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High order Melnikov method: Pendulums

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Abstract

In this paper, we present an explicit integral formula for the high order Melnikov function $D_1(t_0)$ for periodically perturbed pendulum equations. The acquired integral formula is then applied to a specific pendulum equation to offer an example of high frequency perturbation, of which the classical Melnikov function $D_0(t_0)$ fails to dominate the higher order term $\varepsilon D_1(t_0)$.

1. Statement of results

The Poincare/Melnikov method was first introduced by Poincare ([7]) for a pendulum equation, then extended later by Melnikov ([6]) to cover time-periodic Hamiltonian equations. This method has served as a main venue through which the chaos theory, in particular, the theory of Smale horseshoe [9], is applied to the study of ordinary differential equations [4]. Let $D(t_0, \varepsilon)$ be the splitting distance of the stable and unstable manifold of a saddle fixed point of a time-periodic equation. We expand $D(t_0, \varepsilon)$ into a formal power series in ε as

$$D(t_0, \varepsilon) = D_0(t_0) + \varepsilon D_1(t_0) + \cdots + \varepsilon^n D_n(t_0) + \cdots .$$

Poincare and Melnikov derived an explicit integral formula for $D_0(t_0)$, reducing the task of verifying the existence of homoclinic tangles in a given equation to that of a non-tangential zero of $D_0(t_0)$.

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There are, however, two main degenerate cases to which the Poincare/Melnikov method fails to apply. The first is the case of $D_0(t_0) \equiv 0$ (see [2]) and the second is the case of high frequency perturbation assuming ε, the magnitude of perturbation, is in negative power of the forcing frequency ω (see [8], [5], [3], [1], [10]). In both cases, we would need to calculate $D_1(t_0)$ to assert the existence of transversal homoclinic intersections. In a recent paper [2], Chen and Wang introduced a new theory on $D_1(t_0)$ for a class of time-periodic equations. Not only they derived an explicit integral formula for $D_1(t_0)$, but also they introduced a long analytic scheme to calculate $D_1(t_0)$. This computational scheme was also applied to a given time-periodic equation of $D_0(t_0) \equiv 0$, asserting the existence of transversal homoclinic intersections by using $D_1(t_0)$. The theory of [2], unfortunately, does not cover pendulum equations.

The purpose of this paper is to extend the theory of [2] to cover the pendulum equations. We study the non-autonomous second order equation

$$\frac{d^2x}{dt^2} = -\sin x + \varepsilon \cos^2(x/2) \cdot P(t), \tag{1.1}$$

which we rewrite as

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\sin x + \varepsilon \cos^2(x/2) \cdot P(t) \tag{1.2}$$

where $P(t)$ is a periodic function of period T in t and ε is a small parameter representing the magnitude of the perturbation. The unperturbed equation

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\sin x \tag{1.3}$$

has a heteroclinic solution $(x(t), y(t)) = (a(t), b(t))$ where

$$a(t) = 2\sin^{-1}\frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad b(t) = \frac{4}{e^t + e^{-t}}. \tag{1.4}$$

For $\ell = \{(x, y) = (a(t), b(t)) :\ t \in (-\infty, +\infty)\}$, let D_ℓ be an $O(\varepsilon)$-neighborhood of the closure of ℓ and I be the intersection of D_ℓ with the y-axis, which we can properly regard as a small segment of the y-axis around $y = b(0) = 2$. Let t_0 be a given initial time, and $(x(t, t_0), y(t, t_0))$ be the solution of (1.2) such that $(x(t_0, t_0), y(t_0, t_0)) = (0, y_0) \in I$. We say that $(x(t, t_0), y(t, t_0))$ is a *primary* stable solution if $(x(t, t_0), y(t, t_0)) \in D_\ell$ for all $t \in [t_0, +\infty)$ and $\lim_{t\to+\infty}(x(t, t_0), y(t, t_0)) = (\pi, 0)$. Likewise, $(x(t, t_0), y(t, t_0))$ is a *primary* unstable solution if $(x(t, t_0), y(t, t_0)) \in D_\ell$ for all $t \in (-\infty, t_0]$ and $\lim_{t\to-\infty}(x(t, t_0), y(t, t_0)) = (-\pi, 0)$. For a given $t_0 \in (0, T]$, the primary stable and the primary unstable solutions are both uniquely defined assuming ε is sufficiently small. We use $(x^s(t, t_0), y^s(t, t_0))$ for the primary stable solution and $(x^u(t, t_0), y^u(t, t_0))$ for the primary unstable solution. Denote the points of intersection of the primary stable solution and the primary unstable solution with I as $(0, y_0^s(t_0, \varepsilon))$ and $(0, y_0^u(t_0, \varepsilon))$ respectively.

For $(x, y) \in D_\ell$, let

$$E(x, y) = \varepsilon^{-1}\left(\frac{1}{2}y^2 - (1 + \cos x)\right)$$

be the unperturbed energy re-scaled by ε^{-1}.

Definition 1.1. The **splitting distance** $D(t_0, \varepsilon)$ is defined by letting

$$D(t_0, \varepsilon) = E(0, y_0^s(t_0, \varepsilon)) - E(0, y_0^u(t_0, \varepsilon)).$$

Let us now expand the splitting distance into a power series of ε as

$$D(t_0, \varepsilon) = D_0(t_0) + \varepsilon D_1(t_0) + \cdots + \varepsilon^n D_n(t_0) + \cdots . \tag{1.5}$$

The Poincare/Melnikov method offers a simple integral formula for $D_0(t_0)$. It is the classical Melnikov function given by

$$D_0(t_0) = -\frac{1}{4} \int_{-\infty}^{+\infty} b^3(t) P(t + t_0) dt. \tag{1.6}$$

The main results of this paper are as follows.

(A) *An Integral Formula on* $D_1(t_0)$. We obtain an integral formula for $D_1(t_0)$. In what follows, we also let

$$h(t) = \frac{4t - (e^{2t} - e^{-2t})}{2(e^t + e^{-t})^2}; \quad H(t) = \frac{1}{b(t)} \left(b(t)h(t) - b'(t) \right). \tag{1.7}$$

Theorem 1. *For equation (1.2), we have*

$$D_1(t_0) = -\frac{1}{16} \int_{-\infty}^{+\infty} \int_0^{\tau_1} H(\tau_2) b(\tau_2) b^3(\tau_1) \mathcal{P}(\tau_1, \tau_2, t_0) d\tau_2 d\tau_1 \tag{1.8}$$

where

$$\mathcal{P}(\tau_1, \tau_2, t_0) = P_t(\tau_1 + t_0) P(\tau_2 + t_0) + P_t(\tau_2 + t_0) P(\tau_1 + t_0). \tag{1.9}$$

(B) *An Example.* We calculate $D_1(t_0)$ for the pendulum equation

$$\frac{d^2x}{dt^2} = -\sin x + \varepsilon \cos^2(x/2)(\cos 2\omega t + \cos 3\omega t). \tag{1.10}$$

For this equation, it is easy to obtain that

$$D_0(t_0) = \frac{8\pi(8\omega^2 + 1)}{e^{\pi\omega} + e^{-\pi\omega}} \sin 2\omega t_0 + \frac{8\pi(18\omega^2 + 1)}{e^{3\pi\omega/2} + e^{-3\pi\omega/2}} \sin 3\omega t_0. \tag{1.11}$$

Theorem 2. *We have, for equation (1.10)*

$$D_1(t_0, \omega) = -\pi\omega^2 e^{-\pi\omega/2} \left(\frac{1}{36} + O(\omega^{-1}) \right) \sin \omega t_0 + O(e^{-2\pi\omega/3}).$$

