ON THE NON-DOMINANCE OF THE MELNIKOV FUNCTION IN A PERIODICALLY PERTURBED DUFFING EQUATION

ALI OKSASOGLU AND QIUDONG WANG

Abstract. In this paper, we study a periodically perturbed Duffing equation. Let \( \varepsilon \) be a small parameter representing the magnitude of the perturbation, and \( \omega \) be the forcing frequency. We expand the splitting distance \( D(t_0, \varepsilon, \omega) \) into a power series in \( \varepsilon \) as

\[
D(t_0, \varepsilon, \omega) = E_0(t_0, \omega) + \varepsilon E_1(t_0, \omega) + \cdots + \varepsilon^n E_n(t_0, \omega) + \cdots.
\]

We prove that, for the equation studied in this paper, the classic Melnikov function \( E_0(t_0, \omega) \) fails to dominate \( \varepsilon E_1(t_0, \omega) \) assuming \( \omega \) is sufficiently large and \( \varepsilon \) is in polynomial power of \( \omega^{-1} \).

Let

\[
\frac{d^2x}{dt^2} = f(x) + \varepsilon P(x, t)
\]

be a periodically perturbed second order equation. Under the assumption that the unperturbed equation has a saddle fixed point with a homoclinic solution, the splitting distance \( D(t_0, \varepsilon) \) of stable and unstable manifold of the perturbed saddle can be expanded into a power series of \( \varepsilon \) as

\[
D(t_0, \varepsilon) = E_0(t_0) + \varepsilon E_1(t_0) + \cdots + \varepsilon^n E_n(t_0) + \cdots.
\]

An explicit integral formula for \( E_0(t_0) \) was first derived by Poincare for a specific equation [5], then by Melnikov for Hamiltonian equations [4] at a later time. With the derived formula for \( E_0(t_0) \), the existence of homoclinic tangle in a time-periodic equation can then be rigorously verified through the evaluation of a definite integral. This method of verification of homoclinic tangles has been commonly referred to as the Poincare-Melnikov method. The Poincare-Melnikov method has served as a main venue, through which the modern chaos theory – in particular, the theory of horseshoe – is applied to the analysis of ordinary differential equations (see [3]).

There are, however, two main degenerate cases for which the Poincare-Melnikov method fails to apply. The first is when the perturbation is such that \( E_0(t_0) \equiv 0 \). The second is when the frequency of perturbation, which we denote as \( \omega \), is large and \( \varepsilon \) is in polynomial power of \( \omega^{-1} \) (see [6]). In a recent paper [2], Chen and Wang introduce a new theory on \( E_1(t_0) \). They first derive an explicit integral formula for \( E_1(t_0) \). Then, they evaluate the derived integrals to prove the existence of homoclinic tangles in a time-periodic equation for which \( E_0(t_0) \equiv 0 \).

In this paper, we use the integral formula for \( E_1(t_0) \) from [2] to study the second degenerate case, the case of high frequency perturbation. We evaluate \( E_1(t_0) \) for a given equation of high frequency perturbation to prove that, for this equation, the \( C^0 \)-norm of \( E_1(t_0) \) as a function of \( t_0 \) is much larger than that of \( E_0(t_0) \). Our proof is based on the fact that the interactions of perturbations of different frequencies have a clear presence in \( E_1(t_0) \) but not in \( E_0(t_0) \). The dominance of \( E_1(t_0) \) is indeed induced by interactions of this kind.

We note that there has been substantial literature concerning the dominance of the classical Melnikov function \( E_0(t_0) \) over the splitting distance \( D(t_0) \) for equations of high frequency perturbations. See [1] and the references therein. See also [7]. With this study, we offer a concrete example of high frequency perturbation for which the classical Melnikov function \( E_0(t_0) \) fails to dominate \( \varepsilon E_1(t_0) \). We are not aware of any such example in the existing literature.

We also note that, by combining our result with the study presented in [7], we can further establish the dominance of \( E_1(t_0) \) over the rest of the splitting distance \( D(t_0) \) for the equation studied in this paper. The details of this result will be presented in another paper.

1. Statement of Result

We study the periodically perturbed Duffing equation

\[
\frac{d^2x}{dt^2} = x - x^3 + \varepsilon x^2 (\cos 2\omega t + \cos 3\omega t)
\]

(1.1)
where $\varepsilon$ is to represent the magnitude and $\omega$ is to dictate the frequency of the time periodic perturbation. We let $y = dx/dt$ to rewrite equation (1.1) as

\begin{equation}
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3 + \varepsilon x^2 (\cos 2\omega t + \cos 3\omega t).
\end{equation}

In the case of $\varepsilon = 0$, the saddle fixed point $(0, 0)$ of the unperturbed equation has a homoclinic solution $(a(t), b(t))$ where

\begin{equation}
a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}.
\end{equation}

We let $\mathcal{E} = \{(a(t), b(t)) : t \in (-\infty, \infty)\}$ be the homoclinic loop and $D_\ell$ be a small neighborhood of $\mathcal{E}$. For $p = (x, y)$, we let

\begin{equation}
E(p) = \frac{1}{2} y^2 - \frac{1}{2} x^2 + \frac{1}{4} x^4
\end{equation}

be the unperturbed energy.

