Review for Chapter 3

1. On what intervals is the function \( f(z) = z^4 + 2z^3 - 12z^2 - 36z \)
   A. Increasing \((-\sqrt{6}, -1.5) \cup (\sqrt{6}, \infty)\)
   B. Decreasing \((-\infty, -\sqrt{6}) \cup (-1.5, \sqrt{6})\)
   C. concave up \((-\infty, -2) \cup (1, \infty)\)
   D. concave down \((-2, 1)\)
   E. both increasing and concave up \((-\sqrt{6}, -2) \cup (\sqrt{6}, \infty)\)
   F. both increasing and concave down \((-2, -1.5)\)
   G. both decreasing and concave up \((-\infty, -\sqrt{6}) \cup (1, \sqrt{6})\)
   H. both decreasing and concave down \((-1.5, 1)\)
   Verify your results by using your graphing calculator

2. Let \( g(x) = x^4 + x^3 - 3x^2 + 2 \). Using calculus
   A. When is \( g(x) = 0\) (exact values)
      \( g'(x) = 4x^3 + 3x^2 - 6x = x(4x^2 + 3x - 6) \)
      \( g(x) = 4x^3 + 3x^2 - 6x = x(4x^2 + 3x - 6) \)
   B. On what interval(s) is \( g(x) \) increasing? Decreasing?
      increasing \((-\frac{3+\sqrt{105}}{6}, 0)\) and \((-\frac{3+\sqrt{105}}{6}, \infty)\)
      decreasing \((-\infty, -\frac{3-\sqrt{105}}{6})\) and \((-\frac{3+\sqrt{105}}{6}, 0)\)
   C. When is \( g''(x) = 0\)?
      When \( x = -1, 0.5 \)
      \( g''(x) = 12x^2 + 6x - 6 = 6(2x - 1)(x + 1) \)
   D. On what interval(s) is \( g(x) \) concave up? Concave down?
      concave up \((-\infty, -1) \cup (0.5, 1)\) concave down \((-1, 0.5)\)

3. For the following piecewise functions, (i) Is the function continuous everywhere, if not where and why? (ii) Is the function differentiable everywhere, if not where and why? (iii) Write the derivative of the function (piecewise). You may use your derivative rules.

   A. \( T(n) = \begin{cases} 3n^2 - 1 & \text{if } n \leq -2 \\ -2n + 7 & \text{if } n > -2 \end{cases} \)
      (i) \( \lim_{n \to -2^-} (-2n + 7) = (-2)(-2) + 7 = 11 \)
      \( T(-2) = 3(-2)^2 - 1 = 11 \)
      So \( T(n) \) is continuous.
      (ii) \( \frac{d}{dn} (3n^2 - 1) = 6n \)
      \( \frac{d}{dn} (-2n + 7) = -2 \)
      \( \lim_{n \to -2^-} (6n) = 6(-2) = -12 \)
      \( \lim_{n \to -2^+} (-2) = -2 \)
      These values are not equal therefore there is a jump in the derivative at \( n = 2 \).
      (iii) \( T'(n) = \begin{cases} 6n & n < -2 \\ DNE & n = 2 \\ -2 & n > -2 \end{cases} \)
   
   B. \( g(x) = \begin{cases} x^2 + 3 & x < 2 \\ -4x + 15 & x \geq 2 \end{cases} \)
      \( g'(x) = \begin{cases} 2x & x < 2 \\ DNE & x = 2 \\ -4 & x > 2 \end{cases} \)
      \( g'(x) = \begin{cases} 2x & x < 2 \\ DNE & x = 2 \\ -4 & x > 2 \end{cases} \)
      \( f(z) = \begin{cases} 1 & z < 0 \\ 0 & z \geq 0 \end{cases} \)
      \( f'(z) = \begin{cases} 2x & x < 2 \\ DNE & x = 2 \\ -4 & x > 2 \end{cases} \)
      \( f(z) = \begin{cases} 1 & z < 0 \\ 0 & z \geq 0 \end{cases} \)
      \( f'(z) = \begin{cases} 2x & x < 2 \\ DNE & x = 2 \\ -4 & x > 2 \end{cases} \)

   You must show why the derivative does not exist at the point using limits.
4. For the following functions:

(i) Find the exact derivative of each of the following functions at $x = 2$.
(ii) Using this information, write the equation of the tangent line to the graph at $x = 2$.
(iii) Using your tangent line equation estimate $x = 2.1, x = 3$ and $x = 5$.

