon))\psi_e - \|u^m\|_{L^\infty}\|\nabla\psi_e\|_{L^\infty})] \\
&\geq MA\psi_e e^{-(s+a)} = M\gamma_A \geq \tau f(\gamma_A),
\end{aligned}$$

if

$$c \geq M + \|u^m\|_{L^\infty}\left(1 + \left\|\frac{\nabla\psi_e}{\inf\psi_e}\right\|_{L^\infty}\right) + (1+\epsilon). \tag{4.2.3}$$

We defined here

$$M = \sup_{T\in(0,1)} \frac{f(T)}{T}.$$

Hence, γ_A is a super solution for all $A > 0$. If we take A larger than $e^{2a}/(\inf\psi_e)$, then $\gamma_A > 1 \geq T^m$ everywhere in R_a. Hence, we can define

$$A_0 = \inf\{A : \gamma \geq T^m \text{ everywhere in } R_a\}.$$

It follows from compactness of R_a and the continuity of T^m and γ_A that there is a point (s_0, x_0, z_0) such that $T^m(s_0, x_0, z_0) = \gamma_{A_0}(s_0, x_0, z_0)$. The strong maximum principle implies that this cannot be an interior point or a point on the periodic boundary of Ω_p. In addition, the Hopf maximum principle implies that it cannot happen on the boundary $[-a, a] \times B$. Moreover since

$$T^m(a, x, z) = 0 < (\inf\psi_e)e^{-2a} \leq \gamma(a, x, z),$$

it cannot be that $s_0 = a$. Hence, it must be that $s_0 = -a$, from which it follows that $A_0 \leq (\inf\psi_e)^{-1}$.

On the other hand, the normalization condition equation (4.1.10), implies then that $\theta_0 \leq A_0 e^{-a}$. This clearly doesn't hold for a sufficiently large, so it must be that equation (4.2.3) cannot hold. In other words, we must have that

$$c \leq 1 + \epsilon + C\|u^m\|_{L^\infty} + M,$$

as desired.

Now, we prove the lower bound. Let ψ_e' be the principal eigenfunction of the operator

$$-\Delta\psi_e' - 2\partial_x\psi_e' = \mu_e'\psi_e',$$

in Ω_p, with periodic boundary conditions in x and satisfying

$$\eta_1\psi_e + \eta \cdot \nabla\psi_e = 0 \text{ on } B.$$

Again, we know that ψ_e' is positive, we take it to have L^∞ norm 1, and that μ_e' is positive. Let

$$\gamma_A(s,x,z) = 1 - Ae^{s-a}\psi_e'(x,z).$$

Arguing as before, we see that this is a subsolution if $A > 0$ and

$$c \leq -\|u\|_{L^\infty}\left(1 + \left\|\frac{\nabla\psi_e'}{\psi_e'}\right\|_{L^\infty}\right) - (1+\epsilon). \tag{4.2.4}$$

Moreover, as before, if (4.2.4) holds, we can show that we can take A to be at least as small as $(\inf\psi_e')^{-1}$ with $\gamma_A \leq T^m$. Then we have that, by the normalization condition equation (4.1.10),

$$1 - e^{-a} \leq \gamma_1(0,x,z) \leq T^m(0,x,z) \leq \theta_0.$$

This leads to a contradiction if a is large enough. Hence it must be that equation (4.2.4) does not hold when a is large. This implies that our lower bound for c holds. $\square$

Now we will begin collecting L^2 bounds on various objects. The goal here is to eventually close all these bounds, which leads to the proof of Proposition 4.2.1.

A bound on vorticity by temperature

The first is the bound on the vorticity.

Lemma 4.2.3. *There exists a constant $C = C(\epsilon) > 0$ so that if T^m and ω^m satisfy equation* (4.2.1)*-equation* (4.2.2)*, then*

$$\|\omega^m\|_{H^1(D_a)} \leq C\|\tilde{\nabla}T^m\|_{L^2(R_a)}.$$

Proof. We multiply the second equation in equation (4.2.1) by ω^m and integrate in all three coordinates.

$$-c\int_{D_a}\omega^m\omega^m_s dsdxdz - \int_{D_a}\omega^m L_\epsilon\omega^m dsdxdz = \tau\int_{D_a}\omega^m(\hat{e}\cdot\tilde{\nabla}T^m)dsdxdz$$

The first term vanishes because $\omega^m = 0$ on the boundary. Integration by parts (along with the boundary conditions for ω^m), Hölder's inequality, and an application of the Poincaré inequality in the z variable gives us

$$\begin{aligned}\int_{D_a}(|\tilde{\nabla}\omega^m|^2 + \epsilon(\omega^m_s)^2)dsdxdz &\leq C\|\omega^m_z\|_{L^2(D_a)}\cdot\|\tilde{\nabla}T^m\|_{L^2(D_a)} \\ &\leq C\left(\int_{D_a}(|\tilde{\nabla}\omega^m|^2 + \epsilon(\omega^m_s)^2)dsdxdz\right)^{1/2}\|\tilde{\nabla}T^m\|_{L^2(D_a)}\end{aligned}$$

Notice that the Poincaré Inequality can be applied on cross-sections with fixed x and s to give us that $\|\omega^m\|_{L^2(D_a)} \leq C\|\omega_z^m\|_{L^2(D_a)}$. Hence C doesn't depend on a. Then we arrive at

$$\|\omega^m\|_{H^1(D_a)} \leq C\|\tilde{\nabla}T^m\|_{L^2(R_a)}, \tag{4.2.5}$$

and this finishes the proof. □

It follows from the properties of convolutions and the change of variables that we made that this gives us an H^1 bound on $\tilde{\omega}^m$, with an added constant independent of a, ϵ, and δ. In addition, we get H^k bounds on $\tilde{\omega}^m$ for any $k > 1$, though the constants in these will have some δ dependence.

A bound on velocity by vorticity

We now bound the fluid velocity by vorticity. To do this, we will use the fact that convolution with a smooth, compactly supported function is a bounded operator from L^2 to H^k for any k.

Lemma 4.2.4. *For any k, there exists a constant $C = C(\epsilon, \delta, k) > 0$ that depends only on ϵ, δ, and k such that if (c, T^m, u^m) solves equation* (4.2.1)*-equation* (4.2.2) *then*

$$\|\Psi^m\|_{H^k(D_a)} \leq C\|\omega^m\|_{H^1(D_a)}.$$

In addition, the Sobolev embedding theorem gives us that

$$\|u^m\|_{C^k(D_a)} \leq C\|\omega^m\|_{H^1(D_a)}.$$

Proof. First we will obtain an L^2 bound on Ψ^m by multiplying equation (4.2.1) by Ψ^m, integrating by parts, and using the boundary conditions of Ψ^m, we obtain

$$\int_{D_a} \left[\epsilon|\Psi_s^m|^2 + |\nabla\Psi^m|^2\right] dxdzds = -\int_{D_a} \Psi^m\tilde{\omega}^m dxdzds.$$

Then the using the Cauchy-Schwarz and Poincaré inequalities we obtain

$$\|\Psi^m\|_{L^2}^2 \leq C\int_{D_a} \left[\epsilon|\Psi_s^m|^2 + |\nabla\Psi^m|^2\right] dxdzds \leq C\|\Psi^m\|_{L^2}\|\tilde{\omega}^m\|_{L^2},$$

where the constant C depends only on the domain D. Hence, we obtain

$$\|\Psi^m\|_{L^2} \leq C\|\tilde{\omega}^m\|_{L^2}.$$

We will now use the standard elliptic estimates to obtain greater regularity for Ψ^m. In order for these estimates to be useful, we need them to be independent of a. To this end, we argue as in [7], Lemma 2.3. In short, we apply the estimates on sets of the form $[x_0, x_0 + 1] \times \partial\Omega_p$ and sum these to obtain bounds independent of a. For any $k \geq 2$, we've summarized this in the following inequality,

$$\|\Psi^m\|_{H^k} \leq C_\epsilon \left(\|\Psi^m\|_{L^2} + \|\tilde{\omega}^m\|_{H^{k-2}}\right) \leq C_\epsilon \|\tilde{\omega}^m\|_{H^{k-2}}.$$

The constant in this inequality depends only on ϵ and k. To finish, we simply use that convolutions are bounded operators, as discussed above. Hence, we obtain

$$\|\Psi^m\|_{H^k} \leq C_{\epsilon,\delta} \|\omega^m\|_{H^1},$$

where the constant depends only on ϵ, δ, and k. □

A bound on temperature by fluid velocity

Lemma 4.2.5. *There exist constants $C > 0$ and $a_0 > 0$ such that if (c, T^m, u^m) satisfy equation* (4.2.1)*-equation* (4.2.2) *with the normalization equation* (4.1.10)*, and if $a > a_0$, then*

$$\int_{R_a} |\tilde{\nabla} T^m|^2 dxdzds + \epsilon \int_{R_a} (T^m_s)^2 dxdzds + (1+\epsilon) \int_{\Omega_p} T^m_s(a,x,z) dxdz \leq C(1 + \|u^m\|_{L^\infty(R_a)}). \quad (4.2.6)$$

Proof. Recall that T^m satisfies the equation

$$-cT^m_s + u^m \cdot \tilde{\nabla} T^m - L_\epsilon T^m = \tau f(T^m) \quad (4.2.7)$$

with the boundary conditions

$$T^m(-a,x,z) = 1, \quad T^m(a,x,z) = 0, \quad \eta \cdot \tilde{\nabla} T^m(s,x,z) = 0 \text{ on } [-a,a] \times B. \quad (4.2.8)$$

Now if we multiply equation (4.2.7) by $(T^m - 1)$ and integrate over R_a, we obtain

$$\begin{aligned} \frac{-c|\Omega_p|}{2} + \frac{1}{2} \int_{\Omega_p} u^m_1(a,x,z) dxdz + \int_{R_a} |\tilde{\nabla} T^m|^2 dxdzds + \epsilon \int_{R_a} (T^m_s)^2 dxdzds \\ +(1+\epsilon) \int_{\Omega_p} T^m_s(a,x,z) dxdz = \tau \int_{R_a} (T^m - 1) f(T^m) dxdzds. \end{aligned}$$

Notice that the second term is bounded by $(1/2)|\Omega_p| \cdot \|u^m\|_{L^\infty}$. Then, we move the first and second terms from the left hand side to the right hand side. Finally we use Lemma 4.2.2 and that $(T^m - 1) f(T^m) \leq 0$ to obtain

$$\begin{aligned} (1+\epsilon) \int_{\Omega_p} T^m_s(a,x,z) dxdz + \epsilon \int_{R_a} (T^m_s)^2 dxdzds + \int_{R_a} |\tilde{\nabla} T^m|^2 dsdxdz \\ \leq \frac{c|\Omega_p|}{2} + \frac{|\Omega_p|}{2} \|u^m\|_{L^\infty} \leq C(1 + \|u^m\|_{L^\infty}). \end{aligned}$$

□

We need one more lemma to close the inequalities we've accumulated so far. The proof is essentially the same as in [7] with a few extra terms.

