High Order Melnikov Method: Theory and Application

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Unperturbed equation

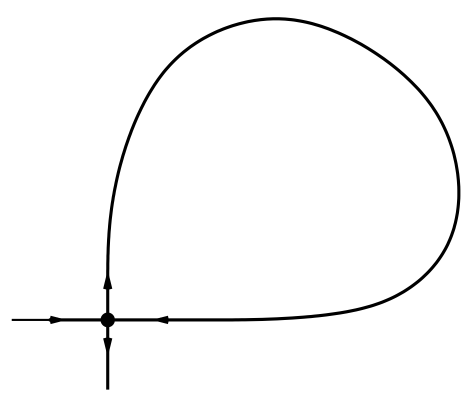
Let $(x, y) \in \mathbb{R}^2$ be the phase variables,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x + g(x).$$

• *Homoclinic solution:* $(0, 0)$ is with a homoclinic solution

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$

• *Assumption on $g(x)$:* $g(0) = g'(0) = 0$ and $g(x)$ is real analytic on D_ℓ, a small neighborhood of ℓ.



Non-autonomously Perturbed equation

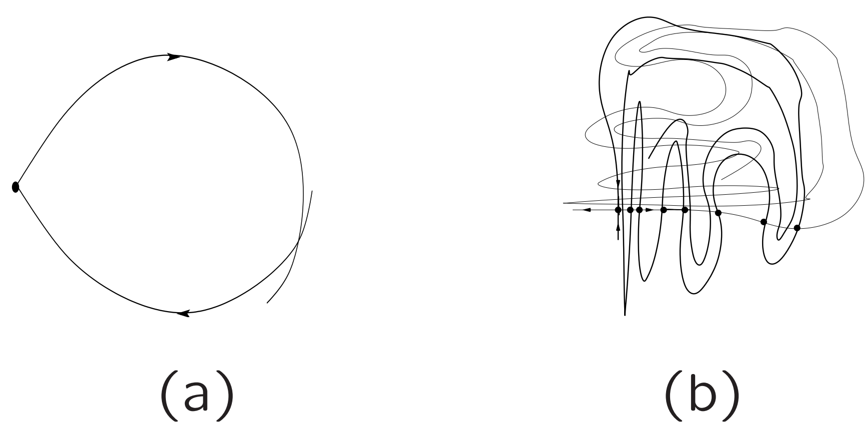
$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = x + g(x) + \varepsilon P(x, y, t).$$

- (A1): $P(t, 0, 0) = 0$, and

$$P(t + T, x, y) = P(t, x, y)$$

for all $(x, y) \in D_\ell$.

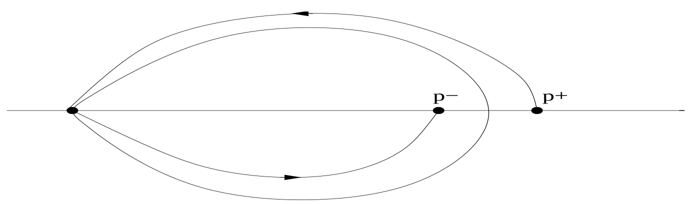
- (A2): $P(t, x, y)$ is C^1 in t, real analytic in (x, y) on D_ℓ for all $t \in [0, T)$.



(a) (b)

Poincare/Melnikov Method

The Splitting Distance



$$E(p) = x^2/2 - \int_0^x g(u)du;$$

$$D(t_0, \varepsilon) = E(p^+) - E(p^-) := \varepsilon D_0(t_0) + \varepsilon^2 D_1(t_0) + \cdots$$

Poincaré/Melnikov Integral:

$$D_0(t_0) = \int_{-\infty}^{+\infty} b(t)P(t + t_0, a(t), b(t))dt$$

Homoclinic tangle exists if (i) $D_0(t_0) = 0$ has a non-degenerate solution. (ii) ε small.

How about $D_1, D_2, \cdots$?

- Fact: Never be calculated.
- (Q1) Can we calculate $D_1(t_0), \cdots$?
- (Q2) Is such calculation useful?