There has been a substantial literature on the study of equations of high frequency perturbations assuming the magnitude of forcing is in negative power of the forcing frequency. Elaborated theories have been developed by a number of authors. This literature strives to prove that, for Hamiltonian equations subjected to high frequency perturbation, $R(t_0) := D(t_0, \varepsilon) - D_0(t_0)$ remains to be dominated by $D_0(t_0)$. Their conclusions are derived through sophisticated analysis of solutions of the Hamilton-Jacobi equation by using techniques borrowed from complex Fourier analysis. We refer the reader to [1] and the references therein for a thorough review on this literature. A rough reformulation of the main conclusions of [1] and its predecessors, to us, is as follows. Let

$$g_{\pm 1} = \int_0^{2\pi\omega^{-1}} D_0(t_0) e^{\pm i\omega t_0} dt_0$$

be the first order Fourier coefficient of $D_0(t_0)$. Under the assumption that

$$g_1^2 + g_{-1}^2 \neq 0, \tag{1.12}$$

the C^1-norm of $D_0(t_0)$ is much larger than that of the remainder $R(t_0) := D(t_0, \varepsilon) - D_0(t_0)$ provided that ω is sufficiently large and ε is in integer power of ω^{-1}. For equation (1.10), however,

$$g_1^2 + g_{-1}^2 = 0.$$

Consequently, neither the Poincare/Melnikov method, nor the theory of [1] and all its predecessors, apply to this example. As a matter of fact, it follows directly from (1.11) and Theorem 2 in the above that the C^1-norm of $D_0(t_0)$ is indeed much smaller than that of $\varepsilon D_1(t_0)$. This is to say that, in pursuing the dominance of the classical Melnikov function $D_0(t_0)$ over the rest of the splitting distance, (1.12) is not only sufficient but also necessary. The lack of dominance of $D_0(t_0)$ in this example is caused by interactions of terms of different forcing frequencies in perturbation. Such interactions do not show up in $D_0(t_0)$, but they have a clear presence in $D_1(t_0)$.

Finally, we note that Theorem 2 alone does not imply the existence of transversal heteroclinic intersection for equation (1.10). In order to prove the existence of homoclinic tangles for equation (1.10), we would need to further acquire upper bound estimates on $D_n(t_0)$ for all $n \geq 1$. In [10], a systematic theory on $D_n(t_0)$ for all $n \geq 1$ was developed for the purpose of acquiring a corresponding upper bounds. Unfortunately, the conclusions of [10] can not be evoked here because the theory of [10] is on a second order Duffing equation of non-Hamiltonian perturbation with a single forcing frequency. To cover equation (1.10), we would need a generalized version of [10], which is not yet available in literature.

2. Proof of Theorem 1

We start the derivation of the acclaimed integral formula for $D_1(t_0)$ by solving the equations of first variations of the unperturbed equation (1.3) in Sects. 2.1 and 2.2. We then drive a set of integral equations for the primary stable solutions in Sect. 2.3. Theorem 1 is then proved in Sect. 2.4.

We have for the heteroclinic solution $(a(t), b(t))$,

$$\sin(a(t)/2) = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad \cos(a(t)/2) = \frac{2}{e^t + e^{-t}}, \quad \sin a(t) = \frac{4(e^t - e^{-t})}{(e^t + e^{-t})^2}. \tag{2.1}$$

It is such that

$$(a(0), b(0)) = (0, 2), \quad \lim_{t\to-\infty}(a(t), b(t)) = (-\pi, 0), \quad \lim_{t\to+\infty}(a(t), b(t)) = (\pi, 0).$$

We also have for $(a(t), b(t))$,

$$a'(t) = b(t), \quad b'(t) = -\sin a(t), \quad b''(t) = -b(t)\cos a(t), \quad b^2(t) = 2(1 + \cos a(t)) \tag{2.2}$$

where the first two equities hold because $(a(t), b(t))$ is a solution of equation (1.3), the third equality is obtained by taking one more derivative on the second, and the fourth is the energy integral.

2.1. Solving the equations of first variations

Let $(x, y) = (a(t), b(t))$ be a solution of the second order autonomous equation

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = f(x).$$

The equations of first variations around $\ell(t) = (a(t), b(t))$ are

$$\frac{d\xi}{dt} = \eta; \qquad \frac{d\eta}{dt} = f'(a)\xi. \tag{2.3}$$

We introduce a specific set of coordinates to solve equation (2.3). Let z_1, z_2 be such that

$$z_1 = \frac{1}{\sqrt{R}}(b'\xi - b\eta); \quad z_2 = \frac{1}{\sqrt{R}}((R - b'g(a))b^{-1}\xi + g(a)\eta) \tag{2.4}$$

where R, g are arbitrary functions. We have in reverse,

$$\xi = \frac{1}{\sqrt{R}}(g(a)z_1 + bz_2); \quad \eta = \frac{1}{\sqrt{R}}(-(R - b'g(a))b^{-1}z_1 + b'z_2). \tag{2.5}$$

Lemma 2.1. *The equations of first variations (2.3) are transformed in z_1, z_2 to*

$$\begin{aligned}\frac{dz_1}{dt} &= -\frac{R'}{2R}z_1\\ \frac{dz_2}{dt} &= Az_1 + \frac{R'}{2R}z_2\end{aligned} \tag{2.6}$$

where

$$A = R'b^{-1}g(a) - g'(a)R - (R - b'g(a))b^{-2}R.$$

Proof. We note that z_1 is the projection of (ξ, η) in the normal direction of the solution (a, b), a variable previously adopted by Melnikov, and z_2 is so designed that the only divisive factor involved in both directions of this change of coordinates is $\sqrt{R}$. The purpose of this design is to use R to control the potential singularities introduced by change of coordinates. The function g is adopted to give us more flexibility in simplifying A.

For z_1 we have

$$\frac{dz_1}{dt} = \frac{1}{\sqrt{R}}(b''\xi + b'\xi' - b'\eta - b\eta') - \frac{R'}{2R}z_1 = -\frac{R'}{2R}z_1.$$

We also have

$$\begin{aligned}
\frac{dz_2}{dt} &= \frac{1}{\sqrt{R}}\Big((R' - b'g'(a)b - b''g(a))b^{-1}\xi - (R - b'g(a))b^{-2}b'\xi + (R - b'g(a))b^{-1}\eta\Big) \\
&\quad + \frac{1}{\sqrt{R}}\left(g'(a)b\eta + g(a)f'(a)\xi\right) - \frac{R'}{2R}z_2 \\
&= \frac{1}{\sqrt{R}}[(R' - b'g'(a)b - b''g(a)))b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a)]\xi \\
&\quad + \frac{1}{\sqrt{R}}[g'(a)b + (R - g(a)b')b^{-1}]\eta - \frac{R'}{2R}z_2.
\end{aligned}$$

To continue, we have

$$\begin{aligned}
\frac{dz_2}{dt} &= \frac{1}{R}[(R' - b'g'(a)b - b''g(a)))b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a)](g(a)z_1 + bz_2) \\
&\quad + \frac{1}{R}\Big([g'(a)b + (R - g(a)b')b^{-1}](-(R - b'g(a))b^{-1}z_1 + b'z_2)\Big) - \frac{R'}{2R}z_2 \\
&= \frac{1}{R}[(R' - b'g'(a)b - b''g(a)))b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a)](g(a)z_1) \\
&\quad + \frac{1}{R}\Big([g'(a)b + (R - g(a)b')b^{-1}](-(R - b'g(a))b^{-1}z_1)\Big) + \frac{R'}{2R}z_2 \\
&= \frac{A}{R}z_1 + \frac{R'}{2R}z_2
\end{aligned}$$

where

$$\begin{aligned}
A &= [(R' - b'g'(a)b - b''g(a)))b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a)](g(a)) \\
&\quad + \Big([g'(a)b + (R - g(a)b')b^{-1}](-(R - b'g(a))b^{-1})\Big) \\
&= R'b^{-1}g(a) - g'(a)R - (R - b'g(a))b^{-2}R. \quad \square
\end{aligned}$$