For a fixed $t_0 \in [0, 2\pi \omega^{-1}]$, equation (1.2) admits a unique stable solution $(x^*(t, \varepsilon, \omega), y^*(t, \varepsilon, \omega))$ in $D_\ell$ satisfying $y^*(t_0, \varepsilon, \omega) = 0$. It also admits a unique unstable solution $(x^u(t, \varepsilon, \omega), y^u(t, \varepsilon, \omega))$ in $D_\ell$ satisfying $y^u(t_0, \varepsilon, \omega) = 0$. Let $p^+ = (x^*(t_0, \varepsilon, \omega), 0)$ for the stable solution and $p^- = (x^u(t_0, \varepsilon, \omega), 0)$ for the unstable solution. The splitting distance $D(t_0, \varepsilon, \omega)$ of the stable and unstable manifold is defined by letting

\begin{equation}
D(t_0, \varepsilon, \omega) = \varepsilon^{-1} \left( E(p^+ (t, \varepsilon, \omega)) - E(p^- (t_0, \varepsilon, \omega)) \right).
\end{equation}

We expand $D(t_0, \varepsilon, \omega)$ into a power series of $\varepsilon$ to write

\begin{equation}
D(t_0, \varepsilon, \omega) = E_0(t_0, \omega) + \varepsilon E_1(t_0, \omega) + \cdots + \varepsilon^n E_n(t_0, \omega) + \cdots.
\end{equation}

Note that $E_0(t_0, \omega)$ is the classic Melnikov function and $E_1(t_0, \omega)$ is the high order Melnikov function of [2]. For equation (1.2), we let

\begin{equation}
P(x, t, \omega) = x^2 (\cos 2\omega t + \cos 3\omega t)
\end{equation}

and denote

\begin{equation}
P(t, t_0, \omega) = P(a(t), t + t_0, \omega), \quad P_t(t, t_0, \omega) = \partial_t P(a(t), t + t_0, \omega).
\end{equation}

We have

\begin{equation}
P(t, t_0, \omega) = a^2(t) (\cos 2\omega(t + t_0) + \cos 3\omega(t + t_0));
\end{equation}

\begin{equation}
P_t(t, t_0, \omega) = - \omega a^2(t) (2 \sin 2\omega(t + t_0) + 3 \sin 3\omega(t + t_0)).
\end{equation}

We also denote

\begin{equation}
H(t) = 3b(t)a(t) \int_0^t a^{-2}(\tau)d\tau + 1.
\end{equation}

Then $E_0(t_0, \omega)$ and $E_1(t_0, \omega)$ for equation (1.2) are as follows according to [2].

1. For $E_0(t_0, \omega)$:

\begin{equation}
E_0(t_0, \omega) = - \int_{-\infty}^{+\infty} b(\tau) P(\tau, t_0, \omega)d\tau.
\end{equation}

2. For $E_1(t_0, \omega)$:

\begin{equation}
E_1(t_0, \omega) = \frac{P(0, t_0, \omega)}{\sqrt{2}} E_0(t_0, \omega) - \int_{-\infty}^{+\infty} \int_0^{\tau_2} f(\tau_1, \tau_2, t_0, \omega)d\tau_1 d\tau_2
\end{equation}

where

\begin{equation}
f(\tau_1, \tau_2, t_0, \omega) = \frac{b(\tau_2) H(\tau_1)}{a(\tau_1)} [P_t(\tau_2, t_0, \omega) P(\tau_1, t_0, \omega) + P_t(\tau_1, t_0, \omega) P(\tau_2, t_0, \omega)].
\end{equation}

To start, we have the following for $E_0(t_0, \omega)$.

**Proposition 1.1.** We have

\begin{equation}
E_0(t_0, \omega) = \frac{4\sqrt{2}\pi \omega (4\omega^2 + 1)}{3(1 + e^{-2\pi \omega})} e^{-\pi \omega} \sin 2\omega t_0 + \frac{2\sqrt{2}\pi \omega (9\omega^2 + 1)}{(1 + e^{-3\pi \omega})} e^{-3\pi \omega/2} \sin 3\omega t_0.
\end{equation}
ON THE NON-DOMINANCE OF THE MELNIKOV FUNCTION IN A PERIODICALLY PERTURBED DUFFING EQUATION

We have, in addition,

**Lemma 2.1.** \(E\) are readily dominated by \(e\) functions on the complex \(t\)-plane taking \((2n + 1)\pi i/2\) for all \(n\) as poles.

The main result of this paper is on \(E_1(t_0, \omega)\).

**Main Theorem.** There exists a sufficiently large \(\omega_0\), so that for all \(\omega > \omega_0\),

\[
\sup_{t_0 \in [0,2\pi]} |E_1(t_0, \omega)| > C_0 \omega^2 e^{-\pi\omega/2}
\]

where \(C_0 = \frac{\pi}{5} \sin \frac{\pi}{5}\). By Proposition 1.1, the two terms in \(E_0(t_0, \omega)\) take \(e^{-\pi\omega}\) and \(e^{-3\pi\omega/2}\) as respective factors, both of which are readily dominated by \(e^{-\pi\omega/2}\), a factor appearing in the lower bound estimation of the Main Theorem for \(E_1(t_0, \omega)\).

2. THE INTEGRALS FOR \(E_1(t_0, \omega)\)

We start by calculating \(P_1(t_2, t_0)P(t_1, t_0)\).