Melnikov's Scheme on D_n

(1) Solve for D_n is the same as solve for stable (and unstable) solution up to order ε^n. Let

$$
\begin{aligned}
X^{(n)} =& a(t) + \varepsilon x_1(t) + \cdots + \varepsilon^n x_n(t) \\
Y^{(n)} =& b(t) + \varepsilon y_1(t) + \cdots + \varepsilon^n y_n(t)
\end{aligned}
$$

be the stable solution up to order of ε^n

(2) $X^{(n+1)}, Y^{(n+1)}$ satisfies the equations of first variations on $X^{(n)}, Y^{(n)}$. This is to say we have

$$
\begin{aligned}
\frac{dX^{(n+1)}}{dt} =& A(t)X^{(n+1)} + B(t)Y^{(n+1)} + f(t) \\
\frac{dY^{(n+1)}}{dt} =& C(t)X^{(n+1)} + D(t)Y^{(n+1)} + g(t)
\end{aligned}
$$

where A, B, C, D, f, g are all explicit in $X^{(n)}, Y^{(n)}$.

(3) Melnikov introduced new variables M, W

and for M, W,

$$\frac{dM}{dt} = \tilde{A}(t)M + \tilde{B}(t)W + \tilde{f}(t)$$
$$\frac{dW}{dt} = \tilde{C}(t)M + \tilde{D}(t)W + \tilde{g}(t)$$

we have

$$\tilde{A}(t) = \tilde{B}(t) = 0.$$

(4) M is perpendicular to $(X^{(n)}, Y^{(n)})$ and W is tangential to it.

Answer to (Q1): One can derive explicit integral formula for D_n in theory. In practice, an integral formula for even D_1, though attainable, is not nearly as simple as D_0.

Answer to (Q2): Our ability to use the formula acquired to study given equations rely on our ability to **evaluate** the integrals of D_n. If the integrals can not be analytically manipulated, then the formula derived is useless.

Our method

(1) In studying differential equations, using correct variables makes all the difference: a good set of variables led to insightful results but improper choice of variables led to puzzling formulas.

(2) Melnikov's choice of W in solving equations of first variation is not the best.

(3) Instead of deriving equations of first variations order by order, we derive an equation for stable solutions in one step.

(4) We convert this equation into an integral equation, then introduce a recursive process (very much like the classical recursion to linearize the equation around a fixed point) to derive integral formula for D_n.

Our Results

Answer to (Q1):

$m :=$ the smallest so that $g^{(m)}(0) \neq 0$
$k := (m+1)/2$
$R(t) := kb^2 - ab'$
$A(t) := \frac{a^2 b''}{a'} - kab' - k(k-1)b^2$
$h(t) = -R(t)\int_0^t A(\tau)R^{-2}(\tau)d\tau$
$H(t) = \frac{1}{\sqrt{R(t)}}(bh(t) + a)$
$\mathcal{P}(t, t_0) := P(t + t_0, a, b)$
$\mathcal{P}_x(t, t_0) := P_x(t + t_0, a, b)$
$\mathcal{P}_y(t, t_0) := P_y(t + t_0, a, b)$.

$$\begin{aligned} D_1(t_0) = & \frac{H(0)\mathcal{P}(0, t_0)}{\sqrt{R(0)}} D_0(t_0) \\ & + \int_{-\infty}^{+\infty} \int_0^{\tau_2} b(\tau_2)\mathcal{P}(\tau_2, t_0)\mathcal{P}_y(\tau_1, t_0) d\tau_1 d\tau_2 \\ & - \int_{-\infty}^{+\infty} \int_0^{\tau_2} \frac{b(\tau_2)H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}_t(\tau_2, t_0)\mathcal{P}(\tau_1, t_0) d\tau_1 d\tau_2 \\ & - \int_{-\infty}^{+\infty} \int_0^{\tau_2} \frac{b(\tau_2)H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}_t(\tau_1, t_0)\mathcal{P}(\tau_2, t_0) d\tau_1 d\tau_2. \end{aligned}$$

On The Integral Functions:

$m :=$ the smallest so that $g^{(m)}(0) \neq 0$
$k := (m+1)/2$
$R(t) := kb^2 - ab'$
$A(t) := \frac{a^2 b''}{a'} - kab' - k(k-1)b^2$
$h(t) = -R(t) \int_0^t A(\tau) R^{-2}(\tau) d\tau$
$H(t) = \frac{1}{\sqrt{R(t)}}(bh(t) + a)$
$\tilde{H}(t) = \frac{1}{\sqrt{R}}(b'h + kb)$

– Equation of first variations around $\ell = (a, b)$:

$$\frac{d\xi}{dt} = \eta; \qquad \frac{d\eta}{dt} = (1 + g_x(a))\xi.$$

– Change of coordinate $(\xi, \eta) \to (\tilde{\xi}, \tilde{\eta})$:

$$\xi = \frac{1}{\sqrt{R}}(a'\tilde{\eta} - \sqrt{R}H\tilde{\xi}); \qquad \eta = \frac{1}{\sqrt{R}}(b'\tilde{\eta} - \sqrt{R}\tilde{H}\tilde{\xi}).$$

– New equation:

$$\frac{d\tilde{\xi}}{dt} = -\frac{R'}{2R}\tilde{\xi}; \qquad \frac{d\tilde{\eta}}{dt} = \frac{R'}{2R}\tilde{\eta}.$$

Answer to (Q2):

High Order Melnikov Method: Evaluate $D_1(t_0)$ to prove the existence of homoclinic tangle when $D_0(t_0) \equiv 0$.

– Apply to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3 + \varepsilon \cos \omega t \cdot (x^2 + \gamma^* x^3).$$

where γ^* is such that $D_0(t_0) \equiv 0$.

– In this case

$$D_1(t_0) = -\int_{-\infty}^{+\infty} \int_0^{\tau_2} \frac{b(\tau_2)H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}_t(\tau_2, t_0)\mathcal{P}(\tau_1, t_0) d\tau_1 d\tau_2 - \int_{-\infty}^{+\infty} \int_0^{\tau_2} \frac{b(\tau_2)H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}_t(\tau_1, t_0)\mathcal{P}(\tau_2, t_0) d\tau_1 d\tau_2.$$

– We need to

(1) Determine γ^*

(2) Evaluate a double integral.

The value of γ^*:

$$a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \qquad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}.$$

$$\begin{aligned} D_0(t_0) &= \int_{-\infty}^{+\infty} b(t) P(t + t_0, a(t), b(t)) dt \\ &= \frac{\pi \omega e^{-\omega\pi/2}}{3} \left(\frac{2\sqrt{2}(\omega^2+1)}{(1+e^{-\omega\pi})} + \gamma^* \cdot \frac{\omega(\omega^2+4)}{(1-e^{-\omega\pi})} \right) \sin \omega t_0. \end{aligned}$$

$D_0(t_0) \equiv 0$ if and only if

$$\gamma^*(\omega) := -\frac{2\sqrt{2}(\omega^2+1)}{\omega(\omega^2+4)} \cdot \frac{(1-e^{-\omega\pi})}{(1+e^{-\omega\pi})}. \tag{1}$$

Result on $D_1(t_0)$:

$$D_1(t_0) = F(\omega) \sin(2\omega t_0)$$

$$F(\omega) = \left(-\frac{32}{3} \pi e^{-\pi\omega} \omega^2 (1 + O(\omega^{-1})) \right).$$

Conclusion: (1) Homoclinic tangle exists for

$$0 < |\varepsilon| \leq K^{-1} \omega |F(\omega)|.$$

(2) $F(\omega) = 0$ can only have finitely many solution.

Exponentially Small Splitting

Equation of Study

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3 + \varepsilon \cos \omega t \cdot y^2.$$

On E_0

$$D_0(t_0, \omega) = \frac{\sqrt{2}\pi}{30} \omega^5 e^{-\omega\pi/2} \left[1 + O(\omega^{-1})\right] \sin \omega t_0$$

Restriction on ε By Poincaré/Melnikov, homoclinic tangle exists for $0 < \varepsilon < Ke^{-\omega\pi/2}$.