We further remove the dependency of the equation for z_2 on z_1 by letting

$$w = hz_1 + z_2$$

We have

$$\frac{dw}{dt} = h'z_1 - h\frac{R'}{2R}z_1 + \frac{A}{R}z_1 + \frac{R'}{2R}z_2.$$

Let $h = h(t)$ be such that

$$h' - \frac{R'}{R}h + \frac{A}{R} = 0. \tag{2.7}$$

We obtain

$$\frac{dw}{dt} = \frac{R'}{2R}w.$$

Re-write z_1 as $\tilde{\xi}$ and w as $\tilde{\eta}$, we have

$$\xi = \frac{1}{\sqrt{R}}\left(b\tilde{\eta} - \sqrt{R}H\tilde{\xi}\right); \quad \eta = \frac{1}{\sqrt{R}}\left(b'\tilde{\eta} - \sqrt{R}\tilde{H}\tilde{\xi}\right) \tag{2.8}$$

where

$$H = \frac{1}{\sqrt{R}}(bh - g(a)); \quad \tilde{H} = \frac{1}{\sqrt{R}}(b'h + (R - b'g(a))b^{-1}). \tag{2.9}$$

In reverse, we have

$$\tilde{\xi} = \frac{1}{\sqrt{R}}(b'\xi - b\eta); \quad \tilde{\eta} = \frac{1}{\sqrt{R}}\left((b'h + (R - b'g(a))b^{-1})\xi + (g(a) - bh)\eta\right). \tag{2.10}$$

In conclusion, we have

Lemma 2.2. *The equations of first variations in new variable $\tilde{\xi}$, $\tilde{\eta}$ are*

$$\frac{d\tilde{\xi}}{dt} = -\frac{R'}{2R}\tilde{\xi}; \quad \frac{d\tilde{\eta}}{dt} = \frac{R'}{2R}\tilde{\eta}. \tag{2.11}$$

We also include an important technical equality for future use.

Lemma 2.3. *We have*

$$\left(\frac{H}{\sqrt{R}}\right)' = \frac{\tilde{H}}{\sqrt{R}}.$$

Proof.

$$
\begin{aligned}
\left(\frac{H}{\sqrt{R}}\right)' &= -\frac{1}{R^2}(bh-g(a))R' + \frac{1}{R}(b'h+bh'-g'(a)b) \\
&= -\frac{1}{R^2}(bh-g(a))R' + \frac{1}{R}\left(b'h+b\left(\frac{R'}{R}h-\frac{A}{R}\right)-g'(a)b\right) \\
&= -\frac{1}{R^2}(bh-g(a))R' \\
&\quad + \frac{1}{R}\left(b'h+b\left(\frac{R'}{R}h-\frac{R'b^{-1}g(a)-g'(a)R-(R-b'g(a))b^{-2}R}{R}\right)-g'(a)b\right) \\
&= \frac{1}{R}(b'h+b\left((R-b'g(a))b^{-1}R\right)) \\
&= \frac{\tilde{H}}{\sqrt{R}}. \quad \square
\end{aligned}
$$

2.2. *Equations of first variations around heteroclinic solution*

We apply the result in the previous subsection on the equations of first variations to the unperturbed equation (1.3) around the heteroclinic solution $(a(t), b(t))$. The equations of first variations are

$$
\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = U \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{2.12}
$$

where

$$
U = \begin{pmatrix} 0 & 1 \\ -\cos a & 0 \end{pmatrix}. \tag{2.13}
$$

For this specific problem, we let

$$
R = b^2(t), \quad g(a) = b'(t) = -\sin a(t). \tag{2.14}
$$

We have, for this specific choice of R and g,

$$
h' - \frac{2b'}{b}h + \frac{(b')^2}{b^2} = 0, \tag{2.15}
$$

and

$$
h(t) = \frac{4t-(e^{2t}-e^{-2t})}{2(e^t+e^{-t})^2}. \tag{2.16}
$$

We also have

$$
H(t) = \frac{1}{b}\left(bh-b'\right), \quad \tilde{H}(t) = \frac{1}{b}\left(\frac{1}{4}b^3+hb'\right). \tag{2.17}
$$

The new variables $\tilde{\xi}, \tilde{\eta}$ are

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = T \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix}$$

where

$$T = \begin{pmatrix} -H & 1 \\ -\tilde{H} & \frac{b'}{b} \end{pmatrix}. \tag{2.18}$$

Lemma 2.4. $\det(T) = 1$.

Proof. We have

$$\begin{aligned} \det(T) = & -\frac{b'}{b}H + \tilde{H} = -\frac{b'}{b}\frac{1}{b}\left(bh - b'\right) + \frac{1}{b}\left(\frac{1}{4}b^3 + hb'\right) \\ = & \frac{(b')^2}{b^2} + \frac{1}{4}b^2 = \frac{\sin^2 a}{2(1+\cos a)} + \frac{1}{2}(1+\cos a) \\ = & 1. \quad \square \end{aligned}$$

This lemma implies,

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = T^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where

$$T^{-1} = \begin{pmatrix} \frac{b'}{b} & -1 \\ \tilde{H} & -H \end{pmatrix}. \tag{2.19}$$

Lemma 2.5. *The equations of first variations (2.12) are transformed in new variables $(\tilde{\xi}, \tilde{\eta})$ to*

$$\tilde{\xi}' = -\frac{b'}{b}\tilde{\xi}; \quad \tilde{\eta}' = \frac{b'}{b}\tilde{\eta}. \tag{2.20}$$

Proof. By definition, we have

$$\begin{pmatrix} \tilde{\xi}' \\ \tilde{\eta}' \end{pmatrix} = B \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix}$$

where

$$B = \left(\left(T^{-1}\right)' T + T^{-1} U T\right).$$

To calculate B, we start with

$$\left(T^{-1}\right)' = \begin{pmatrix} \left(\frac{b'}{b}\right)' & 0 \\ \frac{1}{2}bb' + h'\frac{b'}{b} + h\left(\frac{b'}{b}\right)' & -h' + \left(\frac{b'}{b}\right)' \end{pmatrix}.$$

Note that

$$\left(\frac{b'}{b}\right)' = \frac{b''b - (b')^2}{b^2} = \frac{-(\cos a)b^2 - (\sin a)^2}{b^2} = -\cos a - \frac{(\sin a)^2}{2(1+\cos a)} = -\frac{1}{4}b^2.$$

We have

$$\begin{aligned} \left(T^{-1}\right)' T &= \begin{pmatrix} -\frac{1}{4}b^2 & 0 \\ \frac{1}{2}bb' + h'\frac{b'}{b} - \frac{1}{4}b^2 h & -h' - \frac{1}{4}b^2 \end{pmatrix} \begin{pmatrix} -H & 1 \\ -\tilde{H} & \frac{b'}{b} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}b^2 H & -\frac{1}{4}b^2 \\ -\left(\frac{1}{2}bb' + h'\frac{b'}{b} - \frac{1}{4}b^2 h\right) H + \left(h' + \frac{1}{4}b^2\right)\tilde{H} & \frac{1}{4}bb' - \frac{1}{4}b^2 h \end{pmatrix}. \end{aligned}$$