**Lemma 2.1.** We have

\[
P_1(t_2, t_0)P(t_1, t_0) = g_0(t_1, t_2) - \omega a^2(t_1) a^2(t_2) \sum_{n=1}^{6} \left( f_n(t_1, t_2) \sin n\omega t_0 + g_n(t_1, t_2) \cos n\omega t_0 \right)
\]

where

\[
g_0(t_1, t_2) = \frac{3}{2} \sin 3\omega(t_2 - t_1) + \sin 2\omega(t_2 - t_1)
\]

\[
g_1(t_1, t_2) = \frac{3}{2} \sin(3\omega t_2 - 2\omega t_1) - \sin(3\omega t_1 - 2\omega t_2)
\]

\[
g_2(t_1, t_2) = \sin 2\omega(t_2 + t_1)
\]

\[
g_3(t_1, t_2) = \frac{3}{2} \sin(3\omega t_2 + 2\omega t_1) + \sin(2\omega t_2 + 3\omega t_1)
\]

\[
g_4(t_1, t_2) = \frac{3}{2} \sin 3\omega(t_2 + t_1)
\]

and

\[
f_1(t_1, t_2) = \frac{3}{2} \cos(3\omega t_2 - 2\omega t_1) - \cos(3\omega t_1 - 2\omega t_2)
\]

\[
f_4(t_1, t_2) = \cos 2\omega(t_2 + t_1)
\]

\[
f_5(t_1, t_2) = \frac{3}{2} \cos(3\omega t_2 + 2\omega t_1) + \cos(2\omega t_2 + 3\omega t_1)
\]

\[
f_6(t_1, t_2) = \frac{3}{2} \cos 3\omega(t_2 + t_1)
\]

We have, in addition,

\[
g_2(t_1, t_2) = f_2(t_1, t_2) = g_3(t_1, t_2) = f_3(t_1, t_2) = 0.
\]

**Proof.** This lemma follows from a direct calculation that is completely elementary in nature. \(\square\)

In what follows, we let

\[
I_{n,c} = \int_{-\infty}^{+\infty} \int_{\tau_0}^{\tau_2} a(t_1)H(t_2)b(t_2) a^2(t_2) \left[ g_n(t_1, t_2) + g_n(t_2, t_1) \right] d\tau_1 d\tau_2
\]

\[
I_{n,s} = \int_{-\infty}^{+\infty} \int_{\tau_0}^{\tau_2} a(t_1)H(t_2)b(t_2) a^2(t_2) \left[ f_n(t_1, t_2) + f_n(t_2, t_1) \right] d\tau_1 d\tau_2.
\]

We have, by (1.9) and (1.10),

\[
E_1(t_0, \omega) = \frac{P(0, t_0, \omega)}{\sqrt{2}} E_0(t_0, \omega) - \omega \sum_{n=1}^{6} (I_{n,c} \cos n\omega t_0 + I_{n,s} \sin n\omega t_0).
\]

Our task is then to evaluate \(I_{n,s}, I_{n,c}\) for \(n = 1, 4, 5, 6\).
Lemma 2.2. (1) Assume \( F(\tau_1, \tau_2) \) is even in the sense that \( F(\tau_1, \tau_2) = F(-\tau_1, -\tau_2) \). Then,
\[
\int_0^{+\infty} \int_0^{\tau_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 = \int_0^{-\infty} \int_0^{\tau_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2.
\]
(2) Assume \( F(\tau_1, \tau_2) \) is odd in the sense that \( F(\tau_1, \tau_2) = -F(-\tau_1, -\tau_2) \). Then,
\[
\int_0^{+\infty} \int_0^{\tau_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2 = -\int_0^{-\infty} \int_0^{\tau_2} F(\tau_1, \tau_2) d\tau_1 d\tau_2.
\]
Proof. By changing \((\tau_1, \tau_2)\) to \((-\tau_1, -\tau_2)\). □

We then have, first,
\[
|\tau_1|/2 \leq 2\pi e^{-\pi/2}.
\]

Proposition 3.1. \(|I_{1,s}| \geq 2\pi e^{-\pi/2}沼泽/2\).

Note that by (2.1) and Lemma 2.1,
\[
I_{1,s} = \frac{1}{2} \left( \int_0^{+\infty} \int_0^{\tau_2} H(\tau_1) a(\tau_1) b(\tau_2) a^2(\tau_2) [\cos(3\omega\tau_2 - 2\omega\tau_1) + \cos(3\omega\tau_2 - 2\omega\tau_2)] d\tau_1 d\tau_2.\right)
\]

3.1. Preparations. Let \( G(t) \) be such that
\[
G(t) = H(t) a(t) - \frac{3\pi i}{8} b(t) a^2(t)
\]
where \( a(t), b(t) \) are as in (1.3) and \( H(t) \) is as in (1.7). We also denote
\[
g(t) = G(t + i\pi/2), \quad f(t) = t^4 b(t + i\pi/2) a^2(t + i\pi/2).
\]

Our next lemma is preparatory in nature.

Lemma 3.1. The functions \( g(t), f(t) \) are analytic at \( t = 0 \). In addition, we have (i) \( g(t) \) is odd in \( t \), \( f(t) \) is even in \( t \); (ii) \( g(0) = 0, g'(0) = \sqrt{2}/4; \) and (iii) \( f(0) = -2\sqrt{2}, f'(0) = 0 \).