Lemma 4.2.6. *There exists a constant $C > 0$ and a constant a_0 such that any solution to the system equation* (4.2.1) *- equation* (4.2.2) *with the normalization equation* (4.1.10)*, satisfies, for $a > a_0$*

$$0 \leq -\int_{\Omega_p} T_s^m(a, x, z)dxdz \leq C\left(1 + \|\tilde{\nabla}T^m\|_{L^2(R_a)}\right). \tag{4.2.9}$$

Proof. Define the function

$$I(s) = \frac{1}{|\Omega_p|}\int_{\Omega_p} T^m(s, x, z)dxdz. \tag{4.2.10}$$

Then, integrating the equation for T^m in equation (4.2.1) in x and z and using the boundary conditions, gives us

$$-I_{ss} = G(s),\ I(-a) = 1,\ I(a) = 0. \tag{4.2.11}$$

Here G is the function given by

$$G(s) = \frac{1}{(1+\epsilon)|\Omega_p|}\int_{\Omega_p}(\tau f(T^m) - u^m \cdot \tilde{\nabla}T^m + cT_s^m)dxdz + \frac{2}{(1+\epsilon)|\Omega_p|}\int_B \eta_1 T_s^m dS, \tag{4.2.12}$$

where $\eta = (\eta_1, \eta_2)$ is the unit normal to B. We can solve this equation explicitly as

$$I(s) = -\int_{-a}^{s}(s-r)G(r)dr + As + B \tag{4.2.13}$$

with constants

$$A = -\frac{1}{2a} + \frac{1}{2a}\int_{-a}^{a}(a-r)G(r)dr,$$
$$B = \frac{1}{2} + \frac{1}{2}\int_{-a}^{a}(a-r)G(r)dr$$

which we get from the boundary conditions. Hence, we have that

$$I_s(-a) = A,\ I_s(a) = A - \int_{-a}^{a} G(r)dr. \tag{4.2.14}$$

In addition, the boundary conditions on T^m give us that $I_s(-a) \leq 0$ and $I_s(a) \leq 0$. We wish to get lower bounds on these quantities. Hence, using equation (4.2.12), we have that

$$\begin{aligned}
-I_s(a) &= \frac{1}{2a} + \frac{1}{2a}\int_{-a}^{a}(a+r)G(r)dr \\
&= \frac{1}{2a} + \frac{1}{2a(1+\epsilon)|\Omega_p|}\int_{R_a}(a+r)(\tau f(T^m) - u^m \cdot \tilde{\nabla} T^m + cT^m_s)dxdzdr \\
&+ \frac{1}{a(1+\epsilon)|\Omega_p|}\int_{(-a,a)\times B}(a+r)\eta_1 T^m_s dS(x,z,r).
\end{aligned} \tag{4.2.15}$$

In the right side of this equation, we have three terms. We leave the first as is and we use the fact that η_1 is constant in r to get that the third term is bounded as

$$\frac{1}{a|\Omega_p|}\int_{-a}^{a}\int_{B}(a+r)\eta_1 T^m_s dSdr = -\frac{1}{|\Omega_p|}\int_{B}\eta_1 T^m dS(x,z,r) \leq \frac{|B|}{|\Omega_p|} \leq C. \tag{4.2.16}$$

The second term is a bit more stubborn. Integrating by parts and using the fact that $rf(T^m) \leq 0$ for all $(r,x,z) \in R_a$, gives us

$$\begin{aligned}
&\frac{1}{2a|\Omega_p|}\int_{R_a}(a+r)[\tau f(T^m) - u^m \cdot \tilde{\nabla} T^m + cT^m_s]dxdzdr \\
&\quad = \frac{1}{2|\Omega_p|}\int_{R_a}\tau f(T^m)dxdzdr + \frac{1}{2|\Omega_p|}\int_{R_a}\tau \frac{r}{a} f(T^m)dxdzdr + \frac{1}{2a|\Omega_p|}\int_{R_a}u^m_1 T^m dxdzdr \\
&\quad - \frac{c}{2a|\Omega_p|}\int_{R_a}T^m dxdzdr \\
&\quad \leq \frac{\tau F}{2} + \|u^m\|_{L^\infty} + |c|.
\end{aligned} \tag{4.2.17}$$

Here

$$F = |\Omega_p|^{-1}\int f(T^m)dxdzds.$$

In summary, from equation (4.2.15), equation (4.2.16), and equation (4.2.17),we have

$$0 \leq -I_s(a) \leq \frac{1}{2a} + \|u^m\|_{L^\infty} + |c| + C + \frac{\tau F}{2} \tag{4.2.18}$$

where $C > 0$ is a universal constant given by equation (4.2.16).

Now we look at $I_s(-a)$. As before, we have that

$$0 \leq -I_s(-a).$$

Also, we similarly estimate $-I_s(-a)$ from above as follows.

$$
\begin{aligned}
-I_s(-a) &= \frac{1}{2a} - \frac{1}{2a}\int_{-a}^{a}(a-r)G(r)dr \\
&= \frac{1}{2a} - \frac{\tau}{2a(1+\epsilon)|\Omega_p|}\int_{R_a}(a-r)f(T^m)dxdzdr \\
&+ \frac{1}{2a(1+\epsilon)|\Omega_p|}\int_{R_a}(a-r)u^m\cdot\tilde{\nabla}T^m dxdzdr - \frac{c}{2a(1+\epsilon)|\Omega_p|}\int_{R_a}(a-r)T^m_s dxdzdr \\
&- \frac{1}{a(1+\epsilon)|\Omega_p|}\int_{B_a}(a-r)\eta_1 T^m_s dS \\
&\le \frac{1}{2a} - \frac{1}{(1+\epsilon)|\Omega_p|}\int_{\Omega_p}u^m_1(-a)dxdzds + \frac{1}{2a(1+\epsilon)|\Omega_p|}\int_{R_a}u^m_1 T^m dxdzds \\
&- \frac{c}{2a(1+\epsilon)|\Omega_p|}\int_{R_a}T^m dxdzds - \frac{c}{(1+\epsilon)|\Omega_p|}\int_{\Omega_p}dxdzds \\
&- \frac{1}{a(1+\epsilon)|\Omega_p|}\int_{B_a}\eta_1 T^m dS - \frac{2}{(1+\epsilon)|\Omega_p|}\int_{B}\eta_1(-a,x,z)dS \\
&\le \frac{1}{2a} + 2\|u^m_1\|_{L^\infty} + 2|c| + C,
\end{aligned}
\tag{4.2.19}
$$

where C here is a universal constant. In the computation above, we only used integration by parts and that $(a-r)f(T^m)\ge 0$.

There is one more calculation we need to make before we can finish this proof. Namely, we need to show that the integral of $u\cdot\tilde{\nabla}T^m$ vanishes. To see this, simply integrate by parts as follows

$$
\int_{R_a} u\cdot\tilde{\nabla}T^m dxdzds = -\int_{\Omega_p}u_1(-a,x,z)dxdz = -\int_{\Omega_p}\Psi^m_z(-a,x,z)dxdz = 0.
$$

The first equality is a result of the boundary conditions of T^m and u^m. The last equality is a result of the boundary conditions on Ψ^m. Hence, when we integrate G from $-a$ to a, we get

$$
\int_{-a}^{a}G(s)ds = \frac{\tau F}{(1+\epsilon)} - \frac{c}{(1+\epsilon)} - \frac{1}{(1+\epsilon)|\Omega_p|}\int_B \eta_1(-a,x,z)dS.
$$

To finish the proof we analyze the following three equations:

$$
\begin{gathered}
I_s(a) - I_s(-a) = -\int_{-a}^{a}G(s)ds = \frac{1}{(1+\epsilon)}\left(-\tau F + c + \frac{1}{|\Omega_p|}\int_B \eta_1(-a,x,z)dS\right) \\
0 \le -I_s(a) \le \frac{1}{2a} + \|u^m\|_{L^\infty} + |c| + C + \frac{\tau F}{2} \\
0 \le -I_s(-a) \le \frac{1}{2a} + 2\|u^m\|_{L^\infty} + 2|c| + C.
\end{gathered}
$$

From these, we easily get that

$$\frac{\tau F}{1+\epsilon} \leq (|c| + C) + \left(\|u^m\|_{L^\infty} + \frac{1}{2a} + |c| + C + \frac{\tau F}{2} \right), \tag{4.2.20}$$

and rearranging this, along with the inequalities from Lemmas 4.2.2, 4.2.3 and 4.2.4, gives us that

$$\tau F \leq C(\|\tilde{\nabla} T^m\|_{L^2(R_a)} + 1) \tag{4.2.21}$$

for $a > 1$. Hence using this, the inequalities from Lemmas 4.2.2, 4.2.3, and 4.2.4, and equation (4.2.18) finishes the proof. □

A uniform bound on temperature

We can now combine the previous estimates to obtain the following proposition.

Proposition 4.2.7. *Let* (c, T^m, u^m) *satisfy equation* (4.2.1) - *equation* (4.2.2) *with the normalization equation* (4.1.10). *Then there exists a constant* $C > 0$ *such that*

$$\|\tilde{\nabla} T^m\|_{L^2(R_a)} \leq C.$$

Proof. This is simply given by combining Lemmas 4.2.4, 4.2.5, 4.2.6, and 4.2.3. □

Notice that this closes all of our bounds, giving us a uniform bound on $|c|$, $\|u^m\|_{L^\infty}$ and $\|\omega^m\|_{H^2}$ by using the various estimates from earlier.

Remark 4.2.8. *Notice that we can then use these bounds and the usual elliptic theory to give us that* T^m *is bounded uniformly in* $C^{2,\alpha}(R_a)$. *Of course, this in turn gives us that* ω^m *is bounded uniformly in* $C^{2,\alpha}(D_a)$ *as well. Arguing as before and using Schauder estimates, we can show that* Ψ^m *is also uniformly bounded in* $C^{3,\alpha}(D_a)$. *Hence, we have proven Proposition 4.2.1. These bounds will blow up as* ϵ *tends to zero.*

We will make use of the Leray-Schauder topological degree theory, see e.g. [61], to prove the existence of a solution to our problem on the finite domain. Our a priori bounds make this possible. The proof that follows is similar to that which appears in [7].

Proposition 4.2.9. *For each a sufficiently large, there exists a solution to the system equation* (4.2.1)-*equation* (4.2.2) *which satisfies the normalization condition equation* (4.1.10).

Proof. Let $V = \{(x, T) \in \mathbb{R} \times C^{1,\alpha}(R_a) : |x| + \|T\|_{C^{1,\alpha}} \leq C_0 + 1\}$, where C_0 is a bound to be chosen later. Define the operator $S_\tau : \mathbb{R} \times C^{1,\alpha} \to \mathbb{R} \times C^{1,\alpha}$ as $S_\tau(c, Z) = (c + \theta_0 - \max_{s \geq 0} Z(s, x, z), T)$ where T is the solution to

$$-cT_s + u \cdot \tilde{\nabla} T = L_\epsilon T + \tau f(Z) \text{ on } D_a,$$

where u solves

$$\begin{aligned} -c\omega - L_\epsilon \omega &= \tau \hat{e} \cdot \tilde{\nabla} Z \text{ on } D_a \\ L_\epsilon \Psi &= \tilde{\omega} \text{ on } D_a \\ u &= (\Psi_z, \Psi_x + \Psi_s), \end{aligned}$$

with the boundary conditions as in equation (4.2.2), and with Z extended as before. By the usual elliptic theory, S_τ is continuous and compact. We wish to find a fixed point of map, S_1. To this end, we will show that the degree of our map is non-zero. Notice that by our previous work, we can choose C_0 large enough so that $\mathrm{Id} - S_\tau$ does not vanish on the boundary of V. Then the Leray-Schauder topological degree theory tells us that for all τ,

$$\deg(\mathrm{Id} - S_\tau, V, 0) = \deg(\mathrm{Id} - S_0, V, 0).$$

Hence, we need only show that $\deg(\mathrm{Id} - S_0, V, 0)$ is non-zero. Notice that

$$(\mathrm{Id} - S_0)\,(c, Z) = (\max_{s \geq 0} Z(s,x,z) - \theta_0, Z - T_0^c),$$

where T_0^c is the unique solution to

$$-cT_s - L_\epsilon T = 0,$$

with the boundary conditions as in equation (4.2.2). Since degree is invariant under homotopy, we will find a map, homotopic to $\mathrm{Id} - S_0$, whose degree is easier to compute. To this end, we notice that $\mathrm{Id} - S_0$ is Φ_0 where

$$\Phi_\tau(c, Z) = \left(\max_{s \geq 0} Z - \theta_0, Z - \tau \varphi_c(s) - (1-\tau) T_0^c \right),$$

and where

$$\varphi_c(s) = \frac{e^{-cs} - e^{-ca}}{e^{ca} - e^{-ca}}.$$

Before we calculate the degree of this map, we must first check that Φ_τ does not vanish on the boundary of V for any $0 \leq \tau \leq 1$. This amounts to obtaining an a priori bound on c independent of τ, since any bound on c also provides an a priori bound of T_0^c by standard elliptic theory. To this end, suppose that we have (c, Z) is a zero of the map Φ_τ for some τ. Then

$$(c, Z) = (c, \tau \varphi_c + (1-\tau) T_0^c), \quad \text{and} \quad \max_{s \geq 0} Z = \theta_0.$$

Choose C large enough that if $c \geq C$ then $\varphi_c(0) \leq \theta_0/3$ and if $-c \geq C$ then $\varphi_c(0) \geq \frac{1+\theta_0}{2}$. Notice that, by our work in Lemma 4.2.2, if $c \geq 2$ then $T_0^c \leq A_0 \psi_e e^{-(s+a)}$ for $A_0 = (\inf \psi_e)^{-1}$, and if $c \leq -2$ then $1 - e^{-(s+a)} \leq T_0^c$. We need to rule out the cases when $|c|$ becomes large. Let's first check the case when $c \leq -C$. Here

$$\theta_0 = \max_{s\geq 0} Z(s,x,z) \geq \tau\varphi_c(0) + (1-\tau)(1-e^{-a}) \geq \tau\frac{1+\theta_0}{2} + (1-\tau)(1-e^{-a}).$$

This leads to a contradiction if a is larger than $-\log((1-\theta_0)/2)$. Now we check the case when $c \geq C$. Here

$$\theta_0 = \max_{s\geq 0} Z(s,x,z) \leq \tau\varphi_c(0) + (1-\tau)A_0 e^{-a} \leq \tau\frac{\theta_0}{3} + (1-\tau)A_0 e^{-a}.$$

This leads to a contradiction if a is larger than $-\log(\theta_0/(3A_0))$. Hence, in the definition of V, if we choose C_0 to be larger than C, from above, and the bound in Proposition 4.2.1 then $\mathrm{Id} - S_\tau$ and Φ_τ do not vanish on the boundary of V.