We also have

$$\begin{aligned} T^{-1}UT &= \begin{pmatrix} \frac{b'}{b} & -1 \\ \tilde{H} & -H \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\cos a & 0 \end{pmatrix} \begin{pmatrix} -H & 1 \\ -\tilde{H} & \frac{b'}{b} \end{pmatrix} \\ &= \begin{pmatrix} \frac{b'}{b} & -1 \\ \tilde{H} & -H \end{pmatrix} \begin{pmatrix} -\tilde{H} & \frac{b'}{b} \\ H\cos a & -\cos a \end{pmatrix} \\ &= \begin{pmatrix} -\tilde{H}\frac{b'}{b} - H\cos a & \frac{(b')^2}{b^2} + \cos a \\ -\tilde{H}^2 - H^2\cos a & \frac{b'}{b}\tilde{H} + H\cos a \end{pmatrix}. \end{aligned}$$

This is for us to have

$$B = \left(T^{-1}\right)' T + T^{-1}UT = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where

$$\begin{aligned} B_{11} &= \frac{1}{4}b^2 H - \tilde{H}\frac{b'}{b} - H\cos a \\ B_{12} &= -\frac{1}{4}b^2 + \frac{(b')^2}{b^2} + \cos a \\ B_{21} &= -\left(\frac{1}{2}bb' + h'\frac{b'}{b} - \frac{1}{4}b^2 h\right) H + \left(h' + \frac{1}{4}b^2\right)\tilde{H} - \tilde{H}^2 - H^2\cos a \\ B_{22} &= \frac{1}{4}bb' - \frac{1}{4}b^2 h + \frac{b'}{b}\tilde{H} + H\cos a. \end{aligned}$$

We have

$$
\begin{aligned}
B_{11} &= \frac{1}{4}b^2 H - \tilde{H}\frac{b'}{b} - H\cos a \\
&= \frac{1}{4}b^2\frac{1}{b}\left(bh - b'\right) - \frac{1}{b}\left(\frac{1}{4}b^3 + hb'\right)\frac{b'}{b} - \frac{1}{b}\left(bh - b'\right)\cos a \\
&= \left(\frac{1}{4}b^2 h - \frac{1}{4}bb'\right) - \left(\frac{1}{4}b^2 + h\frac{b'}{b}\right)\frac{b'}{b} - \left(h - \frac{b'}{b}\right)\cos a \\
&= \left(\frac{1}{4}b^2 - \frac{b'}{b}\frac{b'}{b} - \cos a\right)h - \frac{1}{2}bb' - \left(-\frac{b'}{b}\right)\cos a \\
&= \left(\frac{1}{2}(1+\cos a) - \frac{\sin^2 a}{2(1+\cos a)} - \cos a\right)h - \frac{b'}{b}\left(\frac{1}{2}b^2 - \cos a\right) \\
&= -\frac{b'}{b};
\end{aligned}
$$

$$
B_{12} = -\frac{1}{4}b^2 + \frac{(b')^2}{b^2} + \cos a = -\frac{1}{2}(1+\cos a) + \frac{\sin^2 a}{2(1+\cos a)} + \cos a = 0;
$$

$$
\begin{aligned}
B_{21} &= -\left(\frac{1}{2}bb' + h'\frac{b'}{b} - \frac{1}{4}b^2 h - H\cos a\right)H + \left(h' + \frac{1}{4}b^2 - \tilde{H}\right)\tilde{H} \\
&= -\left(\frac{1}{2}bb' + \left(\frac{2b'}{b}h - \frac{(b')^2}{b^2}\right)\frac{b'}{b} - \frac{1}{4}b^2 h - H\cos a\right)H \\
&\quad + \left(\left(\frac{2b'}{b}h - \frac{(b')^2}{b^2}\right) + \frac{1}{4}b^2 - \tilde{H}\right)\tilde{H} \\
&= -\frac{b'}{b}\tilde{H}H + \frac{b'}{b}H\tilde{H} = 0;
\end{aligned}
$$

$$
B_{22} = \frac{1}{4}b^2\left(\frac{b'}{b} - h\right) + \frac{b'}{b}\tilde{H} + H\cos a = -\frac{1}{4}b^2 H + \frac{b'}{b}\tilde{H} + H\cos a = -B_{11}.
$$

This is for us to conclude

$$
B = \begin{pmatrix} -\frac{b'}{b} & 0 \\ 0 & \frac{b'}{b} \end{pmatrix}. \quad \square
$$

Note that we also have

$$
\left(\frac{H}{b}\right)' = \frac{\tilde{H}}{b} \tag{2.21}
$$

from Lemma 2.2.

2.3. Integral equations for primary stable solution

Let t_0 be a given initial time, and $(\hat{x}(t), \hat{y}(t))$ be the primary stable solution of the perturbed equation satisfying $(\hat{x}(t_0), \hat{y}(t_0)) = (0, y_0) \in I$. Let $(x(t), y(t)) = (\hat{x}(t+t_0), \hat{y}(t+t_0))$. Then, $(x(t), y(t))$ is well-defined on $t \in [0, +\infty)$ satisfying

$$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -\sin x + \varepsilon \cos^2(x/2) P(t+t_0) \tag{2.22}$$

and $x(0) = 0$, $y(0) = y_0$. This solution stays inside of D_ℓ for all $t \in [0, +\infty)$ and $(x(t), y(t)) \to (\pi, 0)$ as $t \to +\infty$.

Let X, Y be such that

$$\varepsilon X = x - a; \quad \varepsilon Y = y - b. \tag{2.23}$$

We have

$$\frac{dX}{dt} = Y; \quad \frac{dY}{dt} = -\varepsilon^{-1} \sin(\varepsilon X + a) + \varepsilon^{-1} \sin a + \cos^2(\varepsilon X/2 + a/2) P(t+t_0).$$

This is to have

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = U \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon Q \end{pmatrix} + \begin{pmatrix} 0 \\ \cos^2(\varepsilon X/2 + a/2) P(t+t_0) \end{pmatrix} \tag{2.24}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ -\cos a & 0 \end{pmatrix}; \quad Q(t, \varepsilon, X) = \frac{(\varepsilon X - \sin \varepsilon X)\cos a + (1 - \cos \varepsilon X)\sin a}{\varepsilon^2}.$$

We introduce new variables (M, W) by letting

$$\begin{pmatrix} X \\ Y \end{pmatrix} = T \begin{pmatrix} M \\ W \end{pmatrix}; \quad \begin{pmatrix} M \\ W \end{pmatrix} = T^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}$$

where

$$T = \begin{pmatrix} -H & 1 \\ -\tilde{H} & \frac{b'}{b} \end{pmatrix}; \quad T^{-1} = \begin{pmatrix} \frac{b'}{b} & -1 \\ \tilde{H} & -H \end{pmatrix}.$$

We also recall that

$$H(t) = \frac{1}{b}\left(bh - b'\right), \quad \tilde{H}(t) = \frac{1}{b}\left(\frac{1}{4}b^3 + hb'\right).$$

We have

$$X = -HM + W, \quad Y = -\tilde{H}M + \frac{b'}{b}W; \qquad M = \frac{b'}{b}X - Y, \quad W = \tilde{H}X - HY. \tag{2.25}$$

The new equations for M, W are

$$\frac{d}{dt}\begin{pmatrix} M \\ W \end{pmatrix} = \begin{pmatrix} -\frac{b'}{b} & 0 \\ 0 & \frac{b'}{b} \end{pmatrix}\begin{pmatrix} M \\ W \end{pmatrix} + T^{-1}\begin{pmatrix} 0 \\ \varepsilon Q(t, X, \varepsilon) \end{pmatrix} + T^{-1}\begin{pmatrix} 0 \\ \cos^2(\varepsilon X/2 + a/2)P(t+t_0) \end{pmatrix}. \tag{2.26}$$