Proof. That \( a(t + i\pi/2) \) is odd and \( b(t + i\pi/2) \) is even in \( t \) follows from
\[
a(t + i\pi/2) = -\frac{2\sqrt{2}i}{(e^t - e^{-t})}, \quad b(t + i\pi/2) = \frac{2\sqrt{2}i}{(e^t - e^{-t})}.
\]
We then have,
\[
g(t) = \frac{2\sqrt{2}i}{(e^t - e^{-t})^2} \left( \frac{3(3e^{2t} - e^{-2t} - 4t)}{2(e^t - e^{-t})^2} \right) - \frac{2\sqrt{2}i}{(e^t - e^{-t})}.
\]
This is odd in \( t \). Expanding into power series at \( t = 0 \), we obtain
\[
g(t) = \frac{\sqrt{2}}{5} it + O(t^3).
\]

Next, we have
\[
f(t) = t^4 \frac{2\sqrt{2}i}{(e^t - e^{-t})^2} \left( \frac{2\sqrt{2}i}{(e^t - e^{-t})} \right)^2.
\]
This is even in \( t \). Again, expanding into power series at \( t = 0 \), we obtain
\[
f(t) = -2\sqrt{2} + O(t^2).
\]
All items of this lemma now follow. □
3.2. Reduction. In this subsection, we decompose $I_{1,s}$ into a set of simpler integrals.

**Lemma 3.2.** We have $I_{1,s} = \frac{1}{2} (R_1 + R_2 + E_1 + E_2 + E_3)$ where

\[
R_1 = \int_0^\infty e^{3i\omega s_2} \left( \int_{-\infty}^{+\infty} G(s_1) b(s_1 + s_2) a^2(s_1 + s_2) e^{i\omega s_1} ds_1 \right) ds_2
\]

\[
R_2 = \int_0^\infty e^{-2i\omega s_2} \left( \int_{-\infty}^{+\infty} G(s_1) b(s_1 + s_2) a^2(s_1 + s_2) e^{i\omega s_1} ds_1 \right) ds_2
\]

\[
E_1 = \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{3i\omega \tau} d\tau \left( \int_0^{+\infty} H(\tau) a(\tau) e^{-2i\omega \tau} d\tau \right)
\]

\[
E_2 = \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{-2i\omega \tau} d\tau \left( \int_0^{+\infty} H(\tau) a(\tau) e^{3i\omega \tau} d\tau \right)
\]

\[
E_3 = -\frac{3\pi i}{8} \left( \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{3i\omega \tau} d\tau \right) \left( \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{-2i\omega \tau} d\tau \right).
\]

**Proof.** We use Lemma 2.2(2) to write $I_{1,s}$ as

\[
I_{1,s} = \int_0^{+\infty} \int_{\tau_1}^{\tau_2} H(\tau_1) a(\tau_1) b(\tau_2) a^2(\tau_2) \left[ \cos(3\omega \tau_2 - 2\omega \tau_1) + \cos(3\omega \tau_1 - 2\omega \tau_2) \right] d\tau_1 d\tau_2.
\]

We divide the proof of this lemma into three steps.

**Step 1: Initial Reduction.** We write trigonometric functions in complex form to obtain

\[
I_{1,s} = \frac{1}{2} \int_0^{+\infty} \int_{\tau_1}^{\tau_2} H(\tau_1) a(\tau_1) b(\tau_2) a^2(\tau_2) \left[ e^{i(3\omega \tau_2 - 2\omega \tau_1)} + e^{-i(3\omega \tau_2 - 2\omega \tau_1)} \right] d\tau_1 d\tau_2
\]

\[+ \frac{1}{2} \int_0^{+\infty} \int_{\tau_1}^{\tau_2} H(\tau_1) a(\tau_1) b(\tau_2) a^2(\tau_2) \left[ e^{i(3\omega \tau_1 - 2\omega \tau_2)} + e^{-i(3\omega \tau_1 - 2\omega \tau_2)} \right] d\tau_1 d\tau_2.
\]

We then switch the order of integration from $d\tau_1 d\tau_2$ to $d\tau_2 d\tau_1$ to obtain

\[
I_{1,s} = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} H(\tau_1) a(\tau_1) b(\tau_2) a^2(\tau_2) \left[ e^{i(3\omega \tau_2 - 2\omega \tau_1)} + e^{-i(3\omega \tau_2 - 2\omega \tau_1)} \right] d\tau_2 d\tau_1
\]

\[+ \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} H(\tau_1) a(\tau_1) b(\tau_2) a^2(\tau_2) \left[ e^{i(3\omega \tau_1 - 2\omega \tau_2)} + e^{-i(3\omega \tau_1 - 2\omega \tau_2)} \right] d\tau_2 d\tau_1.
\]

Let $s_1 = \tau_1, s_2 = \tau_2 - \tau_1$. We have

\[
I_{1,s} = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} H(s_1) a(s_1) b(s_1 + s_2) a^2(s_1 + s_2) \left[ e^{i(3\omega s_2 + \omega s_1)} + e^{-i(3\omega s_2 + \omega s_1)} \right] ds_2 ds_1
\]

\[+ \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} H(s_1) a(s_1) b(s_1 + s_2) a^2(s_1 + s_2) \left[ e^{i(\omega s_1 - 2\omega s_2)} + e^{-i(\omega s_1 - 2\omega s_2)} \right] ds_2 ds_1
\]

\[= \frac{1}{2} \left( A^+ + A^- + B^+ + B^- \right)
\]

where

\[
A^\pm = \int_0^{+\infty} H(s_1) a(s_1) e^{\pm i\omega s_1} \int_0^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{\pm 3i\omega s_2} ds_2 ds_1
\]

\[
B^\pm = \int_0^{+\infty} H(s_1) a(s_1) e^{\pm i\omega s_1} \int_0^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{\mp 2i\omega s_2} ds_2 ds_1.
\]