The map Φ_1 is then homotopic to

$$\Pi(c,Z) = (\varphi_c(0) - \theta_0, Z - \varphi_c).$$

Let c_* be the unique values such that $\varphi_{c_*}(0) = \theta_0$. Then Φ_2 is homotopic to

$$\Theta(c,Z) = \left(\varphi_c(0) - \theta_0, Z - \varphi_{c_*}\right).$$

We can calculate the degree of this map. Indeed, its degree is the product of the degrees of the two component functions. The first has degree -1 since φ_c is decreasing in c. The last has degree one. Hence, $\deg(\mathrm{Id} - S_1, V, 0) = -1$, and we conclude that S_1 has a fixed point. This is our desired solution. □

4.3 Solutions on the infinite cylinder

Since we have obtained uniform bounds, we can take weak and strong local limits as $a \to +\infty$, along subsequences if necessary, in the relevant topologies to get the limits c_ϵ, $T^{m,\epsilon}$, $\omega^{m,\epsilon}$, $\Psi^{m,\epsilon}$, and $u^{m,\epsilon}$ that are defined in the infinite cylinder and satisfy the same system of equations. That is, $T^{m,\epsilon}$, $\omega^{m,\epsilon}$, $\Psi^{m,\epsilon}$, and $u^{m,\epsilon}$ are defined on the domain $\mathbb{R}\times\Omega_p$. As before, we omit the ϵ notation whenever there will be no confusion. These functions satisfy the system

$$\begin{aligned} -cT^m_s + u^m \cdot \tilde{\nabla} T^m &= L_\epsilon T^m + f(T^m) \text{ on } \mathbb{R}\times\Omega_p \\ -c\omega^m_s - L_\epsilon \omega^m &= \hat{e}\cdot\tilde{\nabla} T^m \text{ on } \mathbb{R}\times\Omega_p \\ L_\epsilon \Psi^m &= \tilde{\omega}^m \text{ on } \mathbb{R}\times\Omega_p \\ u &= (\Psi^m_z, \Psi^m_s + \Psi^m_x) \end{aligned} \tag{4.3.1}$$

with the boundary conditions

$$\begin{gathered}\eta \cdot \tilde{\nabla} T^m = 0, \quad \Psi^m = 0, \quad \omega^m = 0 \quad \text{ on } \mathbb{R} \times B \\ T^m(s,0,z) = T^m(s,\ell,z), \quad \Psi^m(s,0,z) = \Psi^m(s,\ell,z), \quad \omega^m(s,0,z) = \omega^m(s,\ell,z).\end{gathered} \tag{4.3.2}$$

Since T_a^m satisfies the normalization condition equation (4.1.10) for every a and converges locally uniformly, the limit T^m must satisfy it as well. We make one remark about notation. We will assume that a_n is a sequence tending to infinity along which all the relevant functions converge. We will denote by T_n^m, the function $T_{a_n}^m$, and similarly for c_n and the other functions.

In order to take the limit $\epsilon \downarrow 0$, we need to obtain bounds which are uniform in ϵ. Our bounds from Section 4.2 all used the ellipticity of L_ϵ and, hence, depend on ϵ. Our first step is to show that c_ϵ is positive for all ϵ (though that bound will also depend on ϵ first). This will allow us to prove that T_n^m behaves as we would expect on the right end, namely it and its s-derivative vanish. This will allow us to prove an exponential bound of T^m on the right, which we will need in section 5. Finally, we will obtain L^2 bounds that are uniform in ϵ on our functions and their derivatives on sets of the form $\mathbb{R} \times K$ where $K \subset \Omega$ is bounded. We will use these in the next section to take limits. Finally, we will prove that the front speed, c_ϵ is bounded away from zero by a constant that depends only on δ.

In this section, some of the constants will depend on ϵ. We will denote these by a subscript ϵ since this dependence is important in this section. Many of the constants will also depend on δ, but we will suppress the notation for now.

Before we begin the proof, notice that since u^m is an element of both $C^{0,\alpha}$ and L^2, it must converge uniformly to 0 as s tends to infinity. The same is true for Ψ^m, all of its derivatives, all the first derivatives of T^m, ω^m, and all of its first and second derivatives.

A lower bound on the burning rate

Here we will prove that c_ϵ is positive. Notice, though, that our lower bound degenerates as we take $\epsilon \downarrow 0$. We will address this issue in the following section.

Lemma 4.3.1. *There is a constant $C_\epsilon > 0$ such that*

$$\int_{R_{a_n}} f(T_n^m)dxdzds \geq C_\epsilon$$

for all a sufficiently large.

Proof. The proof is as in [7,24]. We wish to show that there is a universal constant $C > 0$ such that

$$\left(\int f(T_n^m)dxdzds\right)\left(\int |(T_n^m)_s|^2 dxdzds\right) \geq C. \tag{4.3.3}$$

Since we have a uniform upper bound on $\|(T_n^m)_s\|_2$ given by Proposition 4.2.1, then this inequality will give us the desired bound.

First, notice that we can find $(x_0, z_0) \in \Omega_p$ such that

$$\int_{-a_n}^{0} |(T_n^m)_s(s, x_0, z_0)|^2 ds \leq \frac{3}{|\Omega_p|} \int_{R_{a_n}} |(T_n^m)_s(s, x, z)|^2 dx dz ds$$

and

$$\int_{-a_n}^{0} f(T_n^m(s, x_0, z_0)) ds \leq \frac{3}{|\Omega_p|} \int_{R_{a_n}} f(T_n^m) dx dz ds.$$

Then define

$$s_1 = \inf\{s \in (-a_n, 0) : T_n^m(s, x_0, z_0) = r_2\}$$

and

$$s_2 = \inf\{s \in (-a_n, 0) : T_n^m(s, x_0, z_0) = r_1\}.$$

Here, r_1 and r_2 come from equation (4.1.2). Notice that $s_2 > s_1$, and notice that both exist by the normalization condition equation (4.1.10) and the boundary condition on T_n^m at $s = -a_n$. Then for $s_1 \leq s \leq s_2$ we have that $f(x_0, z_0, T_n^m) > C$ for some constant $C > 0$. Hence

$$C|s_2 - s_1| \leq \int_{s_1}^{s_2} f(x_0, z_0, T_n^m(s, x_0, z_0)) ds \leq \frac{3}{|\Omega_p|} \int_{R_{a_n}} f(x, z, T_n^m(s, x, z)) dx dz ds$$

and

$$\frac{(1-\theta_0)^2}{4|s_2 - s_1|} \leq \int_{s_1}^{s_2} |(T_n^m)_s|^2 ds \leq \frac{3}{|\Omega_p|} \int_{R_{a_n}} |(T_n^m)_s|^2 dx dz ds.$$

Multiplying these two inequalities gives us the desired inequality, equation (4.3.3). □

Positivity of the speed and behavior of the temperature on the right

We will use the bound on the burning rate to get a lower bound on the speed of the front. This will be crucial in showing that our solutions are non-trivial up to this point. It will also allow us to make a meaningful change of variables back to the stationary frame where we wish to ultimately show a solution exists. The next two proofs are similar to those found in [7].

Lemma 4.3.2. *The front speed, c_ϵ, of the solution obtained above is strictly positive.*

Proof. We first obtain an equality for use later. Define the function $\Phi_n(s, x, z) = T_n^m(s + a_n, x, z)$ on the set $[-2a_n, 0] \times \Omega_p$. Similarly, define $U_n(s, x, z) = u_n^m(s + a_n, x, z)$, and let V and W be the component functions of U, i.e. let $U = (V, W)$. Using our a priori bounds for T_n^m and u_n^m, we can take the limit, along a subsequence if necessary, to obtain functions Φ and U which satisfy the equation

$$-c_\epsilon \Phi_s + U \cdot \tilde{\nabla} \Phi = L_\epsilon \Phi \tag{4.3.4}$$

on the set $(-\infty, 0] \times \Omega_p$. Moreover, $\Phi(0, x, z) = 0$ for all $(x, z) \in \Omega_p$. By our choice of translation, we get that $0 \leq \Phi \leq \theta_0$. Now, integrating equation (4.3.4) and taking limits as s tends to $-\infty$ gives us

$$c_\epsilon \lim_{s \to -\infty} \int_{\Omega_p} \Phi(s, x, z) dx dz = (1 + \epsilon) \int_{\Omega_p} \Phi_s(0, x, z) dx dz \leq 0. \tag{4.3.5}$$

Now we will proceed with the proof. Notice that by integrating the equation for T_n we arrive at the following

$$\begin{aligned} c_n|\Omega_p| - &\int_{\Omega_p} u_1^m(-a_n, x, z) dx dz \\ &= (1+\epsilon) \int_{\Omega_p} \left[\frac{\partial T_n^m}{\partial s}(a_n, x, z) - \frac{\partial T_n^m}{\partial s}(-a_n, x, z) \right] dx dz + \int f(T_n^m) dx dz ds \\ &\geq (1+\epsilon) \int_{\Omega_p} \frac{\partial T_n^m}{\partial s}(a_n, x, z) + \int f(T_n^m) dx dz ds \end{aligned}$$

The inequality follows from the non-positivity of $\frac{\partial T_n^m}{\partial s}(-a_n, x, z)$, since $T_n^m(-a_n, x, z) = 1$ and since $T_n^m \leq 1$ on R_a, by the maximum principle. First, we show that the second term on the first line is equal to zero. To see this, notice that $u_1^m = \Psi_z^m$. Hence we simply integrate to get

$$\int_{\Omega_p} u_1^m(-a_n, x, z) dx dz = \int_{\partial\Omega_p} \eta_2 \Psi^m(-a_n, x, z) dS(x, z) = 0$$

The last equality follows from the boundary conditions for Ψ^m. Hence we arrive at

$$c_n|\Omega_p| \geq (1+\epsilon) \int_{\Omega_p} \frac{\partial T_n^m}{\partial s}(a_n, x, z) dx dz + \int f(T^m) dx dz ds. \tag{4.3.6}$$

Taking the limit as n tends to infinity in equation (4.3.6) and using the equation (4.3.5), we arrive at

$$\begin{aligned} c_\epsilon|\Omega_p| &\geq c_\epsilon \lim_{s \to -\infty} \int_{\Omega_p} \Phi(s, x, z) dx dz + \int f(T^m) dx dz ds \\ &\geq -|c_\epsilon||\Omega_p|\theta_0 + \int f(T^m) dx dz ds. \end{aligned}$$

Hence, using Lemma 4.3.1 we arrive at

$$c_\epsilon \geq \frac{C_\epsilon}{|\Omega_p|(1-\theta_0)} > 0. \tag{4.3.7}$$

This completes the proof of the lemma. $\square$

Lemma 4.3.3. *There exists a sequence $a_n \to \infty$ such that*

$$\lim_{n\to\infty}\left|\frac{\partial T_n^m}{\partial s}(a_n, x, z)\right| = 0$$

uniformly in x and z. Moreover, we have

$$\lim_{n\to\infty} T_n(a_n - s_0, x, z) = 0$$

for all $s_0 \in \mathbb{R}$.

Proof. Recall the function Φ from Lemma 4.3.2. We will show that $\Phi(s, x, z)$ converges to a constant as $s \to -\infty$. To see this, assume we have two sequences (s_k, x_k, z_k) and (s'_k, x'_k, z'_k) such that $s_k < s'_k < s_{k+1}$, s_k and s'_k tend to $-\infty$, and $T^m(s_k, x_k, z_k)$ converges to θ and $T^m(s'_k, x'_k, z'_k)$ converges to θ'. Then we integrate equation (4.3.4) over $[s_k, s'_k] \times \Omega_p$ to obtain

$$\begin{aligned} c_\epsilon \int_{\Omega_p} (\Phi(s'_k, x, z) - \Phi(s_k, x, z))dxdz + \int_{\Omega_p} (V(s'_k, x, z)\Phi(s'_k, x, z) - V(s_k, x, z)\Phi(s_k, x, z))dxdx \\ = (1+\epsilon)\int_{\Omega_p} (\Phi_s(s'_k, x, z) - \Phi_s(s_k, x, z))dxdz \end{aligned}$$

Notice that both U and $\nabla_{s,x,z}\Phi$ tend uniformly to zero as $s \to -\infty$ since both are uniformly bounded in L^2 and in $C^{0,\alpha}$. Hence taking the limit as k tends to infinity, we obtain

$$c_\epsilon|\Omega_p|(\theta' - \theta) = 0.$$

Hence, since $c_\epsilon > 0$, it follows that Φ converges to a constant on the left, call it Φ^-. Integrating equation (4.3.4) gives us that

$$c_\epsilon|\Omega_p|\Phi^- = 0$$

Hence $\Phi^- = 0$. Then the maximum principle assures us that $\Phi \equiv 0$, finishing the proof. □

The fluid velocity on the right

Here we will show that the fluid velocity on the finite cylinder, u_n^m, converges to zero uniformly in n as $s \to \infty$. This is necessary for proving that the temperature, T_n^m, has the same behavior.