This is to have

$$\begin{aligned} \frac{dM}{dt} &= -\frac{b'}{b}M - \varepsilon Q(t, X, \varepsilon) - \cos^2(\varepsilon X/2 + a/2)P(t+t_0) \\ \frac{dW}{dt} &= \frac{b'}{b}W - \varepsilon H Q(t, X, \varepsilon) - H\cos^2(\varepsilon X/2 + a/2)P(t+t_0) \end{aligned} \tag{2.27}$$

where

$$X = -HM + W.$$

Lastly, we let $(\mathbb{M}, \mathbb{W})$ be such that

$$\mathbb{M} = bM, \quad \mathbb{W} = b^{-1}W.$$

We have, in new variables $\mathbb{M}, \mathbb{W}$,

$$\begin{aligned} \frac{d\mathbb{M}}{dt} &= -\varepsilon b Q(t, X, \varepsilon) - b\cos^2(\varepsilon X/2 + a/2)P(t+t_0) \\ \frac{d\mathbb{W}}{dt} &= -\varepsilon b^{-1} H Q(t, X, \varepsilon) - Hb^{-1}\cos^2(\varepsilon X/2 + a/2)P(t+t_0) \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} Q(t, X, \varepsilon) &= \frac{(\varepsilon X - \sin\varepsilon X)\cos a + (1 - \cos\varepsilon X)\sin a}{\varepsilon^2} \\ X &= -b^{-1}H\mathbb{M} + b\mathbb{W}. \end{aligned}$$

We have

$$\begin{aligned} \mathbb{M}(t) &= \mathbb{M}(0) - \int_0^t \varepsilon b Q(t, X, \varepsilon)dt - \int_0^t b\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt \\ \mathbb{W}(t) &= \mathbb{W}(0) - \int_0^t \varepsilon b^{-1} H Q(t, X, \varepsilon)dt - \int_0^t Hb^{-1}\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt. \end{aligned} \tag{2.29}$$

Lemma 2.6. *We have, for a primary stable solution,*

$$\begin{aligned}\mathbb{M}(t) &= \int_t^{+\infty} \varepsilon b Q(t, X, \varepsilon)dt + \int_t^{+\infty} b\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt \\ \mathbb{W}(t) &= -\int_0^t \varepsilon b^{-1} H Q(t, X, \varepsilon)dt - \int_0^t H b^{-1}\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt.\end{aligned} \tag{2.30}$$

Proof. By definition, we have $X(0) = 0$, $H(0) = 0$. This is to imply

$$\mathbb{W}(0) = b^{-1}(0)W(0) = b^{-1}(0)(\tilde{H}(0)X(0) - H(0)Y(0)) = 0.$$

For $\mathbb{M}(0)$, we note that $X(t), Y(t) \to 0$ as $t \to +\infty$. This is to imply

$$\mathbb{M}(t) = bM = b'X - bY \to 0.$$

as $t \to +\infty$. We then obtain

$$\mathbb{M}(0) = \int_0^{+\infty} \varepsilon b Q(t, X, \varepsilon)dt + \int_0^{+\infty} b\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt.$$

The acclaimed integral formula for $\mathbb{M}(t)$ then follows directly. □

Expanding $\mathbb{M}$, $\mathbb{W}$ into power series of ε as

$$\mathbb{M}(t,\varepsilon) = \mathbb{M}_0(t) + \varepsilon\mathbb{M}_1(t) + \cdots; \quad \mathbb{W}(t,\varepsilon) = \mathbb{W}_0(t) + \varepsilon\mathbb{W}_1(t) + \cdots,$$

we then obtain

$$\mathbb{M}_0(t) = \int_t^{+\infty} b\cos^2(a/2)P(t+t_0)dt; \quad \mathbb{W}_0(t) = -\int_0^t H b^{-1}\cos^2(a/2)P(t+t_0)dt. \tag{2.31}$$

Note that

$$b^2 = 2(1+\cos a) = 4\cos^2(a/2).$$

We have

$$\mathbb{M}_0(t) = \frac{1}{4}\int_t^{+\infty} b^3 P(t+t_0)dt; \qquad \mathbb{W}_0(t) = -\frac{1}{4}\int_0^t HbP(t+t_0)dt. \tag{2.32}$$

2.4. The splitting distance

Let

$$E = \varepsilon^{-1}\left(\frac{1}{2}y^2 - (1+\cos x)\right)$$

be the unperturbed energy scaled by ε^{-1}. We have

$$\begin{aligned}\frac{dE}{dt} &= \varepsilon^{-1}\Big(y(-\sin x + \varepsilon\cos^2(x/2)P(t+t_0)) + y\sin x\Big)\\ &= (\varepsilon Y + b)\cos^2(\varepsilon X/2 + a/2)P(t+t_0)\\ &= b\cos^2(\varepsilon X/2 + a/2)P(t+t_0) + \varepsilon Y\cos^2(\varepsilon X/2 + a/2)P(t+t_0).\end{aligned}$$

This is for us to have

$$E(t) = E(0) + \int_0^t b\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt + \varepsilon\int_0^t Y\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt.$$

For a primary stable solution, we have $E(t) \to 0$ as $t \to +\infty$. It then follows that, for a primary stable solution,

$$E(0) = -\int_0^{+\infty} b\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt - \varepsilon\int_0^{+\infty} Y\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt,$$

which in turn implies that, for primary stable solution,

$$E(t) = -\int_t^{+\infty} b\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt - \varepsilon\int_t^{+\infty} Y\cos^2(\varepsilon X/2 + a/2)P(t+t_0)dt. \quad (2.33)$$

Expanding $E(t)$ into a power series in ε as

$$E(t) = E_0(t, t_0) + \varepsilon E_1(t, t_0) + \cdots.$$

We have

$$E_0(t, t_0) = -\int_t^{+\infty} b\cos^2(a/2)P(t+t_0)dt = -\frac{1}{4}\int_t^{+\infty} b^3 P(t+t_0)dt. \quad (2.34)$$

We also have

Lemma 2.7. *For the primary stable solution, we have*

$$\begin{aligned} E_1(t,t_0) = & \int_t^{+\infty} b\cos(a/2)\sin(a/2)\left(-\frac{H}{b}\mathbb{M}_0 + b\mathbb{W}_0\right)P(t+t_0)dt \\ & - \int_t^{+\infty} \cos^2(a/2)\left(-\frac{\tilde{H}}{b}\mathbb{M}_0 + b'\mathbb{W}_0\right)P(t+t_0)dt. \end{aligned} \tag{2.35}$$

Proof. By using the integral equation (2.33), we have

$$E_1(t,t_0) = \int_t^{+\infty} b\cos(a/2)\sin(a/2)XP(t+t_0)dt - \int_t^{+\infty} Y\cos^2(a/2)P(t+t_0)dt \tag{2.36}$$

where

$$X = \left(-\frac{H}{b}\mathbb{M}_0 + b\mathbb{W}_0\right), \quad Y = \left(-\frac{\tilde{H}}{b}\mathbb{M}_0 + b'\mathbb{W}_0\right). \quad \square$$

Proof of Theorem 1. So far, we have worked, exclusively, on the primary stable solutions. In order to obtain a formula for the splitting distance, we also need a corresponding study on the primary unstable solutions. For this purpose we repeat what has been presented so far for the primary stable solutions, changing the time interval from $[0,+\infty)$ to $(-\infty,0]$, and duly, $+\infty$ to $-\infty$ in all formulas derived. We have

$$E_0^+(0,t_0) = -\frac{1}{4}\int_0^{+\infty} b^3P(t+t_0)dt; \qquad E_0^-(0,t_0) = -\frac{1}{4}\int_0^{-\infty} b^3P(t+t_0)dt.$$