Question: Can we relax the restriction on ε to $0 < \varepsilon < \omega^{-k_0}$?

Answer: Affirmative for Hamiltonian equations by a theory developed by a number of author in last forty some years in a string of long and technically involved paper.

Problem: This theory does not apply here because our equation is not Hamiltonian.

Our Strategy: Derive integral formula for D_n and evaluate E_n for all $n \geq 1$.

Our Result:

(a) for all $n \geq 0$,

$$D_n(t_0, \omega) = \sum_{k=1}^{4(n+1)} A_{k,n}(\omega) \sin k\omega t_0;$$

(b) there exist positive constants κ_0 and ω_0, so that for all $\omega > \omega_0$,

$$|A_{k,n}(\omega)| < \omega^{\kappa_0(n+1)} e^{-\omega\pi/2}.$$

for all $A_{k,n}$.

With the exponentially small factor $e^{-\omega\pi/2}$ for all $A_{k,n}(\omega)$, the dominance of $D_0(t)$ is restored in the power series expansion of

$$D(t_0, \omega, \varepsilon) = D_0(t_0, \omega) + \varepsilon D_1(t_0, \omega) + \cdots$$

High Order Melnikov Integrals: From our theory on D_n, we conclude:

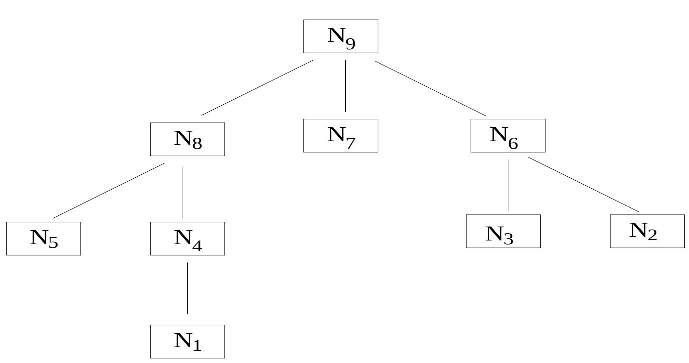
– D_n is a sum of a collection of well-structured multiple integrals,

– The multiplicity of these integrals are in between n and $4n$,

– The number of integrals for D_n is $< K^n$,

– The integral functions of these integrals are explicit in terms of $(a(t), b(t))$ and the others listed for $D_1(t_0)$.

Structure Tree:



Integral Function:

N_j is either an M-node or a W-node. To each node N_j we also assign

(i) an integral variable t_j;

(ii) a kernal function $f_j(t_j)$;

(iii) an interval of integration I_j for t_j.

Details on item (ii): The kernel functions $f_j(t_j)$ is in the form of

$$f_j(t_j) = (\cos \omega(t_j + t_0))^{n_0(j)} d_j(t_j)$$

where $n_0(j)$ is either 0 or 1.

$$d_j(t_j) = b^{m_1} \tilde{H}^{m_2} a^{n_1} A^{n_2} H^{n_3}.$$

Details on item (iii): Let j' be such that N_j is directly branched out of $N_{j'}$. We have $I_j = (t_{j'}, +\infty)$ if N_j is an M-node, but $I_j = (0, t_{j'})$ if N_j is a W-node.

High Order Melnikov Integral:

$$N_p^s = \int_{I_p} f_p(t_p) \left(\cdots \left(\int_{I_1} f_1(t_1) dt_1 \right) \cdots \right) dt_p.$$

Symmetry: For the equation we study

$$d_j(t_j) = -d(-t_j).$$

Analysis: We decompose a give high order Melnikov integral into two collections.

The first collection is exponentially small

The second collection is **not**.

But symmetry $d_j(t_j) = -d(-t_j)$ induces cancellation: the sum of all contributions from the second collection in D_n for all n are canceled out!

Final Remark: Analysis of high order Melnikov integrals are long and hard. The result on D_1 presented earlier is a by-product of this long project.

Preprints for both results are posted on my Website

http://www.math.arizona.edu/ dwang

Thank You!