Lemma 4.3.4. *For each $\mu, \delta, \epsilon > 0$, there exists $R_\epsilon = R(\mu, \delta, \epsilon) < \infty$ such that, for all n and for all $s \geq R_\epsilon$, we have that $|u_n^m(s, x, z)| \leq \mu$ for all $(s, x, z) \in [R_\epsilon, a_n] \times \Omega_p$.*

Proof. We argue by contradiction. Suppose there is some $\mu > 0$ and some sequence $(s_n, x_n, z_n) \in [-a_n, a_n] \times \Omega_p$, with s_n tending to infinity, such that $|u_n^m(s_n, x_n, z_n)| \geq \mu$. Define the recentered functions $\Phi_n(s, x, z) = T_n^m(s + s_n, x, z)$, $W_n(s, x, z) = \omega_n^m(s + s_n, x, z)$, $U_n(s, x, z) = u_n^m(s + s_n, x, z)$ and $S_n(s, x, z) = \Psi_n^m(s + r_n, x, z)$ on $[-a_n - s_n, a_n - s_n] \times \Omega_p$. There are two cases.

Case 1: $a_n - s_n \to \infty$. Since all the recentered functions satisfy the same bounds as the usual functions, we can take limits in all the relevant topologies, to get functions Φ, W, U, and S, which satisfy the following equations on $\mathbb{R} \times \Omega_p$

$$\begin{aligned} -c_\epsilon \Phi_s + u \cdot \tilde{\nabla}\Phi &= L_\epsilon \Phi \\ -c_\epsilon W_s + L_\epsilon W &= \hat{e} \cdot \tilde{\nabla}\Phi \\ L_\epsilon S &= \widetilde{W} \\ U &= (S_z, S_x + S_s) \end{aligned} \tag{4.3.8}$$

with the usual boundary conditions. The first equation is linear since, by our choice of s_n, we have that $0 \leq \Phi \leq \theta_0$. Similarly as in Lemma 4.3.3, we can show that as s tends to $-\infty$ and ∞, $\Phi(s, x, z)$ tends to Φ^- and Φ^+, respectively, uniformly in x and z. Hence integrating the first equation in equation (4.3.8), we arrive at

$$c_\epsilon |\Omega_p| (\Phi^- - \Phi^+) = 0.$$

Since $c_\epsilon > 0$ by Lemma 4.3.2, then we have that the $\Phi^- = \Phi^+$. By the maximum principle, we have that Φ is constant. As a result of this, the Dirichlet boundary conditions, and equation (4.3.8), we have that W and S must be zero. Hence U is equal to 0. However, since U_n converges locally to U in $C^{0,\alpha}$ and $\max_{x,z} U_n(0, x, z) \geq \delta$ for every n, then $U(0, x, z) \geq \delta > 0$ for some $(x, z) \in \Omega_p$ as well. This is a contradiction.

Case 2: $a_n - s_n \to b \in [0, \infty)$. By Lemma 4.3.3, $\Phi \equiv 0$. Hence, we argue as above to conclude that $U \equiv 0$. This is a contradiction, as before. □

Corollary 4.3.5. *For every ϵ sufficiently small, every n large enough, and $0 < \alpha \leq c_\epsilon/8$, there are constants $C_\alpha, R_\epsilon > 0$, which depend only on α and ϵ, respectively, such that*

$$T_n^m(s, x, z) \leq C_\alpha e^{-\alpha(s - R_\epsilon)}.$$

As a result

$$T^{m,\epsilon} \leq C_\alpha e^{-\alpha(s - R_\epsilon)}.$$

Proof. Recall that $\epsilon < 1/2$. Choose n large enough that $|c_\epsilon - c_{\epsilon, a_n}| < c_\epsilon/2$. Then, as we did in the proof of Lemma 4.2.2, let ψ_e be the principal eigenfunction of the operator $-\Delta + 2\alpha\partial_x$ on Ω_p with periodic boundary conditions on P and satisfying $-\alpha\eta_1\psi_e + \eta \cdot \nabla\psi_e = 0$ on B. Then we choose R such that

$$\|u^m\|_{L^\infty([R, a_n] \times \Omega_p)} \leq \min\left\{ \frac{c_\epsilon}{16}, \frac{\alpha c_\epsilon}{16 \| \frac{\nabla \psi_e}{\psi_e} \|_{L^\infty}} \right\}.$$

Then, letting $\gamma_A(s,x,z) = Ae^{-\alpha(s-R_\epsilon)}$, we see that this is a supersolution on $[R, a_n] \times \Omega_p$ since

$$\begin{aligned}
&-c_{\epsilon,a_n}(\gamma_A)_s + u\cdot\tilde{\nabla}\gamma_A - L_\epsilon\gamma_A \\
&\qquad = Ae^{-\alpha(s-R_\epsilon)}[(c_{\epsilon,a_n,\delta}\alpha - (1+\epsilon)\alpha^2)\psi_e - \alpha u_a\psi_e + u\cdot\nabla\psi_e - \Delta\psi_e + 2\alpha(\psi_e)_x] \\
&\qquad \geq Ae^{-\alpha(s-R_\epsilon)}\left[\left(\frac{c_\epsilon}{2} - (1+\epsilon)\alpha\right)\alpha\psi_e - \alpha u_a\psi_e + u\cdot\nabla\psi_e\right] \\
&\qquad \geq Ae^{-\alpha(s-R_\epsilon)}\left[\alpha\psi_e\left(\frac{c_\epsilon}{2} - \frac{c_\epsilon}{4}\right) - \alpha\psi_e\frac{c_\epsilon}{16} - \psi_e\frac{\alpha c_\epsilon}{16}\right] \\
&\qquad = Ae^{-\alpha(s-R_\epsilon)}\psi_e\alpha\frac{c_\epsilon}{8} \geq 0.
\end{aligned}$$

As in Lemma 4.2.2, we can show that if A is large enough, then $T^m \leq \gamma_A$ on $[R, a_n] \times \Omega_p$. We can also argue that this inequality holds for all $A \geq A_0$ where $A_0 = \|\psi_e^{-1}\|_\infty$. Hence $T_n^m \leq \gamma_{A_0}$ on $[R, a_n] \times \Omega_p$ when n is sufficiently large, which finishes the proof. □

The left and right limits of the temperature

We will now show that T^m converges to constants uniformly as $s \to \pm\infty$.

Lemma 4.3.6. *There exists limits $\theta_\pm$ such that*

$$\lim_{s\to\pm\infty} \|T^m(s,x,z) - \theta_\pm\|_{L^\infty(\Omega_p)} = 0. \tag{4.3.9}$$

Moreover, the limit on the right hand side is zero and the limit on the left hand side is positive.

Proof. The limit on the right hand side follows from Corollary 4.3.5. Hence we need only consider the limit on the left hand side. The proof of this is handled exactly as in Lemma 4.3.3.

The fact that this limit is positive follows from integrating equation (4.3.1) to obtain

$$c_\epsilon|\Omega_p|\theta_- = \int_{\mathbb{R}\times\Omega_p} f(T^m)dxdzds. \tag{4.3.10}$$

As c_ϵ is positive, positivity of θ_- is equivalent to the positivity of the integral on the right. However, if the integral were zero, T^m would solve an elliptic problem and achieves a maximum, θ_0, in the interior of the domain. This contradicts the maximum principle. Hence, it must be that

$$\int_{\mathbb{R}\times\Omega_p} f(T^m)dxdzds > 0,$$

and thus θ_- is positive. □

Bounds in the stationary frame

We will now change variables back to the stationary frame. We will obtain uniform bounds in ϵ in order to pass to the limit $\epsilon \downarrow 0$. Then we will only need to deal with the δ-dependence and the boundary conditions in the next section in order to finish the proof of Theorem 4.1.1.

Define the function $T^\epsilon(t,x,z) = T^m(x - c_\epsilon t, x, z)$ on $\mathbb{R}\times\Omega$. Make the same change of coordinates to define the functions ω^ϵ, u^ϵ, and Ψ^ϵ. We first get some local, uniform bounds on these functions. Here we follow the work of Berestycki and Hamel in [9].

Lemma 4.3.7. *For every compact set $K \subset \Omega$, there is a constant $C = C(\mathrm{diam}(K)) > 0$ that does not depend on ϵ such that*

$$\int_{\mathbb{R}\times K} |\nabla T^\epsilon|^2 dt dx dz \leq C. \tag{4.3.11}$$

Proof. We begin by noting that $0 \leq T^m \leq 1$ and that T^m_x, T^m_s, and T^m_z tend to zero uniformly as s tends to infinity since T^m is bounded in $C^{1,\alpha}$ and $\nabla_{s,x,z} T^m$ is bounded in L^2 (by constants possibly depending on ϵ). Hence we can multiply equation (4.3.1) by T^m and integrate to obtain

$$c_\epsilon |\Omega_p| \frac{\theta_-^2}{2} + \epsilon \int_{\mathbb{R}\times\Omega_p} (T^m_s)^2 dx dz ds + \int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla} T^m|^2 dx dz ds = \int_{\mathbb{R}\times\Omega_p} T^m f(T^m) dx dz ds.$$

Now, using equation (4.3.10), we get

$$\epsilon \int_{\mathbb{R}\times\Omega_p} (T^m_s)^2 dx dz ds + \int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla} T^m|^2 dx dz ds \leq c_\epsilon |\Omega_p| \left(\theta_- - \frac{\theta_-^2}{2}\right).$$

Ignoring the first term and changing variables appropriately, we arrive at

$$\int_{\mathbb{R}\times\{(x,z)\in\Omega : x \in (k\ell,(k+1)\ell)\}} |\nabla T|^2 dx dz \leq |\Omega_p| \left(\theta_- - \frac{\theta_-^2}{2}\right),$$

for any integer k. To finish the proof, simply notice that given a compact set $K \subset \Omega$, we can find $k_1, \ldots, k_n \in \mathbb{Z}$, where n depends only on the diameter of K, such that

$$K \subset \{(x,z) : (x,z) \in \Omega, x \in [k_i L, (k_i+1)L] \text{ for some } i\}.$$

$\square$

Lemma 4.3.8. *For every compact set $K \subset \Omega$, there is a constant $C = C(\mathrm{diam}(K)) > 0$ that does not depend on ϵ such that*

$$\int_{\mathbb{R}\times K} |\nabla \omega^\epsilon|^2 dt dx dz + \int_{\mathbb{R}\times K} |\omega^\epsilon_t|^2 dt dx dz \leq C_K. \tag{4.3.12}$$

Proof. Since $\omega^m \in H^1(\mathbb{R}\times\Omega_p) \cap C^{1,\alpha}(\mathbb{R}\times\Omega_p)$, then ω^m and its first derivatives tend to zero uniformly as s tends to infinity. Hence, multiplying equation (4.3.1) by ω^m and integrating over

$\mathbb{R} \times \Omega_p$ yields

$$\epsilon \int_{\mathbb{R}\times\Omega_p} (\omega_s^m)^2 dsdxdz + \int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla}\omega^m|^2 dxdzds = \int_{\mathbb{R}\times\Omega_p} \omega^m(\hat{e}\cdot\tilde{\nabla}T^m)dsdxdz$$

Using Poincaré's inequality and the non-negativity of $(\omega_s^m)^2$, we arrive at

$$\int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla}\omega^m|^2 dxdzds \leq \left(\int_{\mathbb{R}\times\Omega_p} (\omega_z^m)^2 dsdxdz\right)^{1/2} \left(\int_{\mathbb{R}\times\Omega_p} (\tilde{\nabla}T^m)^2 dsdxdz\right)^{1/2}$$

Changing variables and using Lemma 4.3.7, finishes the bound on the spatial gradient.