Consequently, the classical Melnikov function is

$$D_0(t_0) = E_0^+(0,t_0) - E_0^-(0,t_0) = -\frac{1}{4}\int_{-\infty}^{+\infty} b^3P(t+t_0)dt.$$

We also have

$$\begin{aligned} E_1^+(0,t_0) = & \int_0^{+\infty} b\cos(a/2)\sin(a/2)\left(-\frac{H}{b}\mathbb{M}_0 + b\mathbb{W}_0\right)P(t+t_0)dt \\ & - \int_0^{+\infty} \cos^2(a/2)\left(-\frac{\tilde{H}}{b}\mathbb{M}_0 + b'\mathbb{W}_0\right)P(t+t_0)dt. \end{aligned}$$

We now use the conclusion of Lemma 2.3 to obtain

$$E_1^+(0,t_0) = \int_0^{+\infty} \left(b\cos(a/2)\sin(a/2)\left(-\frac{H}{b}\right)P(t+t_0) - \cos^2(a/2)\left(-\frac{H}{b}\right)' P(t+t_0) \right) \mathbb{M}_0^+ dt$$
$$+ \int_0^{+\infty} \left(b^2\cos(a/2)\sin(a/2)P(t+t_0) - b'\cos^2(a/2)P(t+t_0) \right) \mathbb{W}_0^+ dt$$
$$= \int_0^{+\infty} \left(\left(\cos^2(a/2)\left(\frac{H}{b}\right)P(t+t_0) \right)' - \cos^2(a/2)\left(\frac{H}{b}\right)P_t(t+t_0) \right) \mathbb{M}_0^+ dt$$
$$- \int_0^{+\infty} \left(\left(b\cos^2(a/2)P(t+t_0) \right)' - b\cos^2(a/2)P_t(t+t_0) \right) \mathbb{W}_0^+ dt$$

where

$$\mathbb{M}_0^+(t) = \frac{1}{4}\int_t^{+\infty} b^3 P(t+t_0)dt; \qquad \mathbb{W}_0^+(t) = -\frac{1}{4}\int_0^{t} HbP(t+t_0)dt.$$

This is to have

$$E_1^+(0,t_0) = \int_0^{+\infty} \left(\cos^2(a/2)\left(\frac{H}{b}\right)P(t+t_0) \right)' \mathbb{M}_0^+ dt - \int_0^{+\infty} \left(b\cos^2(a/2)P(t+t_0) \right)' \mathbb{W}_0^+ dt$$
$$- \int_0^{+\infty} \left(\cos^2(a/2)\left(\frac{H}{b}\right)P_t(t+t_0) \right) \mathbb{M}_0^+ dt + \int_0^{+\infty} \left(b\cos^2(a/2)P_t(t+t_0) \right) \mathbb{W}_0^+ dt$$
$$= - \int_0^{+\infty} \cos^2(a/2)\left(\frac{H}{b}\right)P(t+t_0) \left(\mathbb{M}_0^+\right)' dt + \int_0^{+\infty} b\cos^2(a/2)P(t+t_0) \left(\mathbb{W}_0^+\right)' dt$$
$$- \int_0^{+\infty} \left(\cos^2(a/2)\left(\frac{H}{b}\right)P_t(t+t_0) \right) \mathbb{M}_0^+ dt + \int_0^{+\infty} \left(b\cos^2(a/2)P_t(t+t_0) \right) \mathbb{W}_0^+ dt.$$

Note that the first two integrals cancel each other. We have, in conclusion

$$E_1^+(0,t_0) = - \int_0^{+\infty} \left(\cos^2(a/2)\left(\frac{H}{b}\right)P_t(t+t_0) \right) \mathbb{M}_0^+ dt + \int_0^{+\infty} \left(b\cos^2(a/2)P_t(t+t_0) \right) \mathbb{W}_0^+ dt$$
$$= - \int_0^{+\infty} \left(\cos^2(a(\tau_1)/2)\left(\frac{H(\tau_1)}{b(\tau_1)}\right)P_t(\tau_1+t_0) \right)$$

$$
\begin{aligned}
&\times \left(\int_{\tau_1}^{+\infty} b(\tau_2)\cos^2(a(\tau_2)/2)P(\tau_2+t_0)d\tau_2 \right) d\tau_1 \\
&- \int_0^{+\infty} \Big(b(\tau_1)\cos^2(a(\tau_1)/2)P_t(\tau_1+t_0) \Big) \\
&\times \left(\int_0^{\tau_1} H(\tau_2)b^{-1}(\tau_2)\cos^2(a(\tau_2)/2)P(\tau_2+t_0)d\tau_2 \right) d\tau_1 \\
= &- \int_0^{+\infty}\int_{\tau_1}^{+\infty} \frac{H(\tau_1)}{b(\tau_1)} b(\tau_2)\cos^2(a(\tau_1)/2)\cos^2(a(\tau_2)/2)P_t(\tau_1+t_0)P(\tau_2+t_0)d\tau_2 d\tau_1 \\
&- \int_0^{+\infty}\int_0^{\tau_1} \frac{H(\tau_2)}{b(\tau_2)} b(\tau_1)\cos^2(a(\tau_2)/2)\cos^2(a(\tau_1)/2)P_t(\tau_1+t_0)P(\tau_2+t_0)d\tau_2 d\tau_1 \\
= &- \int_0^{+\infty}\int_0^{\tau_2} \frac{H(\tau_1)}{b(\tau_1)} b(\tau_2)\cos^2(a(\tau_1)/2)\cos^2(a(\tau_2)/2)P_t(\tau_1+t_0)P(\tau_2+t_0)d\tau_1 d\tau_2 \\
&- \int_0^{+\infty}\int_0^{\tau_1} \frac{H(\tau_2)}{b(\tau_2)} b(\tau_1)\cos^2(a(\tau_2)/2)\cos^2(a(\tau_1)/2)P_t(\tau_1+t_0)P(\tau_2+t_0)d\tau_2 d\tau_1.
\end{aligned}
$$

Rewrite τ_1 as τ_2 and τ_2 as τ_1 in the first integral, we finally obtain

$$
E_1^+(0,t_0) = - \int_0^{+\infty}\int_0^{\tau_1} f(\tau_1,\tau_2)\left(P_t(\tau_1+t_0)P(\tau_2+t_0) + P_t(\tau_2+t_0)P(\tau_1+t_0)\right) d\tau_2 d\tau_1 \quad (2.37)
$$

where

$$
f(\tau_1,\tau_2) = \frac{H(\tau_2)}{b(\tau_2)}\cos^2(a(\tau_2)/2)b(\tau_1)\cos^2(a(\tau_1)/2).
$$

We have in parallel,

$$
E_1^-(0,t_0) = - \int_0^{-\infty}\int_0^{\tau_1} f(\tau_1,\tau_2)\left(P_t(\tau_1+t_0)P(\tau_2+t_0) + P_t(\tau_2+t_0)P(\tau_1+t_0)\right) d\tau_2 d\tau_1.
$$

In conclusion,

$$
D_1(t_0) = E_1^+(0,t_0) - E_1^-(0,t_0)
$$

$$= - \int_{-\infty}^{+\infty} \int_{0}^{\tau_1} f(\tau_1, \tau_2) \left(P_t(\tau_1 + t_0) P(\tau_2 + t_0) + P_t(\tau_2 + t_0) P(\tau_1 + t_0) \right) d\tau_2 d\tau_1 .$$