**Step 2: Extend integral bounds.** We have

\[
A^+ = \int_0^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_0^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1
\]

\[= \int_0^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_{-\infty}^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1
\]

\[= \int_0^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_{-\infty}^{0} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1.
\]

\[
B^+ = \int_0^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_{-\infty}^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{2i\omega s_2} ds_2 ds_1.
\]
Here, we first change the lower bound of the inner integral from 0 to $-\infty$, then nullify the effect of this replacement by subtracting the second integral. We do the same to the lower bound of the outer integral to obtain

$$A^+ = (I) + (II) + (III)$$

where

$$(I) = \int_0^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_{-\infty}^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1$$

$$(II) = - \int_{-\infty}^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_0^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1$$

$$(III) = \int_{-\infty}^{0} H(s_1) a(s_1) e^{i\omega s_1} \int_{-\infty}^{0} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1.$$

For (I), we let $\tau = s_1 + s_2$ to obtain

$$\int_0^{+\infty} H(s_1) a(s_1) e^{-2i\omega s_1} \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{3i\omega \tau} d\tau ds_1 = E_1$$

where $E_1$ is as in (3.3). We also note that $(III) = -A^-$. This is for us to conclude that

$$A^+ + A^- = \int_{-\infty}^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_0^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1 + E_1.$$  

In parallel, we have

$$B^+ + B^- = \int_{-\infty}^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_0^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{-2i\omega s_2} ds_2 ds_1 + E_2$$

where $E_2$ is as in (3.3). It then follows from (3.5) and (3.6) that

$$I_{1,s} = -\frac{\omega}{2} \left( \hat{R}_1 + \hat{R}_2 + E_1 + E_2 \right)$$

where

$$\hat{R}_1 = \int_{-\infty}^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_{-\infty}^{+\infty} b(s_1 + s_2) a^2(s_1 + s_2) e^{3i\omega s_2} ds_2 ds_1$$

$$\hat{R}_2 = \int_{-\infty}^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} \int_{-\infty}^{0} b(s_1 + s_2) a^2(s_1 + s_2) e^{-2i\omega s_2} ds_2 ds_1.$$

**Step 3: Final Reduction.** Switching the order of integration from $ds_2 ds_1$ to $ds_1 ds_2$, we obtain

$$\hat{R}_1 = \int_{-\infty}^{+\infty} e^{3i\omega s_2} \int_{-\infty}^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} b(s_1 + s_2) a^2(s_1 + s_2) ds_1 ds_2$$

$$\hat{R}_2 = \int_{-\infty}^{+\infty} e^{-2i\omega s_2} \int_{-\infty}^{+\infty} H(s_1) a(s_1) e^{i\omega s_1} b(s_1 + s_2) a^2(s_1 + s_2) ds_1 ds_2.$$

Recall that, by definition,

$$H(t) a(t) = G(t) + \frac{3\pi i}{8} b(t) a^2(t).$$

This is for us to have

$$\hat{R}_1 + \hat{R}_2 = R_1 + R_2 + W_1 + W_2$$

where $R_1, R_2$ are as in (3.3), and

$$W_1 = \frac{3\pi i}{8} \int_0^{+\infty} e^{3i\omega s_2} \left( \int_{-\infty}^{+\infty} b(s_1) a^2(s_1) b(s_1 + s_2) a^2(s_1 + s_2) e^{i\omega s_1} ds_1 \right) ds_2$$

$$W_2 = \frac{3\pi i}{8} \int_0^{+\infty} e^{-2i\omega s_2} \left( \int_{-\infty}^{+\infty} b(s_1) a^2(s_1) b(s_1 + s_2) a^2(s_1 + s_2) e^{i\omega s_1} ds_1 \right) ds_2.$$
We work on \( W_1 + W_2 \). Letting \( t_1 = s_1 + s_2 \), we have
\[
W_1 = \frac{3\pi i}{8} \int_{-\infty}^{\infty} e^{2i\omega s_2} \left( \int_{-\infty}^{+\infty} b(t_1 - s_2)a^2(t_1 - s_2)b(t_1)a^2(t_1)e^{i\omega t_1}dt_1 \right) ds_2
\]
\[
= -\frac{3\pi i}{8} \int_{0}^{+\infty} e^{-2i\omega t_2} \left( \int_{-\infty}^{+\infty} b(t_1 + t_2)a^2(t_1 + t_2)b(t_1)a^2(t_1)e^{i\omega t_1}dt_1 \right) dt_2
\]
where the second equality is obtained by letting \( t_2 = -s_2 \). It then follows that \( W_1 + W_2 = E_3 \) where \( E_3 \) is as in (3.3).

### 3.3. A generic setting

In this subsection, we introduce a generic setting to facilitate the computation of \( R_1 \) and \( R_2 \). Let
\[
(3.9) \quad K_{g,f}(z,t) = \frac{e^{i\omega z}g(z)f(t+z)}{(t+z)^4}
\]
where \( t \neq 0 \) is a real parameter and \( z \) is a complex variable. We assume

(A1) the functions \( f(z), g(z) \) are independent of the forcing frequency \( \omega \),
(A2) \( f(z), g(z) \) are real analytic, non-constant functions and \( f(0) \neq 0 \),
(A3) \(|f^{(k)}(t)|, |g^{(k)}(t)| \leq C_6 e^{-c_0|b|} \) for all \( 0 \leq k < 4 \) where both \( C_6 \) and \( c_0 \) are positive constants.