For the bound on the time derivative, first notice that ω_s^m is zero on one part of the boundary, and periodic on the other, and tends to zero as s tends to zero. If we multiply equation (4.3.1) by ω_s^m, we integrate to obtain

$$\begin{aligned} c_\epsilon \int_{\mathbb{R}\times\Omega_p} (\omega_s^m)^2 dsdxdz &= -\int_{\mathbb{R}\times\Omega_p} \omega_s^m(\hat{e}\cdot\tilde{\nabla}T^m)dsdxdz - \int_{\mathbb{R}\times\Omega_p} \omega_s^m L_\epsilon \omega^m dsdxdz \\ &\leq \left(\int_{\mathbb{R}\times\Omega_p} (\omega_s^m)^2 dxdzds\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla}T^m|^2 dxdzds\right)^{\frac{1}{2}} \\ &\quad + \int_{\mathbb{R}\times\Omega_p} \left(\tilde{\nabla}\omega^m\cdot\tilde{\nabla}\omega_s^m + (1+\epsilon)\omega_s\omega_{ss}\right) dxdsdz. \end{aligned} \tag{4.3.13}$$

The last line comes from applying the Cauchy-Schwarz inequality to the first term and integrating the second term by parts. Since the integrand of the last term is

$$\frac{1}{2}\left(\partial_s|\tilde{\nabla}\omega^m|^2 + \partial_s(\omega_s^m)^2\right),$$

it integrates to zero. Hence, equation (4.3.13) becomes

$$\int_{\mathbb{R}\times\Omega_p} (c_\epsilon\omega_s^m)^2 dxdzds \leq \int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla}T^m|^2 dxdzds.$$

Changing variables (which removes the factor of c_ϵ) and using Lemma 4.3.7 finishes the proof. □

Next we will prove bounds on the stream function and the fluid velocity using our estimates above. However, since the equation for the stream function degenerates in the t variable, we will first prove a general result concerning Hölder norms for degenerate families of elliptic equations.

Lemma 4.3.9. *Let U be a smooth open domain in $\mathbb{R}^n$, and let $\alpha \in (0,1)$. Suppose that ϕ is a*

$C^2(\mathbb{R}\times U)$ solution to

$$\begin{aligned}\beta^2\phi_{tt}+\Delta\phi &= g \qquad \textit{on } \mathbb{R}\times U\\ \phi &= 0 \qquad \textit{on } \mathbb{R}\times\partial U\end{aligned}$$

where g is in bounded in $C^{0,\alpha}(\mathbb{R}\times U)$ and where $0<\beta<\beta_0$. Then there exists a constant C, which depends only on β_0, α, and the smoothness of the boundary of U, such that

$$\|\phi(t,\cdot)\|_{C^{2,\alpha}(U)}\le C\left(\|\phi\|_{C^0(\mathbb{R}\times U)}+\|g\|_{C^{0,\alpha}(\mathbb{R}\times U)}\right),$$

for every $t\in\mathbb{R}$.

Proof. Define $\phi_\beta(t,x)=\phi(\beta t,x)$ and $g_\beta(t,x)=g(\beta t,x)$. Then ϕ_β satisfies

$$\begin{aligned}\Delta_{t,x}\phi_\beta &= g_\beta \qquad \text{on } \mathbb{R}\times U\\ \phi_\beta &= 0 \qquad \text{on } \mathbb{R}\times\partial U.\end{aligned}$$

The usual Schauder estimates tell us that

$$\|\phi_\beta\|_{C^{2,\alpha}}\le C\left(\|\phi\|_{C^0(\mathbb{R}\times U)}+\|g_\beta\|_{C^{0,\alpha}(\mathbb{R}\times U)}\right).$$

Notice that

$$\|g_\beta\|_{C^{0,\alpha}(\mathbb{R}\times U)}\le(1+\beta^\alpha)\|g\|_{C^{0,\alpha}(\mathbb{R}\times U)}.$$

Hence we get that

$$\|\phi(t,\cdot)\|_{C^{2,\alpha}(U)}\le C\left(\|\phi\|_{C^0(\mathbb{R}\times U)}+(1+\beta_0)^\alpha\|g\|_{C^{0,\alpha}(\mathbb{R}\times U)}\right).$$

□

In order to use the result above in the equation for Ψ, we need to show that ϵ/c_ϵ^2 is bounded uniformly. We do this below. This will also be a crucial Lemma when we eventually show that the front speed, c_ϵ, is bounded away from 0 uniformly in ϵ.

Lemma 4.3.10. *There exists a universal constant $C>0$ such that*

$$\frac{c_\epsilon^2}{\epsilon}\ge C$$

Proof. To begin, assume n is large enough so that

$$\frac{c_\epsilon}{2}\le c_{n,\epsilon}\le\frac{3c_\epsilon}{2}.$$

Recall from Lemma 4.3.2 that for each n,

$$\left(\int f(T_n^{m,\epsilon})dxdzds\right)\left(\int |(T_n^{m,\epsilon})_s|^2 dxdzds\right) \geq C.$$

Multiplying equation (4.2.1) by $(T_n^m - 1)$, integrating over R_{a_n}, and using that $(T_n^m - 1)f(T_n^m) \leq 0$, we obtain

$$\epsilon \int |(T_n^{m,\epsilon})_s|^2 dxdzds \leq \frac{c_\epsilon|\Omega_p|}{2} - (1+\epsilon)\int_{\Omega_p} (T_n^{m,\epsilon})_s(a_n, x, z)dxdz.$$

Since, by Lemma 4.3.3, the second term here tends to zero uniformly, then by choosing n large enough, we can combine the above equations to get

$$\frac{c_\epsilon}{\epsilon}\int f(T_n^{m,\epsilon})dxdzds \geq C.$$

To get a bound on the reaction term, we simply take the equation for $T_n^{m,\epsilon}$ in equation (4.2.1) and integrate it to get

$$c_{\epsilon,n}|\Omega_p| + (1+\epsilon)\int_{\Omega_p} \left((T_n^{m,\epsilon})_s(-a_n, x, z) - (T_n^{m,\epsilon})_s(a_n, x, z)\right) dxdz = \int f(T_n^{m,\epsilon})dxdzds.$$

The first integral has two terms in it. One is non-positive and the other tends uniformly to zero as n tends to zero by Lemma 4.3.3. Hence we combine this with the above equation to obtain the desired inequality. □

Because of the degeneracy in the equation for Ψ, it will be convenient for us to use partial Sobolev norms in the following lemma. We define these below.

Definition 4.3.11. *If $U \subset \mathbb{R}\times\mathbb{R}^n$ is a simply connected domain with a Lipschitz boundary, then a function f is in the space $X^{i,j}(U) \subset L^2(U)$ if for every multi-index β, f satisfies*

$$\int_U |\partial_t^{\beta_0}\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n} f(x)|^2 dx < \infty,$$

where $\beta_0 \leq i$ and $|\beta| \leq j$. We endow the space with the norm

$$\|f\|^2_{X^{i,j}(U)} = \sum_{\substack{|\beta|\leq j,\\ \beta_1 \leq i}} \int_U |\partial_{x_1}^{\beta_1}\cdots\partial_{x_{n+1}}^{\beta_{n+1}} f(x)|^2 dx.$$

Lemma 4.3.12. *For every compact set $K \subset \Omega$, there is a constant $C = C(\mathrm{diam}(K)) > 0$ that does not depend on ϵ such that*

$$\int_{\mathbb{R}\times K} |\nabla u^\epsilon|^2 dtdxdz + \int_{\mathbb{R}\times K} |u_t^\epsilon|^2 dtdxdz + \int_{\mathbb{R}\times K} |\nabla\Psi^\epsilon|^2 dtdxdz + \int_{\mathbb{R}\times K} |\Psi_t^\epsilon|^2 dtdxdz \leq C_K. \quad (4.3.14)$$

In addition, there is a constant $C_\delta > 0$, depending only on δ, such that the following bounds hold on $\mathbb{R} \times \Omega$:

$$\|u^\epsilon\|_{L^\infty} \leq C_\delta.$$

Proof. We first prove the Sobolev bounds. Multiply equation equation (4.3.1) by $\Psi^{m,\epsilon}$ and integrate by parts. The boundary terms vanish because $\Psi^{m,\epsilon}$ and its derivatives vanish at infinity. We obtain

$$\int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla}\Psi^{m,\epsilon}|^2 dsdxdz \leq C\|\Psi^m\|_{L^2(\mathbb{R}\times\Omega_p)}\|\tilde{\omega}^{m,\epsilon}\|_{L^2(\mathbb{R}\times\Omega_p)}.$$

Then we change variables and use the Poincaré inequality to finish the bound on $\nabla\Psi^\epsilon$. We get a similar estimate for $\nabla\Psi^\epsilon_t$ by differentiating equation (4.3.1) in s and then arguing in the same manner as above, which gives us the bound on u^ϵ_t. The bound on Ψ^ϵ_t follows then from the Poincaré inequality again. Finally, we need to get the bound on ∇u^ϵ. Note that the usual elliptic estimates, combined with our work above, give us that

$$\|\Psi^\epsilon\|_{H^3(\mathbb{R}\times K)} \leq C\frac{c_\epsilon^2}{\epsilon}\|\tilde{\omega}^\epsilon\|_{H^1(\mathbb{R}\times K)}.$$

The constant, C, in the equation above is universal. In particular, we have that $(\epsilon/c_\epsilon^2)\Psi^\epsilon_{tt}$ is bounded in L^2. Since Ψ^ϵ satisfies

$$\Delta\Psi^\epsilon = \tilde{\omega}^\epsilon - \frac{\epsilon}{c_\epsilon^2}\Psi^\epsilon_{tt},$$

we obtain, via the usual elliptic estimates in the spatial coordinates,

$$\|\Psi^\epsilon\|_{X^{0,3}(\mathbb{R}\times K)} \leq C\left(\|\tilde{\omega}^\epsilon\|_{H^1(\mathbb{R}\times K)} + \left\|\frac{\epsilon}{c_\epsilon^2}\Psi^\epsilon_{tt}\right\|_{H^1(\mathbb{R}\times K)}\right) \leq C\|\tilde{\omega}^\epsilon\|_{H^1(\mathbb{R}\times K)}.$$

The last bound comes from differentiating equation (4.3.1) in s, multiplying it by Ψ_s, integrating by parts, and using the Cauchy-Schwarz and Poincaré inequalities to obtain

$$\int_{\mathbb{R}\times\Omega_p} |\tilde{\nabla}\Psi^\epsilon_s|^2 dxdzds \leq C\int_{\mathbb{R}\times\Omega_p} |\tilde{\omega}^\epsilon_s|^2 dxdzds.$$

Changing variables and arguing as we did above gives us

$$\|\Psi^\epsilon_t\|_{X^{0,2}(\mathbb{R}\times K)} \leq C\left(\|\tilde{\omega}^\epsilon_t\|_{L^2(\mathbb{R}\times K)} + \left\|\frac{\epsilon}{c_\epsilon^2}\Psi^\epsilon_{ttt}\right\|_{L^2(\mathbb{R}\times K)}\right) \leq C\|\tilde{\omega}^\epsilon\|_{H^1(\mathbb{R}\times K)}$$

Combining this with our estimates above yields

$$\|\Psi^\epsilon\|_{X^{1,3}(\mathbb{R}\times K)} \leq C\|\omega^\epsilon\|_{H^1(\mathbb{R}\times K)},$$

where C is a universal constant.

Notice that our work above gave us an L^∞ bound on Ψ^ϵ by Theorem 2.2.6 from [49]. Moreover, Ψ^ϵ satisfies the equation

$$\frac{\epsilon}{c_\epsilon^2}\Psi^\epsilon_{tt} + \Delta\Psi^\epsilon = \tilde{\omega}^\epsilon.$$

Hence, applying Lemma 4.3.9 and noting Lemma 4.3.10 finishes the proof for us.

□

Now that we have a bound for u which is independent of ϵ, we may get an upper bound on $c_{\epsilon,\delta}$ which is independent of ϵ.