Finally, we note that because

$$b^2 = 2(1 + \cos a) = 4 \cos^2(a/2),$$

we have

$$D_1(t_0) = - \frac{1}{16} \int_{-\infty}^{+\infty} \int_{0}^{\tau_1} H(\tau_2) b(\tau_2) b^3(\tau_1) \mathcal{P}(\tau_1, \tau_2, t_0) d\tau_2 d\tau_1$$

where

$$\mathcal{P}(\tau_1, \tau_2, t_0) = P_t(\tau_1 + t_0) P(\tau_2 + t_0) + P_t(\tau_2 + t_0) P(\tau_1 + t_0).$$

This proves Theorem 1. □

3. Evaluation of $D_1(t_0)$: an example

In this section, we evaluate $D_1(t_0)$ for

$$\frac{d^2 x}{dt^2} = - \sin x + \varepsilon \cos^2(x/2) \left(\cos 2\omega t + \cos 3\omega t \right).$$

We have, for this equation,

$$D_1(t_0, \omega) = - \frac{1}{16} \int_{-\infty}^{+\infty} \int_{0}^{\tau_1} H(\tau_2) b(\tau_2) b^3(\tau_1) \mathcal{P}(\tau_1, \tau_2, t_0) d\tau_2 d\tau_1 \tag{3.1}$$

where

$$\mathcal{P}(\tau_1, \tau_2, t_0) = P_t(\tau_1 + t_0) P(\tau_2 + t_0) + P_t(\tau_2 + t_0) P(\tau_1 + t_0),$$

and

$$P(t) = \cos 2\omega t + \cos 3\omega t.$$

3.1. The integrals for $D_1(t_0, \omega)$

We first calculate $P_t(\tau_2 + t_0)P(\tau_1, +t_0)$.

Lemma 3.1. *We have*

$$P_t(\tau_2 + t_0)P(\tau_1 + t_0) = -\omega\left(g_0(\tau_1, \tau_2) + \sum_{n=1}^{6}(g_n(\tau_1, \tau_2)\cos n\omega t_0 + f_n(\tau_1, \tau_2)\sin n\omega t_0)\right)$$

where

$$\begin{aligned}
g_0(\tau_1, \tau_2) &= \frac{3}{2}\sin 3\omega(\tau_2 - \tau_1) + \sin 2\omega(\tau_2 - \tau_1) \\
g_1(\tau_1, \tau_2) &= \frac{3}{2}\sin(3\omega\tau_2 - 2\omega\tau_1) - \sin(3\omega\tau_1 - 2\omega\tau_2) \\
g_4(\tau_1, \tau_2) &= \sin 2\omega(\tau_2 + \tau_1) \\
g_5(\tau_1, \tau_2) &= \frac{3}{2}\sin(3\omega\tau_2 + 2\omega\tau_1) + \sin(2\omega\tau_2 + 3\omega\tau_1) \\
g_6(\tau_1, \tau_2) &= \frac{3}{2}\sin 3\omega(\tau_2 + \tau_1)
\end{aligned}$$

and

$$\begin{aligned}
f_1(\tau_1, \tau_2) &= \frac{3}{2}\cos(3\omega\tau_2 - 2\omega\tau_1) - \cos(3\omega\tau_1 - 2\omega\tau_2) \\
f_4(\tau_1, \tau_2) &= \cos 2\omega(\tau_2 + \tau_1) \\
f_5(\tau_1, \tau_2) &= \frac{3}{2}\cos(3\omega\tau_2 + 2\omega\tau_1) + \cos(2\omega\tau_2 + 3\omega\tau_1) \\
f_6(\tau_1, \tau_2) &= \frac{3}{2}\cos 3\omega(\tau_2 + \tau_1).
\end{aligned}$$

We have, in addition,

$$g_2(\tau_1, \tau_2) = f_2(\tau_1, \tau_2) = g_3(\tau_1, \tau_2) = f_3(\tau_1, \tau_2) = 0.$$

Proof. This lemma follows directly from an elementary calculation. □

In what follows, we let

$$\begin{aligned}
I_{n,c} &= \int_{-\infty}^{+\infty}\int_{0}^{\tau_1} H(\tau_2)b(\tau_2)b^3(\tau_1)\,[g_n(\tau_1, \tau_2) + g_n(\tau_2, \tau_1)]\,d\tau_2 d\tau_1 \\
I_{n,s} &= \int_{-\infty}^{+\infty}\int_{0}^{\tau_1} H(\tau_2)b(\tau_2)b^3(\tau_1)\,[f_n(\tau_1, \tau_2) + f_n(\tau_2, \tau_1)]\,d\tau_2 d\tau_1.
\end{aligned} \tag{3.2}$$

We have

$$D_1(t_0,\omega) = -\frac{\omega}{16} I_{0,c} - \frac{\omega}{16}\sum_{n=1}^{6}\left(I_{n,c}\cos n\omega t_0 + I_{n,s}\sin n\omega t_0\right). \tag{3.3}$$

Our task is then to evaluate $I_{0,c}$ and $I_{n,c}, I_{n,s}$ for $n = 1, 4, 5, 6$.

Lemma 3.2. *(1) Assume $F(\tau_1, \tau_2)$ is even in the sense that $F(\tau_1, \tau_2) = F(-\tau_1, -\tau_2)$. Then,*

$$\int_0^{+\infty}\int_0^{\tau_1} F(\tau_1,\tau_2)d\tau_2 d\tau_1 = \int_0^{-\infty}\int_0^{\tau_1} F(\tau_1,\tau_2)d\tau_2 d\tau_1.$$

(2) Assume $F(\tau_1, \tau_2)$ is odd in the sense that $F(\tau_1, \tau_2) = -F(-\tau_1, -\tau_2)$. Then,

$$\int_0^{+\infty}\int_0^{\tau_1} F(\tau_1,\tau_2)d\tau_2 d\tau_1 = -\int_0^{-\infty}\int_0^{\tau_1} F(\tau_1,\tau_2)d\tau_2 d\tau_1.$$

Proof. By changing (τ_1, τ_2) to $(-\tau_1, -\tau_2)$. □

We have, as a direct corollary,

Corollary 3.1. *For $n = 0$ to 6, $I_{n,c} = 0$.*

Proof. By Lemma 3.1, $g_n(\tau_1, \tau_2) + g_n(\tau_2, \tau_1)$ are odd functions of τ_1, τ_2. Note that $H(t), a(t)$ are odd, but $b(t)$ is even in t. It then follows that the integral function for $I_{n,c}$,

$$H(\tau_2)b(\tau_2)b^3(\tau_1)\left[g_n(\tau_1,\tau_2) + g_n(\tau_2,\tau_1)\right],$$

is an even function of τ_1, τ_2. By Lemma 3.2(1), we have $I_{n,c} = 0$ for all n. □

In conclusion, we have

$$E_1(t_0,\omega) = -\frac{\omega}{16}\sum_{n=1}^{6} I_{n,s}\sin n\omega t_0. \tag{3.4}$$

We also have $I_{2,s} = I_{3.s} = 0$ by Lemma 3.1. We need to further evaluate $I_{1,s}, I_{4,s}, I_{5,s}, I_{6,s}$.

3.2. Evaluation of $I_{1,s}$

By (3.2) and Lemma 3.1, we have

$$I_{1,s} = \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^{\tau_1} H(\tau_2)b(\tau_2)b^3(\tau_1)\left[\cos(3\omega\tau_2 - 2\omega\tau_1) + \cos(3\omega\tau_1 - 2\omega\tau_2)\right]d\tau_1 d\tau_2.$$

We calculate $I_{1,s}$ in this subsection to prove

Proposition 3.1.

$$I_{1,s} = \pi\omega e^{-\pi\omega/2}\left(\frac{4}{9} + O(\omega^{-1})\right) + O(e^{-2\pi\omega/3}).$$

The proof of this proposition is rather technical. We divide into a few sub-steps.

