What Have We Learned from Calculus Reform? The Road to Conceptual Understanding

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Goals

In order to evaluate the impact of calculus reform, we first need to recall its goals. Although different people may phrase it differently, everyone involved would agree that they were trying to improve the teaching of calculus. Some would say they wanted more student involvement; others would say they wanted to take advantage of technology; others would say they wanted to emphasize problem solving and modeling. Most would agree that they wanted to improve conceptual understanding. What has been the impact of this effort?

Background to calculus reform: Rationale for change[1]

The teaching of calculus came under scrutiny in the 1980s for several reasons. One was concern over the students' apparent lack of understanding of the subject, especially when asked to use it in an unfamiliar situation. Faculty outside mathematics frequently complained that students could not apply the concepts they had been taught. In some instances, ideas were being used in other fields in ways that were sufficiently different from the way they are used in mathematics that it was not surprising that students did not make the connection. For example, the minimization of average cost has was done symbolically in mathematics, if at all, whereas it is usually done graphically in economics.

However, students also had difficulty recognizing mathematical ideas that were presented the same way as in mathematics. A small difference in notation or the absence of familiar clues—such as "largest" or "smallest" in an optimization problem—easily threw students off. This striking difficulty in transferring knowledge between fields suggested that students' understanding was not sufficiently robust.

In addition, many students came to college believing that mathematics centers on manipulative techniques, rather than interpretation and understanding [2]. These students spent little energy thinking about where ideas came from or

[1] Parts of this paper are taken from "Calculus at the Start of the New Millenium" by Deborah Hughes Hallett, the *Proceedings of the International Conference on Technology in Mathematis Education*, Beirut, Lebanon, July 2000.

[2] See, for example, Deborah Hughes Hallett "Are We Encouraging Our Students to Think Mathematically?" in *How to Teach Mathematics by Steven G. Kranz, 2nd edition, American Mathematical Society, 1999.*

how they were used. Besides being a disappointment to faculty, these students never saw the power of mathematics to unite disparate fields.

Materials used in the middle 1980s suggested that mathematicians were doing little to challenge students' views that equated mathematics to applying formulas. Exam questions were often of the form "Use method X to do Y"; problems in the text were usually to be done by the formula most recently presented. Consequently students got little experience on choosing a method. Even the choice of variables seldom needed thought. It was not uncommon for an entire set of exercises in a text to be written as functions in terms of x, with at most a couple that involved t or θ. Since an unfamiliar variable is a real stumbling block to students and x is virtually never used outside mathematics, this lack of variety significantly limited students' ability to apply their mathematics.

Changes in the teaching of calculus

When efforts to improve the teaching of calculus started in the late 1980s, there was great variation amongst the projects that were undertaken. Some were technology-driven, redesigning curriculum and pedagogy using the power of the new computer tools—for example, *Calculus&Mathematica*, from the University of Illinois, and Project CALC, from Duke. Others used no technology, but incorporated extended applications, group work, and collaborative learning—for example, the New Mexico State Project. Over the next decade, there was remarkable convergence among the projects. Effective components in one project were adapted and incorporated by other projects—and eventually by mainstream authors. The most fundamental change made by the new calculus texts in the 1990s was the introduction of many more nonstandard problems. By the mid 1980s, before calculus reform got underway, such problems had been relegated to the end of the exercise sets in the texts and were few enough in number that many students—and many instructors—ignored them. Newer calculus texts have a much wider variety of problems, and fewer "template" problems that can be solved by mimicking a worked example in the text.

Technology is incorporated in many current calculus courses. The Advanced Placement (AP) exam taken by many high school students now requires a graphing calculator, as does the International Baccalaureate (IB). Most high school calculus courses, and hence most high school precalculus courses, are taught with a graphing calculator. College courses are more varied. Some have not moved beyond the memorization and multiple choice of the 1980s; others are now Web or computer based.

Although using technology in calculus is the most visible, and perhaps the most controversial, change in the teaching of calculus, it is not the one that will have the most impact on student learning. Expecting conceptual understanding on homework and exams is more important. Although faculty often spend more time designing lectures than homework, most students learn more from homework than from lectures. Changes in homework and exams have a larger effect on student learning than changing lecture content. Requiring thinking is central to establishing the idea that mathematics is more than applying formulas. To the surprise of faculty, students often described the new courses as being "more theoretical" than the old. Although they are not using the word "theoretical" in the usual mathematical sense, from a student perspective they are right. The new courses require more reasoning, justification, and explanation. Just getting an answer is no longer enough.

Along with the increase in nonstandard problems and the use of technology, many new calculus courses emphasize open-ended problems that require extensive writing, often in cooperative groups. In the early 1990s, several books of calculus labs were published; the IB now requires all students to submit a portfolio as part of their final assessment.

Thus there is now both more variety in calculus courses and more emphasis on conceptual understanding.

Cooperation with client disciplines

An unexpected side benefit of calculus reform has been increased cooperation between mathematics and other fields, such as as engineering, biology, physics, and economics. In their quest to learn how students used calculus, mathematicians talked to their colleagues in other departments, looked at their texts and exams, and listened carefully to their needs. The result was some new texts that genuinely reflected the needs of the sciences—for example, those from the University of Iowa and Iowa State—and much improved calculus texts for the social sciences. Students take a subject more seriously—and learn it better—if faculty from more than one field "conspire" (in the students' words) to teach it to them.

However, more important than the improved texts was the good will generated by these conversations. Efforts to make the teaching of calculus more responsive to other fields lead directly to the MAA's Curriculum Foundations Project. In this project, faculty from outside mathematics were invited to have direct input into the current MAA curriculum review. Perhaps the most sobering aspect of the Curriculum Foundation Project was how surprised and grateful the faculty from other fields were to be contacted. This is an effort whose time was long overdue.

Impact of calculus reform

The most obvious measure of the impact of calculus reform is that the features initiated in the 1990s are now commonplace in "traditional" courses. Some of the new features have been transformed—in some cases their originators would say beyond recognition—but many have been adopted "as is" by standard textbooks. There is a wider variety of problems than before, and the "Rule of Three," which originated in calculus reform, has found its way into a large number of calculus texts. The wider variety of problems and the use of multiple representations (graphical, numerical, analytical, and verbal) make it harder for students simply to memorize template problems—though still not impossible—and hence encourage conceptual understanding.

Most texts now allow the use of technology, although often as an add-on, to accommodate a variety of faculty preferences. Open-ended problems and extended applications are found in many books, although often as an add-on at the end of the chapter. Students are now expected to write more often in calculus courses than they were in the 1980s, although seldom in university lecture courses with large enrollments. However instructors who get their students to write report that writing both deepens conceptual understanding and provides a window into their students' thinking processes.

Summary

The impact of calculus reform has been substantial. In spite of objections—sometimes vociferous—to any one particular aspect of the new courses (technology, conceptual, rather than formal, understanding), many aspects of calculus reform are now so embedded in the mainstream that they are considered mainstream.

The changes that have had the most impact are not those that were originally considered to be the most profound. For example, the topics covered have not been greatly impacted, in spite of the call for a "lean and lively" curriculum. However, the pedagogy and types of problems solved has been impacted. During the 1990s essentially every math department made some changes to their calculus courses. Some of these changes have persisted, some have not, but all have made the teaching of calculus a subject of discussion in many math departments where this was not the case previously.

In the long run, the largest impact of calculus reform is likely to be the creation of a community of mathematicians who innovate and reflect on their teaching—and who do so in collaboration with faculty in other disciplines and across institutional boundaries.