Corollary 4.3.13. *There exists a universal constant $C > 0$ such that*

$$c_{\epsilon,\delta} \leq C(1 + \|u^{\epsilon,\delta}\|_{L^\infty}).$$

Proof. We can assume without loss of generality that $c_\epsilon \geq 24$ for otherwise there is nothing to prove. Choose n large enough such that $c_{n,\epsilon} \geq c_\epsilon/2$. After possibly passing to a subsequence, we may assume that $u_n^{m,\epsilon}$ satisfies

$$\|u_n^{m,\epsilon}\|_{L^\infty([-n,n]\times\Omega_p)} \leq 2\|u^{m,\epsilon}\|_{L^\infty}.$$

Let $\gamma(s,x,z) = \psi_e(x,z)e^{-(s+n)}$ where ψ_e to be the principal eigenfunction of the operator

$$-\Delta + 2\partial_x,$$

with the boundary conditions $\eta_1\psi_e - \eta\cdot\nabla\psi_e = 0$ on B and periodic on the boundary P. Letting $A_0 = \|\psi_e^{-1}\|_\infty$ and arguing as before, we have that $\gamma_{A_0}(s) \geq T_n^m$ on $[-n,n]\times\Omega_p$ as long as the following conditions hold

$$\begin{aligned} c_{n,\epsilon} &\geq M + 2 + \|u_n^m\|_{L^\infty([-n,n]\times\Omega_p)}\left(1 + \left\|\frac{\nabla\psi_e}{\psi_e}\right\|_{L^\infty\Omega_p}\right) \\ A_0e^{-2n} &\geq T_n^m(n,x,z). \end{aligned} \tag{4.3.15}$$

Notice that the second equation holds for large enough n, by simply taking $\alpha = 3$ in Corollary 4.3.5. Notice that we can apply Corollary 4.3.5 because of the assumption that $c_\epsilon \geq 24$. Hence, if the first equation holds then we obtain

$$A_0e^{-n} \geq \gamma(0) \geq \max T_n^m(0,x,z) = \theta_0.$$

This is clearly not true for n large. Hence it must be that the first inequality in equation (4.3.15) is false. This leads to the inequality

$$c_\epsilon \le 2c_{n,\epsilon} \le 2\left(M + 2 + 2\|u^m\|_{L^\infty(\mathbb{R}\times\Omega_p)}\left(1 + \left\|\frac{\nabla\psi_e}{\psi_e}\right\|_{L^\infty\Omega_p}\right)\right).$$

This finishes the proof. □

Finally, we get an L^2 bound on the time derivative of the temperature.

Lemma 4.3.14. *For every compact set $K \subset \Omega$, there is a constant $C_\delta = C(\delta, \mathrm{diam}(K)) > 0$, which depends only on δ and the diameter of K, such that*

$$\int_{\mathbb{R}\times K} (T_t^\epsilon)^2 \le C_\delta. \tag{4.3.16}$$

Proof. Multiply equation (4.3.1) by T_s^m, integrate over $[-A, A]\times\Omega_p$, and integrate by parts to obtain

$$\begin{aligned}
-c_\epsilon \int_{[-A,A]\times\Omega_p} (T_s^m)^2 ds dx dz - (1+\epsilon)\int_{[-A,A]\times\Omega_p} T_{ss}^m T_s^m dx dz ds \\
+ \int_{[-A,A]\times\Omega_p} \nabla_{x,y}T^m \cdot \nabla_{x,y}T_s^m dx dz ds + \frac{1+\epsilon}{2}\int_{\Omega_p} [(T_s^m)^2]_{-A}^A dx dz \\
+ \int_{[-A,A]\times\Omega_p} (u^m\cdot\tilde{\nabla}T^m)T_s^m dx dz ds = \int_{[-A,A]\times\Omega_p} T_s^m f(T) dx dz ds.
\end{aligned} \tag{4.3.17}$$

First notice that, as usual, the boundary terms will tend to zero as A tends to infinity. The second and third terms will tend to zero as we take A to infinity since we can write the integrands as $\partial_s(T_s^m)^2$ and $\partial_s|\tilde{\nabla}T^m|^2$, respectively. Hence, there is a function $\eta(A)$ which tends to zero as A tends to infinity which bounds the second, third, and fourth term. We write the last term as

$$\int_{[-A,A]\times\Omega_p} T_s^m f(T^m) dx dz ds = \int_{\Omega_p} [F(T^m)]_{-A}^A dx dz$$

where

$$F(t) = \int_0^t f(\tau)d\tau.$$

Let C be a uniform bound on $f(t)$. Then, combining this with equation (4.3.17), we arrive at

$$\begin{aligned}
c_\epsilon \int_{[-A,A]\times\Omega_p} (T_s^m)^2 ds dx dz &\le \eta(A) - \int_{[-A,A]\times\Omega_p} (u^m\cdot\tilde{\nabla}T^m)T_s^m dx dz ds - \int_{\Omega_p} [F(T^m)]_{-A}^A dx dz \\
&\le \eta(A) - \int_{[-A,A]\times\Omega_p} (u^m\cdot\tilde{\nabla}T^m)T_s^m dx dz ds + C \\
&\le \eta(A) + \int_{[-A,A]\times\Omega_p} \frac{\|u^m\|_{L^\infty}}{2}\left(\alpha|\tilde{\nabla}T^m|^2 + \frac{(T_s^m)^2}{\alpha}\right) dx dz ds + C.
\end{aligned} \tag{4.3.18}$$

Here, we used the boundedness of F and the bound on u^m that is uniform in ϵ. The following holds for any choice of α which is positive. If $u^m \equiv 0$ then the desired inequality holds by simply skipping the last step in the calculations above. Otherwise, let $\alpha = \|u^m\|_{L^\infty}/c_\epsilon$. Then, taking $A \to \infty$, we arrive at

$$\frac{c_\epsilon}{2}\int_{\mathbb{R}\times\Omega_p}(T_s^m)^2 dxdzds \leq C + \int_{\mathbb{R}\times\Omega_p}\frac{\|u^m\|_\infty^2}{2c_\epsilon}|\tilde{\nabla}T^m|^2 dxdzds.$$

From Lemma 4.3.7, we get that

$$\int_{\mathbb{R}\times\Omega_p}(c_\epsilon T_s^m)^2 dxdzds \leq C_\delta c_\epsilon.$$

Changing variables and arguing as in Lemma 4.3.7, finishes the proof. □

In order to take the limit as ϵ tends to zero we need a lower bound on the front speed c_ϵ. We obtain that here, in the next lemmas. Recall that we have already eliminated the troublesome case where c_ϵ tends to zero faster than $\sqrt{\epsilon}$ in Lemma 4.3.10.

A lower bound on the front speed

Now we will complete the proof that c_ϵ is uniformly bounded away from zero as ϵ tends to zero.

Proposition 4.3.15. *There exists a constant $C_\delta > 0$ that depends only on $\delta > 0$ so that $c_\epsilon > C_\delta$ for all ϵ sufficiently small.*

Proof. We will prove this by contradiction. Assume that there is a sequence $\epsilon_n \to 0$ such that $c_{\epsilon_n} \to 0$ as well. For simplicity, we drop the n from the notation. Notice that ϵ/c_ϵ^2 is uniformly bounded so it must converge, along a subsequence if necessary, to a constant $\kappa \geq 0$. In order to come to a contradiction, we build new solutions to our equations as follows. Let

$$s_n = \sup\left\{s : \frac{1+\theta_0}{2} = \min_{r\leq s}\frac{1}{|\{(x,z)\in\Omega_p : x\in[0,c_\epsilon]\}|}\int_0^{c_\epsilon}\int_{\Omega_p(x)}T_n^m(r,x,z)dxdz\right\}, \qquad (4.3.19)$$

where $\Omega_p(x) = \{z \in \mathbb{R} : (x,z) \in \Omega_p\}$, and define re-centered functions

$$\Phi_n^m(s,x,z) = T_n^m(s+s_n,x,z), \quad U_n(s,x,z) = u(s+s_n,x,z).$$

These functions satisfy the same equations and bounds as before on $[-a_n - s_n, a_n - s_n]\times\Omega_p$ so we can take the limit as n tends to infinity. We let

$$b_\epsilon = \lim_{n\to+\infty}(-a_n - s_n).$$

For each ϵ, passing to the limit $n \to +\infty$ we also get solutions $\Phi_\epsilon^m, U_\epsilon^m$ to equation (4.3.1) on $(-b_\epsilon,\infty)\times\Omega_p$. Changing variables to the stationary frame, we arrive at solutions Φ^ϵ, U^ϵ, which solve the equation

$$\Phi_t^\epsilon + U^\epsilon \cdot \nabla\Phi^\epsilon = \Delta\Phi^\epsilon + \frac{\epsilon}{c_\epsilon^2}\Phi_{tt}^\epsilon + f(\Phi^\epsilon).$$

Let $\Omega(x) = \{z : (x, z) \in \Omega\}$. Notice that our choice of s_n in equation (4.3.19) gives us that

$$\frac{1}{\int_0^1 |\Omega(c_\epsilon t)|dt}\int_0^1\int_{\Omega(c_\epsilon t)} \Phi^\epsilon(t, c_\epsilon t, z)dzdt = \frac{1+\theta_0}{2} \tag{4.3.20}$$

and that

$$\frac{1}{\int_0^1 |\Omega(x_0 + c_\epsilon t)|dt}\int_0^1\int_{\Omega(x_0+c_\epsilon t)} \Phi^\epsilon(t, x_0 + c_\epsilon t, z)dzdt \geq \frac{1+\theta_0}{2} \tag{4.3.21}$$

as long as $b_\epsilon \leq x_0 \leq 0$. Since we have a uniform H^1_{loc} bound on both functions, we can take ϵ to zero (along a subsequence if necessary) to get functions Φ, U which weakly solve the equation

$$\Phi_t + U \cdot \nabla\Phi = \Delta\Phi + \kappa\Phi_{tt} + f(\Phi). \tag{4.3.22}$$

Also, taking a subsequence if necessary, b_ϵ converges as $\epsilon \to 0$ to either a finite number or $-\infty$, and we set

$$b_0 = \lim_{\epsilon\to 0} b_\epsilon.$$

Equation (4.3.22) is posed on the set $\Omega^{b_0} = \{(t, x, z) \in \mathbb{R} \times \Omega : x \geq b_0\}$. Also, by using the trace theorem, we get, from equation (4.3.20) and equation (4.3.21) that

$$\frac{1}{|\Omega(0)|}\int_0^1\int_{\Omega(0)} \Phi^\epsilon(t, 0, z)dzdt = \frac{1+\theta_0}{2} \tag{4.3.23}$$

and that

$$\frac{1}{|\Omega(x_0)|}\int_0^1\int_{\Omega(x_0)} \Phi^\epsilon(t, x_0, z)dzdt \geq \frac{1+\theta_0}{2} \tag{4.3.24}$$

as long as $b_0 \leq x_0 \leq 0$. The last ingredient that we need is a global L^2 bound on the derivatives of some functions. Looking at how we obtained the L^2 gradient bounds earlier in this section, notice that if we had changed variables in a different way, we have, for each ϵ, bounds of the form

$$\begin{aligned}
\int_0^{\ell/c_\epsilon}\int_{\Omega^{b_\epsilon+c_\epsilon t}} |\Phi_t^\epsilon|^2 + |\nabla\Phi^\epsilon|^2 dxdzdt &= \int_{-\ell/c_\epsilon}^0\int_{\Omega^{b_\epsilon+c_\epsilon t}} |\Phi_t^\epsilon|^2 + |\nabla\Phi^\epsilon|^2 dxdzdt \leq C,\\
\int_0^{\ell/c_\epsilon}\int_{\Omega^{b_\epsilon+c_\epsilon t}} |U_t^\epsilon|^2 + |\nabla U^\epsilon|^2 dxdzdt &= \int_{-\ell/c_\epsilon}^0\int_{\Omega^{b_\epsilon+c_\epsilon t}} |U_t^\epsilon|^2 + |\nabla U^\epsilon|^2 dxdzdt \leq C
\end{aligned} \tag{4.3.25}$$

Similarly, from the identity $\int_{\mathbb{R}\times\Omega_p} f(T^{m,\epsilon})dxdzds \leq Cc_\epsilon$, we arrive at

$$\int_0^{\ell/c_\epsilon}\int_{\Omega^{b_\epsilon+c_\epsilon t}} f(\Phi^\epsilon)dxdzdt = \int_{-\ell/c_\epsilon}^0\int_{\Omega^{b_\epsilon+c_\epsilon t}} |f(\Phi^\epsilon)dxdzdt \leq C. \tag{4.3.26}$$

Now taking ϵ to zero in equations equation (4.3.25) and equation (4.3.26) gives us the following bounds

$$\int_{\mathbb{R}}\int_{\Omega^{b_0}}\left[f(\Phi)+|\nabla\Phi|^2+|\Phi_t|^2+|\nabla U|^2+|U_t|^2\right]dxdzdt\le C. \tag{4.3.27}$$

By parabolic or elliptic regularity (depending on whether $\kappa=0$ or $\kappa>0$, respectively), it follows that Φ has a uniform $C^{1,\alpha}$ bound in both space and time, see e.g. [34, 48, 49]. Let $0<\nu<\frac{1-\theta_0}{2}$ and choose μ small enough that $f(T)<\mu$ implies that $T\le\theta_0+\nu$ or $T\ge 1-\nu$. Then take R large enough such that if $|t|+|x|\ge R$, we have that $f(\Phi(t,x,z))<\mu$. This exists since $f(\Phi)$ satisfies a global L^2 bound and a global $C^{1,\alpha}$ bound. Since $\{|t|+|x|\ge R\}$ is connected and since $f(\Phi)$ is continuous then either $\Phi(t,x,z)\ge 1-\nu$ on $\{|t|+|x|\ge R\}$ or $\Phi(t,x,z)\le\theta_0+\nu$ on $\{|t|+|x|\ge R\}$.