A. $g(x) = 5x^3 + 6x - 12$
B. $r(x) = \left(\frac{1}{3}\right)^x$
C. $f(r) = r(\sin(r))$
D. $j(m) = 4e^{(-m^2+1)}$

Solutions:

A. Slope: $g'(2) = 66$
   
   $g'(x) = 15x^2 + 6$
   
   $g'(2) = 15(2)^2 + 6 = 66$

   Point: $(2, g(2)) = (2, 40)$
   
   $g(2.1) \approx 66(2.1 - 2) + 40 = 46.6$
   
   $g(3) \approx 66(3 - 2) + 40 = 106$
   
   $g(5) \approx 66(5 - 2) + 40 = 238$

B. Slope: $r'(2) = \ln\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^x$ and $r'(2) = \ln\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$
   
   Point: $(2, r(2)) = (2, \frac{1}{9})$
   
   $y = \frac{1}{9}\ln\left(\frac{1}{3}\right)(x - 2) + \frac{1}{9}$ or $y = -\frac{1}{9}\ln(3)(x - 2) + \frac{1}{9}$

   $r(2.1) \approx -\frac{1}{9}\ln(3)(2.1 - 2) + \frac{1}{9} \approx 0.0989$
   
   $r(3) \approx -\frac{1}{9}\ln(3)(3 - 2) + \frac{1}{9} \approx -0.0109569$
   
   $r(5) \approx -\frac{1}{9}\ln(3)(5 - 2) + \frac{1}{9} \approx -0.25509$

C. Slope $f''(2) = \sin(2) + 2\cos(2) \approx 0.770$
   
   Point $(2, 1.81859)$
   
   $f(2) = 2\sin(2) \approx 1.81859$

   $y = \sin(2) + 2\cos(2)(r - 2) + 2\sin(2)$ or $y = 0.77(r - 2) + 1.82$

   $y(2.1) = 0.77(1.1) + 2\sin(2) \approx 1.8263$
   
   $y(3) = 0.77(1) + 2\sin(2) \approx 1.8956$
   
   $y(5) = 0.77(3) + 2\sin(2) \approx 2.0496$

D. Slope $j''(2) = 4e^{(-2+1)}(-2(2)) \approx -7.966$
   
   Point $j(2) = 4e^{(-2+1)} \approx 0.1991$

   $j(2.1) \approx 0.11944$
   
   $j(3) \approx -0.5975$
   
   $j(5) \approx -2.1907$

5. Which of the following phrases best describes the functions below?

**Concave up**  **Concave Down**  **A mixture of concave up and concave down**

You absolutely must clearly state your answer and explain your answer using calculus to receive credit. There will be no credit given for correct choice of phrase without a clear calculus explanation.

A. $f(x) = x^4 + 4x$
B. $g(x) = \cos(x)$
C. The smooth function illustrated by the table of a values below

<table>
<thead>
<tr>
<th>$x$</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1.1</td>
<td>1.5</td>
<td>1.7</td>
<td>1.8</td>
<td>1.85</td>
<td>1.851</td>
</tr>
</tbody>
</table>

(the values in the table representative of the properties of the function.)

Solution: A. concave up  B. concave up and down  C. concave down
6. A. What effect, if any, would a vertical shift in the graph of \( f(x) = \tan(x) \) have on its derivative? \( g(x) = f(x) + k \). Explain by finding the derivative of each function.

   B. What effect, if any, would a horizontal expansion in the graph of \( f(x) = \sin(x) \) have on its derivative? \( j(x) = f(3x) \). Explain by finding the derivative of each function.