Letting \( t \neq 0 \) be fixed, we regard \( K_{g,f}(z,t) \) as a function of \( z \). The function \( K_{g,f}(z,t) \) has a pole of order 4 on the real \( z \)-axis at \( z = -t \) (with the possible exception of countably many values of \( t \) so that \( g(t) = 0 \)). Denote the residue of this pole as \( \partial_z K_{g,f}(z,t) \).

Denote the residue of this pole as \( \partial_z K_{g,f}(z,t) \).

\[
\text{Lemma 3.3. We have}
\]
\[
(3.10) \quad I_{g,f}^{(m)}(\omega) = \int_{0}^{\infty} e^{im\omega t}R(t)dt.
\]
\[
(3.11) \quad I_{g,f}^{(m)}(\omega) = \frac{(i\omega)^2}{3!m} f(0)g(0) + \frac{i\omega}{2m} f(0)g'(0) + f'(0)g(0) - \frac{i\omega}{3!m^2} f(0)g''(0) + O(1).
\]
\[
\text{Proof. Denote}
\]
\[
(3.12) \quad K(z) = e^{i\omega z}g(z)
\]
to write \( K_{g,f}(z,t) \) as
\[
(3.13) \quad K_{g,f}(z,t) = \frac{K(z)f(t+z)}{(t+z)^4}.
\]
We have
\[
R(t) = \frac{1}{3!} \partial_z \partial_z (f(z)K(z-t))|_{z=0}.
\]
To compute \( R(t) \), we start with Leibniz’s formula (product rule)
\[
(3.14) \quad \partial_z^3(h_1(z)h_2(z)) = \sum_{\alpha=0}^{3} \frac{3!}{\alpha!(3-\alpha)!} \partial_z^{3-\alpha}h_1(z) \cdot \partial_z^{\alpha}h_2(z).
\]
We have, by using (3.14),
\[
R(t) = \sum_{\alpha=0}^{3} \frac{1}{\alpha!(3-\alpha)!} f^{(\alpha)}(0) \partial_z^{3-\alpha}K(\tau)
\]
where
\[
\tau = -t.
\]
We use \( K(\tau) = e^{i\omega \tau}g(\tau) \) to obtain
\[
K^{(3-\alpha)}(\tau) = e^{i\omega \tau} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}(3-\alpha)!}{\gamma!(3-\alpha-\gamma)!} g^{(\gamma)}(\tau).
\]
This is then for us to have
\[
R(t) = \sum_{\alpha=0}^{3} \frac{1}{\alpha!(3-\alpha)!} f^{(\alpha)}(0)e^{i\omega t} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}(3-\alpha)!}{\gamma!(3-\alpha-\gamma)!} g^{(\gamma)}(\tau)
\]
\[
e^{i\omega t} \sum_{\alpha=0}^{3} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}}{\alpha!\gamma!(3-\alpha-\gamma)!} f^{(\alpha)}(0)g^{(\gamma)}(\tau)
\]
\[
e^{-i\omega t} \sum_{\alpha=0}^{3} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}}{\alpha!\gamma!(3-\alpha-\gamma)!} f^{(\alpha)}(0)g^{(\gamma)}(-t).
\]
This is for us to have
\[
I_{g,f}^{(m)} = \sum_{\alpha=0}^{3} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}}{\alpha!\gamma!(3-\alpha-\gamma)!} f^{(\alpha)}(0)T_{g}^{\alpha,\gamma,(m)}
\]
where
\[
T_{g}^{\alpha,\gamma,(m)} = \int_{0}^{-\infty} e^{-im\omega t} g^{(\gamma)}(\tau) dt.
\]
We have
\[
T_{g}^{\alpha,\gamma,(m)} = -\int_{0}^{+\infty} e^{im\omega t} g^{(\gamma)}(t) dt = -\frac{1}{im\omega} \int_{0}^{+\infty} g^{(\gamma)}(t) de^{im\omega t}
\]
\[
= \frac{1}{im\omega} g^{(\gamma)}(0) + \frac{1}{(im\omega)^{2}} \int_{0}^{+\infty} g^{(\gamma+1)}(t) de^{im\omega t}
\]
\[
= \frac{1}{im\omega} g^{(\gamma)}(0) - \frac{1}{(im\omega)^{2}} g^{(\gamma+1)}(0) + O(\omega^{-3}).
\]
It then follows from (3.15) that
\[
I_{g,f}^{(m)} = \frac{(i\omega)^{2}}{3im} f(0)g(0) + \frac{i\omega}{2im} (f(0)g'(0) + f'(0)g(0)) - \frac{i\omega}{3im^2} f(0)g''(0) + O(1).
\]
\[\square\]

3.4. **Evaluating $R_1$ and $R_2$.** For a continuous curve $\ell$ in the complex $z$-plane, we let
\[
I_{\ell}(t) = \int_{\ell} e^{i\omega z} G(z)b(t+z)a^{2}(t+z)dz
\]
where $t \neq 0$ is a real parameter. Let
\[
\ell_{1}(t) = \{ z = t_{1} + is_{1}, \quad t_{1} \in (-\infty, +\infty), \quad s_{1} = 0 \}
\]
\[
\ell_{2}(t) = \{ z = t_{1} + is_{1}, \quad t_{1} \in (-\infty, +\infty), \quad s_{1} = i\rho \}
\]
where $\rho = 3\pi/2 - \omega^{-1}$ for $\ell_{2}$. First, we work on $I_{\ell_{2}}$.