We claim that the first possibility holds. To see this, we look at the two different cases separately: either $b_0=-\infty$ or $0\le -b_0<\infty$. In the first case, we use equation (4.3.24). In the second case we use the fact that $T(t,b_0,z)=1$ for all $t\in\mathbb{R}$ and $z\in\Omega(b_0)$.

To review, we have that if $|t|+|x|\ge R$ then $T(t,x,z)\ge 1-\nu$. In addition, there is some $t_0\in(0,1)$ and $z_0\in\Omega(0)$ such that $T(t_0,0,z_0)=\frac{1+\theta_0}{2}$. We claim that this leads to a contradiction. Look at the domain $[-R,R]\times\{(x,z)\in\Omega^{b_0}:|x|\le R\}$. Then the Neumann boundary conditions on T and the strong maximum principle (either for parabolic or elliptic equation, depending on whether $\kappa=0$ or $\kappa>0$, respectively) tells us that $T(t,x,z)\ge 1-\nu$ for every (t,x,z) in this domain. This is a contradiction since $T(t_0,0,z_0)=\frac{1+\theta_0}{2}$. $\square$

A direct result of this lemma is that we can estimate $\tilde{\omega}$ in any Sobolev norm by the L^2 norm of ω. This allows us to conclude that Ψ is smooth and hence, that its derivatives decay at infinity. We will need this in the next section in order to justify integration by parts when we obtain new estimates on our functions and to get a bound on the L^∞ norm of the ∇T.

Corollary 4.3.16. *For every compact set $K\subset\Omega$, there is a constant $C_\delta=C(k,\delta,\mathrm{diam}(K))>0$, which depends only on k, δ and the diameter of K, such that*

$$\|\Psi\|_{H^k(\mathbb{R}\times K)}\le C_\delta. \tag{4.3.28}$$

Proof. First notice that for any natural number k,

$$\|\tilde{\omega}\|_{H^k(\mathbb{R}\times K)}\le C\|\omega\|_{L^2(\mathbb{R}\times K)}.$$

The constant above depends only on δ. Then arguing as in Lemma 4.3.12 we see that

$$\int_{\mathbb{R}\times K}|\nabla\partial_t^j\Psi|^2dxdzdt\le C\int_{\mathbb{R}\times K}|\partial_t^j\tilde{\omega}|^2dxdzdt$$

and

$$\|\Psi\|_{H^{k+2}(\mathbb{R}\times K)}\le C\frac{c_\epsilon^2}{\epsilon}\left(\|\psi\|_{L^2(\mathbb{R}\times K)}+\|\tilde{\omega}\|_{H^k(\mathbb{R}\times K)}\right)\le C\frac{c_\epsilon^2}{\epsilon}\|\omega\|_{L^2(\mathbb{R}\times K)}.$$

Fix j and k and notice that $\partial_t^j \Psi$ satisfies

$$\begin{aligned}
\Delta \partial_t^j \Psi &= \partial_t^j \tilde{\omega} - \frac{\epsilon}{c_\epsilon^2} \partial_t^{j+2} \Psi \quad \text{ on } \mathbb{R} \times \Omega \\
\partial_t^j \Psi &= 0 \quad \text{ on } \mathbb{R} \times \partial\Omega
\end{aligned}$$

Hence, the usual elliptic estimates tell us that

$$\begin{aligned}
\|\partial_t^j \Psi\|_{X^{0,k}} &\leq C \left(\|\partial_t^j \psi\|_{L^2(\mathbb{R}\times K)} + \|\partial_t^j \tilde{\omega} - \frac{c_\epsilon^2}{\epsilon} \partial_t^{j+2} \Psi\|_{H^k(\mathbb{R}\times K)} \right) \\
&\leq C \left(\|\tilde{\omega}\|_{H^{j+k}(\mathbb{R}\times K)} + \frac{\epsilon}{c_\epsilon^2} \|\Psi\|_{H^{j+k+2}(\mathbb{R}\times K)} \right) \\
&\leq C \|\omega\|_{L^2(\mathbb{R}\times K)}.
\end{aligned}$$

The estimate of Ψ is finished by noting that $\|\omega\|_{L^2(\mathbb{R}\times K)}$ is uniformly bounded in ϵ. □

4.4 Solution of the unregularized equation

The results of the previous section allow us to take the limit as ϵ tends to zero in all the relevant topologies. Thus, we arrive at $T_\delta \in H^1_{loc}(\mathbb{R}\times\Omega)$, $u_\delta \in C^{0,\alpha}(\mathbb{R}\times\Omega) \cap H^1_{loc}(\mathbb{R}\times\Omega)$, $\omega_\delta \in H^1_{loc}(\mathbb{R}\times\Omega)$, and $\Psi_\delta \in H^1_{loc}(\mathbb{R}\times\Omega)$. These satisfy the system

$$\begin{aligned}
(T_\delta)_t + u_\delta \cdot \nabla T_\delta - \Delta T_\delta &= f(T_\delta) \\
(\omega_\delta)_t - \Delta \omega_\delta &= \hat{e} \cdot \nabla T_\delta \\
\Delta \Psi_\delta &= \tilde{\omega}_\delta \\
u_\delta &= \nabla^\perp \Psi_\delta
\end{aligned} \tag{4.4.1}$$

on $\mathbb{R}\times\Omega$ with boundary conditions

$$\begin{aligned}
\frac{\partial T_\delta}{\partial \eta} &= 0 \\
\Psi_\delta &= 0 \\
\omega_\delta &= 0 \\
u_\delta \cdot \eta &= 0
\end{aligned} \tag{4.4.2}$$

on $\partial\Omega$.

The last remaining step is to pass to the limit $\delta \to 0$. Before we do that, we will need to check a few properties for each fixed δ. First, we will check that, for each δ, the functions we have obtained at this point are pulsating fronts as in equation equation (4.1.5). This is necessary to eventually show that the functions we obtain as we take δ to zero are pulsating fronts. Then we will show that

T_δ^m can be bounded by an exponential as in Corollary 4.3.5. Finally, we will discuss some uniform in δ estimates on our functions. This will allow us to take the limit as δ tends to zero. Finally, we will finish the proof of Theorem 4.1.1.

Here we will show that the T_δ satisfies the normalization condition, as in equation (4.1.10). We will need this later, to show that this condition holds in the limit as δ tends to zero.

Lemma 4.4.1. *For T_δ, as constructed above,*

$$\max\{T_\delta(t,x,z) : (t,x,z) \in \mathbb{R} \times \Omega : x - c_\delta t \leq 0\} = \theta_0.$$

Proof. The main problem here is that the convergence of T_δ^ϵ to T_δ is in H^1, which is not strong enough to guarantee that this condition carries over. Certainly the inequality $T_\delta(t,x,z) \leq \theta_0$ holds as a result of our construction of T, so what remains is to show that T_δ takes the value θ_0 somewhere on the set $\{(t,x,z) \in \mathbb{R} \times \Omega : x - c_\delta t \leq 0\}$. Here the front speed, c_δ, is the limit of $c_{\epsilon,\delta}$ as ϵ tends to zero.

To this end, we use the bounds in Corollary 4.3.16 and the results of Berestycki and Hamel in [10] to get a uniform gradient bound on T_δ^ϵ. This paper implies that since u is bounded in C^1, there is a constant $C > 0$ which is independent of ϵ, though which does depend on δ, such that $\|\nabla_{x,z} T_\delta^\epsilon\|_\infty < C$. This, along with the trace theorem, gives us the existence of a point such that $T_\delta^\epsilon(t,x,z) = \theta_0$ and such that $x - c_\delta t \leq 0$. □

Now we have to worry about the limits as $t \to \pm\infty$. Here it will be convenient to use the functions in the moving frame. This is justified since the front speed c_δ is positive, and the norms in the moving and stationary frames are equivalent, up to a factor of c_δ.

We first show that on the right, T^m tends to zero. In order to prove this, we wish to show that T^m is bounded by an exponential function on $[R,\infty) \times \Omega_p$ for some R as we did in Lemma 4.3.4 and Corollary 4.3.5. The proof is the same in spirit but altered slightly since we do not begin with the finite domain problem here.

Lemma 4.4.2. *Let*

$$R_\epsilon = \inf\{r \in [0,\infty) : c_\delta/10 \geq \max_{\substack{(x,z)\in\Omega_p,\\ s \geq r}} |u^{m,\epsilon}(s,x,z)|\}.$$

For ϵ sufficiently small, there is a constant $C_\delta > 0$, which does not depend on ϵ, such that

$$R_\epsilon < C_\delta.$$

Proof. Suppose that there is a subsequence $\epsilon_n \downarrow 0$ such that $R_{\epsilon_n} \to \infty$. For ease of notation, we drop the "n" notation. In this case, we recenter our equations so that

$$\Phi^{m,\epsilon}(s,x,z) = T^{m,\epsilon}(s+R_\epsilon,x,z), \;\; U^{m,\epsilon}(s,x,z) = u^{m,\epsilon}(s+R_\epsilon,x,z),$$
$$W^{m,\epsilon}(s,x,z) = \omega^{m,\epsilon}(s+R_\epsilon,x,z), \;\; S^{m,\epsilon}(s,x,z) = \Psi^{m,\epsilon}(s+R_\epsilon,x,z).$$

These functions satisfy the same bounds as before so we can take limits as ϵ tends to zero, to obtain function Φ^m, U^m, W^m, and S^m. Since $R_\epsilon \to \infty$, we get that the $\Phi^m \le \theta_0$. Then arguments similar to those in Lemma 4.3.3 give us that there are limits θ_- and θ_+ such that

$$\theta_\pm = \lim_{s\to\pm\infty} \Phi^m(s,x,z),$$

where the limit is uniform in Ω_p. Hence integrating

$$-c_\delta \Phi^m_s + U \cdot \tilde{\nabla}\Phi^m - L\Phi^m = 0$$

gives us that

$$c_\delta(\theta_- - \theta_+) = 0.$$

Since c_δ is positive by Proposition 4.3.15, we get that $\theta_- = \theta_+$. In the stationary frame, where

$$\Phi(t,x,z) = \Phi^m(x - c_\delta t, x, z), \;\; U(t,x,z) = U^m(x - c_\delta t, x, z),$$

notice that Φ satisfies

$$\Phi_t + U \cdot \nabla\Phi = \Delta\Phi.$$

Hence, the Hopf maximum principle along with the fact that $\theta_- = \theta_+$ implies that $\Phi \equiv \theta_-$. This implies that Φ^m takes a constant value.

Hence we have that W^m satisfies

$$-c_\delta W^m_s - LW^m = 0,$$

with Dirichlet boundary conditions on one boundary of Ω_p and periodic boundary conditions on the other boundary. Integrating this over $\mathbb{R} \times \Omega_p$ and using a Poincaré inequality gives us that $W^m \equiv 0$. Therefore, S^m satisfies

$$-LS^m = 0,$$

with Dirichlet boundary conditions on one boundary of Ω_p and periodic boundary conditions on the other boundary, whence $S \equiv 0$. This finally implies that U must be zero because of its relationship to S. This contradicts the fact that $\max_{\Omega_p} U^m(0,x,z) = c_\delta/10 > 0$. Hence it must be that R_ϵ is bounded. $\square$

This result allows us to bound T_δ^m above by an exponential. Eventually, we will show that the parameters in this exponential are bounded. Hence, when we eventually take the limit $\delta \to 0$, this bound will be preserved.

Corollary 4.4.3. *For every δ and every $0 < \alpha \le c_\delta/8$, there are constants $C_\alpha, R_\delta > 0$, which depend only on α and δ, respectively, such that*

$$T_\delta^m(s,x,z) \le C_\alpha e^{-\alpha(s-R_\delta)}.$$

Proof. This is simply a result of combining the conclusions of Corollary 4.3.5 and Lemma 4.4.2. □

The last property that we need to check is that the functions satisfy the first condition of equation (4.1.5). Later, we will use this to show that this still holds in the limit $\delta \to 0$.