   **Solution:**
   
a) none: since \( \frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)} \) and \( \frac{d}{dx} (\tan(x) + k) = \frac{1}{\cos^2(x)} \)

   b) the derivatives would be different \( j'(x) = 3 \cos(3x) \neq f'(x) = \cos(x) \)

7. Write the derivative of the function (piecewise). Justify for derivative at 1 and e.

   \( r(z) = \begin{cases} 
   \sin(z - 1) & z < 1 \\
   \ln(z) & 1 \leq z \leq e \\
   z^0 & z > e 
   \end{cases} \)

   \( r'(z) = \begin{cases} 
   \cos(z - 1) & z < 1 \\
   1 & z = 1 \\
   \frac{1}{z} & 1 < z < e \\
   DNE & z = e \\
   0 & z > e 
   \end{cases} \)

   You must verify the derivative at \( z = 1 \) & \( e \) by using the limit functions.

8. For the following
   (i) Write the equation of the tangent line at the point given.
   (ii) Write the equation of the best fit quadratic equation at the point indicated, the function values are the same; first and second derivatives are the same at.

   A. \( f(x) = e^{x-1} \) \( x = 1 \)
   B. \( g(x) = \sin(x) \) \( x = \frac{\pi}{6} \)
   C. \( j(x) = \ln(x) \) at \( x = 1 \).

   **Answer:**
   
   A. (i) \( y = x \) (ii) \( y = \frac{1}{2}x^2 + \frac{1}{2} \)
   B. (i) \( y = \frac{\sqrt{3}}{2} (x - \frac{\pi}{6}) + \frac{1}{2} \) (ii) \( y = -\frac{1}{4}x^2 + \left(\frac{\sqrt{3}}{2} + \frac{\pi}{12}\right)x + \left(\frac{72-\pi^2-12\sqrt{3}}{144}\right) \)
   C. (i) \( y = 1(x - 1) \) (ii) \( y = -\frac{1}{2}x^2 + 2x + \frac{3}{2} \)

9. Use the table and the fact that \( f(x) \) is invertible and differentiable everywhere to find \((f^{-1})'(3) = \frac{1}{5}\)

   \[ \begin{array}{c|c|c}
   x & f(x) & f'(x) \\
   \hline
   3 & 1 & 7 \\
   6 & 2 & 10 \\
   9 & 3 & 5 \\
   \end{array} \]

   **Solution:** \( \frac{1}{5} \)

10. Find \( \frac{dy}{dx} \) given \( 3\sqrt{x} + 5x^4y^3 - \frac{x}{3} = 23 + 7y \).

   **Solution:** \( \frac{dy}{dx} = \frac{2x^3 - 9 - 120x(2y^3)}{90x^2y^2 - 42x^2} \) must not have negative exponents or compound fraction.

11. Find the local linearization to \( y = \cos(x) \) at \( x = \frac{\pi}{4} \).

   **Solution:**
   \[ y = \frac{-1}{\sqrt{2}}(x - \frac{\pi}{4}) + \frac{1}{\sqrt{2}} \]
12. Consider the function \( y = f(x) \) given by the graph

Answer the following questions giving an explanation for your answer (you still need to justify by using the derivative)

A. Is \( g(x) = (f(x))^2 \) an increasing function? **no**
B. Is \( h(x) = (f(x))^3 \) an increasing function? **yes**
C. Is \( j(x) = \frac{1}{f(x)} \) an increasing function? **no (always decreasing)**
D. Is \( T(x) = f'(x) \) an increasing function? **no**

13. Suppose \( f'(x) \) is differentiable and decreasing function for all \( x \)? Describe what the following symbols mean, and which has a larger value and why.

\[ f(5 + \Delta x) \quad \text{and} \quad f(5) + f'(5)\Delta x \]

Solution:

\( f(5 + \Delta x) \) is the function value at \( x = 5 + \Delta x \).
\( f(5) + f'(5)\Delta x \) is the value on the tangent line at \( x = 5 + \Delta x \).
\( f(5) + f'(5)\Delta x > f(5 + \Delta x) \) since \( f'(x) \) is decreasing then \( f(x) \) is concave down so the tangent line is above the curve at \( x = 5 + \Delta x \).

14. Find values for \( A \) and \( B \) so that the function is differentiable. \( R(\theta) = \begin{cases} \sin(\theta) & \theta < \pi \\ A\theta + B & \theta \geq \pi \end{cases} \)

Answer: \( A = -1 \) and \( B = \pi \)