**Lemma 3.4.** There exists a constant $C$ independent of $\omega$ such that
\[
|I_{\ell_{2}}| < C\omega^{4}e^{-3\pi\omega/2}e^{-|t|}.
\]

**Proof.** Recall that $\omega = 3\pi/2 - \omega^{-1}$. First, by the asymptotic behavior of $G(t), b(t), a(t)$ on $\ell_{2}$, we have
\[
|G(t + i\rho)| \leq Ce^{-|t|}, \quad |b(t + i\rho)a^{2}(t + i\rho)| \leq C\omega^{4}e^{-3|t|}.
\]
We have a factor $\omega^{4}$ in the second estimate because the order of the pole of the function $b(t)a^{2}(t)$ at $t = 3i\pi/2$ is four. Note that the distance from $\ell_{2}$ to the pole located at $-t + 3i\pi/2$ is $\geq \omega^{-1}$. We have
\[
I_{\ell_{2}} = e^{-3\pi\omega/2 + 1} \int_{-\infty}^{+\infty} e^{i\omega t_{1}} G(t_{1} + i\rho)b(t + t_{1} + i\rho)a^{2}(t + t_{1} + i\rho)dt_{1}
\]
\[
e^{-3\pi\omega/2 + 1} e^{-i\omega t} \int_{-\infty}^{+\infty} e^{i\omega s_{1}} G(s_{1} - t + i\rho)b(s_{1} + i\rho)a^{2}(s_{1} + i\rho)ds_{1}.
\]
This implies
\[ |I_{\ell}| \leq e^{-3\pi\omega/2+1} \int_{-\infty}^{+\infty} |G(s_1 - t + \text{i} \rho)b(s_1 + \text{i} \rho)a^2(s_1 + \text{i} \rho)| \, ds_1 \]
\[ \leq C\omega^4 e^{-3\pi\omega/2} \int_{-\infty}^{+\infty} e^{-|s_1-t|^{4}} e^{-3|s_1|} \, ds_1 \]
\[ \leq C\omega^4 e^{-3\pi\omega/2} \int_{-\infty}^{+\infty} e^{-|t|-|s_1|} e^{-3|s_1|} \, ds_1 \]
\[ \leq C\omega^4 e^{-3\pi\omega/2} e^{-|t|}. \]

For the second inequality here, we use (3.19); for the third, we use \(|s_1 - t| \geq |t| - |s_1|\).

\[ \square \]

**Lemma 3.5.** We have
\[ R_1 + R_2 = 2\pi\omega e^{-\pi\omega/2} \left( \frac{31}{270} + O(\omega^{-1}) \right) + O(\omega^4 e^{-3\pi\omega/2}). \]

**Proof.** We have, by definition,
\[ R_1 = \int_{0}^{\infty} e^{3i\omega t} I_{\ell}, \quad R_2 = \int_{0}^{\infty} e^{-2i\omega t} I_{\ell}, \]

By the residue theorem,
\[ I_{\ell} = I_{\ell} + 2\pi i \text{ Res} (e^{i\omega z} G(z) b(t + z) a^2(t + z)) \big|_{z=-t+i\pi/2} = I_{\ell} + 2\pi i e^{-\pi\omega/2} R(t) \]

where
\[ R(t) = \text{ Res} (e^{i\omega z} G(z) b(t + z) a^2(t + z)) \big|_{z=-t+i\pi/2} = \text{ Res} \left( \frac{e^{i\omega z_1} g(z_1) f(z_1 + t)}{(z + z_1)^{4}} \right) \bigg|_{z_1=-t}. \]

We note that \( z_1 \) in the second equality is such that \( z_1 = z - i\pi/2 \) and the functions \( f, g \) are as in (3.2). We then have by Lemma 3.4,
\[ R_1 = \int_{0}^{\infty} e^{3i\omega t} I_{\ell}(t) \, dt = \int_{0}^{\infty} e^{3i\omega t} \left( I_{\ell} + 2\pi i e^{-\pi\omega/2} R(t) \right) \, dt = 2\pi i e^{-\pi\omega/2} I_{g,f}^{(3)} + O(\omega^4 e^{-3\pi\omega/2}). \]

In parallel, we have
\[ R_2 = 2\pi i e^{-\pi\omega/2} I_{g,f}^{(-2)} + O(\omega^4 e^{-3\pi\omega/2}). \]

By using Lemmas 3.3 and 3.1,
\[ I_{g,f}^{(m)}(\omega) = \left( \frac{1}{2m} \right) \left( \frac{1}{6m^2} \right) (i\omega) f(0) g'(0) + O(1). \]

This gives us
\[ I_{g,f}^{(3)}(\omega) + I_{g,f}^{(-2)}(\omega) = -\frac{31\omega i}{54 \times 5} + O(1). \]

We finally conclude
\[ R_1 + R_2 = 2\pi\omega e^{-\pi\omega/2} \left( \frac{31}{270} + O(\omega^{-1}) \right) + O(\omega^4 e^{-3\pi\omega/2}). \]

\[ \square \]