Lemma 4.4.4. *The functions T_δ and u_δ satisfy the condition equation* (4.1.5)*.*

Proof. The argument for this is the same as that given in [9]. We fix any positive real number B and any compact set $K \subset \Omega$. Notice that T^ϵ converges in L^2 to T on $[-A,A]\times K$. Hence, to show that

$$\int_{[-A,A]\times K} \left[T_\delta(t+\frac{\ell}{c_\delta},x,z) - T_\delta(t,x-\ell,z)\right]^2 dxdzdt = 0 \tag{4.4.3}$$

it suffices to prove that

$$\lim_{\epsilon\to 0}\int_{[-A,A]\times K} \left[T_\delta^\epsilon(t+\frac{\ell}{c_\delta},x,z) - T_\delta^\epsilon(t,x-\ell,z)\right]^2 dxdzdt = 0.$$

To this end, we first notice that $T_\delta^\epsilon(t+\frac{\ell}{c_{\epsilon,\delta}},x,z) = T_\delta^\epsilon(t,x-\ell,z)$. Hence we calculate:

$$\begin{aligned}
&\int_{[-A,A]\times K} \left[T_\delta^\epsilon(t+\frac{\ell}{c_\delta},x,z) - T_\delta^\epsilon(t,x-\ell,z)\right]^2 dxdzdt \\
&\qquad = \int_{[-A,A]\times K} \left[T_\delta^\epsilon(t+\frac{\ell}{c_\delta},x,z) - T_\delta^\epsilon(t+\frac{\ell}{c_{\epsilon,\delta}},x,z)\right]^2 dxdzdt \\
&\qquad \le \left|\frac{\ell}{c_{\epsilon,\delta}} - \frac{\ell}{c_\delta}\right|^2 \int_{\mathbb{R}\times K} |(T_\delta^\epsilon)_t|^2 dxdzdt \\
&\qquad \le C\cdot\left|\frac{\ell}{c_{\epsilon,\delta}} - \frac{\ell}{c_\delta}\right|^2.
\end{aligned}$$

Since $c_\epsilon \to c$, then we have that equation (4.4.3) holds. Hence $T_\delta(t+\ell/c,x,z) = T_\delta(t,x-\ell,z)$ holds almost everywhere. Of course, by parabolic regularity, T_δ is continuous. Hence, the equality holds everywhere. The same argument works for u_δ as well. □

Finally, in order to take the limit as δ tends to zero, we need to have some uniform bounds on our functions. We will do that here. First, notice that the L^2 gradient bounds we obtained in Section 4.3 did not depend on δ, except for the bound on the time derivative of T_δ. These bounds, along with parabolic regularity, are enough to get upper bounds on the $X^{1,2}$ norm of ω. This gives us an upper bound on the $C^{0,\alpha}$ norm of u_δ as in Lemma 4.3.12. Arguing as in Corollary 4.3.13 will then provide an upper bound on the front speed, c_δ, which, in turn, provides an upper bound on the L^2 norm of the time derivative of T_δ. From here, parabolic and elliptic regularity gives us Sobolev and Hölder bounds independent of δ that we summarize in the lemma below.

Lemma 4.4.5. *There exists a constant $C > 0$, independent of δ such that*

$$|c_\delta| + \|T_\delta\|_{C^{1+\alpha,2+\alpha}} + \|\omega_\delta\|_{C^{1+\alpha,2+\alpha}} + \|\Psi_\delta\|_{C^{1+\alpha,2+\alpha}} + \|u_\delta\|_{C^{1+\alpha,2+\alpha}} \leq C.$$

Moreover, for every compact set $K \subset \Omega$, there exists a constant $C_K = C(\mathrm{diam}(K)) > 0$, which depends only on the diameter of K such that

$$\|T_\delta\|_{X^{1,2}(\mathbb{R}\times K)} + \|\omega_\delta\|_{X^{1,2}(\mathbb{R}\times K)} + \|\Psi_\delta\|_{X^{1,2}(\mathbb{R}\times K)} + \|u_\delta\|_{X^{1,2}(\mathbb{R}\times K)} \leq C_K.$$

In order to finish Theorem 4.1.1, we need to take the limit $\delta \to 0$. Afterwards, we need to check that the front speed, c, is positive and that our solutions are non-trivial, satisfy condition equation (4.1.10), and satisfy equation (4.1.5). Most of these claims are proved similarly as the analogous results from this section.

Proof of Theorem 4.1.1. Notice that Lemma 4.4.5 gives us uniform bounds in Sobolev and Hölder spaces which are independent of δ. As a result, we can take limits in these spaces to obtain functions T, u, Ψ, and ω. By the theory of convolutions, we get that $\tilde{\omega}_\delta$ and ω_δ converge to the same function, ω. Hence, these functions satisfy the stream function formulation of our problem, which in turn implies that they satisfy the system equation (4.1.3) - equation (4.1.4).

Arguing exactly as in Proposition 4.3.15, we see that c_δ is bounded away from zero. Moreover, arguing as in Corollary 4.3.13 shows that it is bounded uniformly above. Hence c_δ converges to a positive number c.

Since we have convergence in the Hölder norms as δ tends to zero, and since, for each δ, these functions satisfy the condition equation (4.1.5) and normalization condition equation (4.1.10), then our limiting functions will satisfy these as well.

Hence we need only check their limits as $x \to \pm\infty$. To do this it is convenient to look in the moving frame, where the equivalent limit to check is what happens when $s \to \pm\infty$. The limits are easiest to prove for Ψ, u, and ω. The bounds in Lemma 4.4.5 imply that these functions are globally bounded in L^2 and $C^{1,\alpha}$ in the moving frame. Hence as $s \to \pm\infty$, these functions tend to zero.

Now we will check the limits of T. First, arguing exactly as in Lemmas 4.4.2, we can find a universal constant $R > 0$, which does not depend on δ such that

$$c/10 \geq \max_{\substack{(x,z)\in\Omega_p,\\ s\geq R}} u_\delta^m(s,x,z).$$

Hence arguing as in Corollary 4.3.5, we can bound T^m by an exponential. It follows that $T^m(s,x,z) \to 0$ as $s \to \infty$. This is equivalent to $T(t,x,z) \to 0$ as $x \to \infty$.

To get the limit as s tends to $-\infty$, one simply argues as in Lemma 4.3.3. To see that θ_- is positive, we notice that integrating the equation for T^m implies that

$$c|\Omega_p|\theta_- = \int_{\mathbb{R}\times\Omega_p} f(T^m)dxdzds.$$

Hence if $\theta_- = 0$, then $T \leq \theta_0$ on $\mathbb{R}\times\Omega$. In this case, the parabolic maximum principle applied in the stationary frame, along with Lemma 4.4.1 imply that $T \equiv \theta_0$. This cannot be true since $T(t,x,z) \to 0$ as $t \to -\infty$. Hence $\theta_- > 0$. Notice that this also implies that

$$\int_{\mathbb{R}\times\Omega_p} f(T^m)dxdzds > 0,$$

and, hence, T takes values larger than θ_0.

Finally, we show that if equation (4.1.11) is satisfied, then $\theta_- = 1$. This proof is similar, though not identical, to the proofs in [7, 51]. First, we assume that $p \in (2,3]$ since if $p > 3$ then $(T-\theta_0)_+^p \leq (T-\theta_0)_+^3$. Hence, in this case, we may assume condition equation (4.1.11) holds with $p = 3$. To begin notice that since

$$\int_{\mathbb{R}\times\Omega_p} f(T^m)dxdzds < \infty,$$

then $\theta_- \in (0,\theta_0] \cup \{1\}$. Hence we suppose that $\theta_- \leq \theta_0$. Now let $K = \{(x,z) \in \Omega : x \in [0,\ell]\}$. Define

$$M(t) = \max_{(x,z)\in K} T(t,x,z) \quad \text{and} \quad m(t) = \min_{(x,z)\in K} T(t,x,z),$$

for any $t \in \mathbb{R}$. First notice that for any $t \in \mathbb{R}$, we have

$$\begin{aligned}
M(t) - m(t) &= M(t) - \frac{1}{|K|}\int_K T(t,x,z)dxdz + \frac{1}{|K|}\int_K T(t,x,z)dxdz - m(t) \\
&\leq 2\left\| T(t,\cdot,\cdot) - |K|^{-1}\int_K T(t,x,z)dxdz \right\|_{L^\infty(K)} \\
&\leq C\left\| T(t,\cdot,\cdot) - |K|^{-1}\int_K T(t,x,z)dxdz \right\|_{W^{1,p}(K)} \\
&\leq C\|\nabla T\|_{L^p(K)}.
\end{aligned} \tag{4.4.4}$$

The second inequality comes from the Sobolev inequality and the third inequality comes from the Poincaré inequality. We need a bound on the L^p norm of ∇T, which we will obtain by getting a bound on the L^3 norm of ∇T and interpolating with the L^2 bound of ∇T. To this end we follow the development in [51]. First notice that the work in Lemma 4.3.14 shows that if $\theta_- \leq \theta_0$ then we have that

$$\int_{\mathbb{R}\times K} |T_t|^2 dxdzdt \leq C \int_{\mathbb{R}\times K} |\nabla T|^2 dxdzdt.$$

Multiplying equation (4.1.3) by $T|\nabla T|$ and integrating over $\mathbb{R}\times K$ gives us

$$\begin{aligned}\left|\int T|\nabla T|\Delta T dtdxdz\right| &= \left|\int T|\nabla T|T_t + \int (u\cdot\nabla T)T|\nabla T| - \int f(T)T|\nabla T|\right| \\ &\leq (C + \|u\|_{L^\infty})\|\nabla T\|_{L^2}^2 + \int f(T)|\nabla T| \\ &\leq C\|\nabla T\|_{L^2}^2 + C\int f(T).\end{aligned}$$

To understand the right hand side, we notice that

$$\int |\nabla T|^3 = -\int T|\nabla T|\Delta T - \int T(\nabla T\cdot\nabla|\nabla T|).$$

We need to understand the last term on the right. To this end, the argument in Lemma 3.4 of [51] implies that

$$\|\nabla^2 T\|_{L^2} \leq C(\|\Delta T\|_{L^2} + \|\nabla T\|_{L^2}).$$

Also, equation (4.1.3) implies that

$$\|\Delta T\|_{L^2} \leq \|T_t\|_{L^2} + \|u\cdot\nabla T\|_{L^2} + \|f(T)\|_{L^2}.$$

Combining all this gives us

$$\int |\nabla T|^3 \leq C\left(\int |\nabla T|^2 + \int f(T)\right). \tag{4.4.5}$$

We also observe that

$$\theta_- = \int_{\mathbb{R}\times K} f(T)\frac{dtdxdz}{|K|}, \quad \frac{\theta_-^2}{2} + \int_{\mathbb{R}\times K} |\nabla T(t,x,z)|^2\frac{dtdxdz}{|K|} = \int_{\mathbb{R}\times K} Tf(T)\frac{dtdxdz}{|K|}$$

so that

$$\int_{\mathbb{R}\times K} |\nabla T|^2 dtdxdz = \int_{\mathbb{R}\times K}\left(T - \frac{\theta_-}{2}\right) f(T)dtdxdz \geq \frac{\theta_0}{2}\int f(T) \tag{4.4.6}$$

Hence equations equation (4.4.4), equation (4.4.5), and equation (4.4.6) give us that

$$\int_{\mathbb{R}} (M(t) - m(t))^p dt \leq C \int_{\mathbb{R}\times K} |\nabla T|^2 dtdxdz. \tag{4.4.7}$$

Define $C_{\Omega,p}$ in equation (4.1.11) to be the reciprocal of the constant in the above equation, namely

$$C_{\Omega,p} = \frac{1}{C}$$

with C as in equation (4.4.7).

We claim that $m(t)$ is non-decreasing in t. To see this, one can first show it on the finite domain using the maximum principe since $T^{\epsilon}_{n,\delta}$ satisfies an elliptic equation. Then this property will hold through the limits since when we take $n \to \infty$ and $\delta \to 0$ we have convergence in Hölder norms and when we take $\epsilon \to 0$ we have bounds on the L^2 and L^∞ norms of T as in Lemma 4.4.1. Because m is non-decreasing and T is non-constant, then it follows that $m(t) < \theta_0$ for all t.

Since $m(t) < \theta_0$, $M(t) \geq T(t, x, z)$ and by equation (4.4.7), we have that

$$\int_{\mathbb{R}\times K} |\nabla T(t,x,z)|^2 \geq C_{\Omega,p} \int_{\mathbb{R}} (M(t) - m(t))^p dt \geq C_{\Omega,p} \int_{\mathbb{R}\times K} (T(t,x,z) - \theta_0)_+^p dtdxdz.$$

Hence using equation (4.1.11) we obtain

$$\begin{aligned} C_{\Omega,p} \int_{\mathbb{R}\times K} \left(T - \frac{\theta_-}{2}\right)(T-\theta_0)_+^p dtdxdz &\geq \int_{\mathbb{R}\times K} \left(T - \frac{\theta_-}{2}\right) f(T) dtdxdz \\ &= \int_{\mathbb{R}\times K} |\nabla T|^2 dtdxdz \\ &\geq C_{\Omega,p} \int_{\mathbb{R}\times K} (T-\theta_0)_+^p dtdxdz. \end{aligned}$$

The left hand side is smaller than the right hand side unless $T \equiv \theta_0$. This can't occur because of Corollary 4.4.3. Hence, we have reached a contradiction, implying that $\theta_- > \theta_0$. We then conclude that $\theta_- = 1$. $\square$

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