**Proof of Proposition 3.1.** Let
\[ I_{m_1,m_2,n} = \int_{-\infty}^{+\infty} e^{i\omega t} a^{m_1} m_2(t) \, dt. \]

Recall that \( a(t), b(t) \) are meromorphic function in the complex \( t \)-plane and both functions take \((2n + 1)\pi i/2\) for all \( n \) as poles. By a direct application of the residue theorem, we obtain
\[ |I_{m_1,m_2,n}| \leq K\omega^{m_1+2m_2-1} e^{-\omega \pi/2}. \]

We leave the details of this estimate to the reader. Applying (3.20) to \( E_1, E_2, E_3, \) we have
\[ |E_1| \leq K\omega^{3} e^{-3\pi\omega/2}, \quad |E_2| \leq K\omega^{3} e^{-\pi\omega}, \quad |E_3| \leq K\omega^{6} e^{-3\pi\omega/2}. \]
Proposition 3.1 then follows from Lemma 3.2 by combining these estimates with the conclusion of Lemma 3.5.

4. PROOF OF THE MAIN THEOREM

First, we prove

Lemma 4.1. \(|I_{4,s}| < K\omega^3 e^{-\pi\omega}, \ |I_{6,s}| < K\omega^3 e^{-3\pi\omega/2}\).

Proof. We start with \(I_{4,s}\). The process of Sect. 3.2 is now closely followed to decompose \(I_{4,s}\) into a collection of integrals to obtain

\[ I_{4,s} = R_1 + R_2 + E_1 + E_2 + E_3 \]

where

\[
R_1 = -\int_{-\infty}^{0} e^{2i\omega s_2} \left( \int_{-\infty}^{+\infty} e^{4i\omega_1} G(s_1) b(s_1 + s_2) a^2(s_1 + s_2) ds_1 \right) ds_2
\]

\[
R_2 = -\int_{-\infty}^{0} e^{-2i\omega s_2} \left( \int_{-\infty}^{+\infty} e^{-4i\omega_1} G(s_1) b(s_1 + s_2) a^2(s_1 + s_2) ds_1 \right) ds_2
\]

\[
E_1 = \left( \int_{0}^{+\infty} a(\tau) H(\tau) e^{2i\omega \tau} d\tau \right) \left( \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{2i\omega \tau} d\tau \right)
\]

\[
E_2 = \left( \int_{0}^{+\infty} a(\tau) H(\tau) e^{-2i\omega \tau} d\tau \right) \left( \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{-2i\omega \tau} d\tau \right)
\]

\[
E_3 = -\frac{3\pi i}{8} \left( \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{2i\omega \tau} d\tau \right) \left( \int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{2i\omega \tau} d\tau \right).
\]

In contrast to \(R_1, R_2\) for \(I_{1,s}\), the triangular part of the inner integrals for \(R_1, R_2\) here is respectively \(e^{4i\omega_1}\) and \(e^{-4i\omega_1}\). Consequently, applying the residue theorem to the inner integral would induce a factor \(e^{-2\pi\omega}\) for \(I_{4,s}\) instead of \(e^{-\pi\omega/2}\) for \(I_{1,s}\). It then follows directly that we have

\[ |R_1|, |R_2| < K\omega^3 e^{-2\pi\omega}. \]

We further have, for \(E_1, E_2, E_3\),

\[ |E_1|, |E_2|, |E_3| < K\omega^3 e^{-\pi\omega}. \]

In conclusion, we have

\[ |I_{4,s}| < K\omega^3 e^{-\pi\omega}. \]

The proof for \(I_{6,s}\) is similar. 

By Lemma 2.1 and Corollary 2.1, we have

\[ E_1(t_0, \omega) = \frac{P(0, t_0, \omega)}{\sqrt{2}} E_0(t_0, \omega) - \omega (I_{1,s} \sin \omega t_0 + I_{4,s} \sin 4\omega t_0 + I_{5,s} \sin 5\omega t_0 + I_{6,s} \sin 6\omega t_0). \]

To continue, we let \(t_0 = \omega^{-1} \pi/5\). We have

\[ \sup_{t_0 \in [0, 2\pi\omega^{-1}]} |E(t_0, \omega)| \geq E(\omega^{-1} \pi/5, \omega), \]

and

\[ E(\omega^{-1} \pi/5, \omega) = \frac{P(0, \omega^{-1} \pi/5, \omega)}{\sqrt{2}} E_0(\omega^{-1} \pi/5, \omega) - \omega (I_{1,s} \sin \pi/5 + I_{4,s} \sin 4\pi/5 + I_{6,s} \sin 6\pi/5). \]

It then follows from Proposition 1.1, Proposition 3.1 and Lemma 4.1 that

\[ \sup_{t_0 \in [0, 2\pi\omega^{-1}]} |E(t_0, \omega)| \geq C_0\omega^2 e^{-\pi\omega/2} \]

where \(C_0 = \frac{\pi}{6} \sin \frac{\pi}{6}\) provided that \(\omega\) is sufficiently large. 

\[ \square \]
ON THE NON-DOMINANCE OF THE MELNIKOV FUNCTION IN A PERIODICALLY PERTURBED DUFFING EQUATION

References


(Ali Oksasoglu) Honeywell Corporation, 11100 N. Oracle Rd., Tucson, AZ 85737
Email address: ali.oksasoglu@honeywell.com

(Qidong Wang) Dept. of Mathematics, University of Arizona, Tucson, AZ 85750, U.S.A
Email address: dwang@math.arizona.edu