3.2.1. Preparations

Let $G(t)$ be such that

$$G(t) = H(t)b(t) - \frac{i\pi}{16}b^3(t). \tag{3.5}$$

We also denote

$$g(t) = G(t + i\pi/2), \quad f(t) = t^3 b^3(t + i\pi/2) \tag{3.6}$$

Our next lemma is preparatory in nature.

Lemma 3.3. *The functions $g(t)$, $f(t)$ are even in t and are analytic at $t = 0$. In addition, we have (i) both $g(t)$ and $f(t)$ are even in t; (ii) $g(0) = \frac{2}{3}i$, $g'(0) = 0$; and (iii) $f(0) = 8i$, $f'(0) = 0$.*

Proof. By definition, we have

$$\sin(a(t)/2) = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad \cos(a(t)/2) = \frac{2}{e^t + e^{-t}}, \quad \sin a(t) = \frac{4(e^t - e^{-t})}{(e^t + e^{-t})^2},$$

$$b(t) = \frac{4}{e^t + e^{-t}}, \quad h(t) = \frac{4t - (e^{2t} - e^{-2t})}{2(e^t + e^{-t})^2}.$$

This is for us to have

$$\sin a(t + i\pi/2)/2 = \frac{e^t + e^{-t}}{e^t - e^{-t}}, \quad \cos a(t + i\pi/2)/2 = \frac{-2i}{e^t - e^{-t}}, \quad \sin a(t + i\pi/2) = -\frac{4i(e^t + e^{-t})}{(e^t - e^{-t})^2}$$

$$b(t + i\pi/2) = -\frac{4i}{e^t - e^{-t}}, \quad h(t + i\pi/2) = \frac{4(t + i\pi/2) + (e^{2t} - e^{-2t})}{2(ie^t - ie^{-t})^2}.$$

We also recall that

$$H(t)b(t) = bh - b' = hb - b' = hb + \sin a$$

We then have, first,

$$
\begin{aligned}
g(t) &= \left(\frac{4t + (e^{2t} - e^{-2t})}{2(ie^t - ie^{-t})^2} + \frac{(e^t + e^{-t})}{(e^t - e^{-t})} \right) \frac{4i}{e^t - e^{-t}} \\
&= \left(\frac{-4t - (e^{2t} - e^{-2t})}{2(e^t - e^{-t})^2} + \frac{(e^t + e^{-t})}{(e^t - e^{-t})} \right) \frac{4i}{e^t - e^{-t}} \\
&= \left(\frac{-4t + (e^{2t} - e^{-2t})}{2(e^t - e^{-t})^2} \right) \frac{4i}{e^t - e^{-t}} \\
&= \frac{2}{3} i + O(t^2).
\end{aligned}
$$

Next, we have

$$
f(t) = t^3 \frac{4^3 i}{(e^t - e^{-t})^3}.
$$

This is even in t. Again, expanding into power series at $t = 0$, we obtain

$$
f(t) = 8i + O(t^2).
$$

All items of this lemma now follow. □

3.2.2. Reduction

In this subsection, we decompose $I_{1,s}$ into a set of simpler integrals.

Lemma 3.4. *We have* $I_{1,s} = \frac{1}{2}(\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$ *where*

$$
\begin{aligned}
\mathcal{R}_1 &= \int_0^{-\infty} e^{3i\omega s_2} \left(\int_{-\infty}^{+\infty} G(s_1) b^3(s_1 + s_2) e^{i\omega s_1} ds_1 \right) ds_2 \\
\mathcal{R}_2 &= \int_0^{-\infty} e^{-2i\omega s_2} \left(\int_{-\infty}^{+\infty} G(s_1) b^3(s_1 + s_2) e^{i\omega s_1} ds_1 \right) ds_2 \\
\mathcal{E}_1 &= \left(\int_{-\infty}^{+\infty} b^3(\tau) e^{3i\omega\tau} d\tau \right) \left(\int_0^{+\infty} H(\tau) b(\tau) e^{-2i\omega\tau} d\tau \right) \\
\mathcal{E}_2 &= \left(\int_{-\infty}^{+\infty} b^3(\tau) e^{-2i\omega\tau} d\tau \right) \left(\int_0^{+\infty} H(\tau) b(\tau) e^{3i\omega\tau} d\tau \right) \\
\mathcal{E}_3 &= -\frac{\pi i}{16} \left(\int_{-\infty}^{+\infty} b^3(\tau) e^{3i\omega\tau} d\tau \right) \left(\int_{-\infty}^{+\infty} b^3(\tau) e^{-2i\omega\tau} d\tau \right).
\end{aligned}
\tag{3.7}
$$

Proof. We use Lemma 3.2(2) to write $I_{1,s}$ as

$$I_{1,s} = \int_0^{+\infty}\int_0^{\tau_1} H(\tau_2)b(\tau_2)b^3(\tau_1)\left[\cos(3\omega\tau_2 - 2\omega\tau_1) + \cos(3\omega\tau_1 - 2\omega\tau_2)\right] d\tau_2 d\tau_1.$$

We divide the proof of this lemma into three steps.

Step 1: Initial Reduction. We write trigonometric functions in complex form to obtain

$$\begin{aligned} I_{1,s} =& \frac{1}{2}\int_0^{+\infty}\int_0^{\tau_1} H(\tau_2)b(\tau_2)b^3(\tau_1)\left[e^{i(3\omega\tau_2-2\omega\tau_1)} + e^{-i(3\omega\tau_2-2\omega\tau_1)}\right] d\tau_2 d\tau_1 \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{\tau_1} H(\tau_2)b(\tau_2)b^3(\tau_1)\left[e^{i(3\omega\tau_1-2\omega\tau_2)} + e^{-i(3\omega\tau_1-2\omega\tau_2)}\right] d\tau_2 d\tau_1. \end{aligned}$$

We then switch the order of integration from $d\tau_1 d\tau_2$ to $d\tau_2 d\tau_1$ to obtain

$$\begin{aligned} I_{1,s} =& \frac{1}{2}\int_0^{+\infty}\int_{\tau_2}^{+\infty} H(\tau_2)b(\tau_2)b^3(\tau_1)\left[e^{i(3\omega\tau_2-2\omega\tau_1)} + e^{-i(3\omega\tau_2-2\omega\tau_1)}\right] d\tau_1 d\tau_2 \\ &+ \frac{1}{2}\int_0^{+\infty}\int_{\tau_2}^{+\infty} H(\tau_2)b(\tau_2)b^3(\tau_1)\left[e^{i(3\omega\tau_1-2\omega\tau_2)} + e^{-i(3\omega\tau_1-2\omega\tau_2)}\right] d\tau_1 d\tau_2. \end{aligned}$$

Let $s_1 = \tau_2, s_2 = \tau_1 - \tau_2$. We have

$$\begin{aligned} I_{1,s} =& \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} H(s_1)b(s_1)b^3(s_1+s_2)\left[e^{i(3\omega s_2+\omega s_1)} + e^{-i(3\omega s_2+\omega s_1)}\right] ds_2 ds_1 \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} H(s_1)b(s_1)b^3(s_1+s_2)\left[e^{i(\omega s_1-2\omega s_2)} + e^{-i(\omega s_1-2\omega s_2)}\right] ds_2 ds_1 \\ =& \frac{1}{2}\left(A^+ + A^- + B^+ + B^-\right) \end{aligned}$$

where

$$\begin{aligned} A^{\pm} &= \int_0^{+\infty} H(s_1)b(s_1)e^{\pm i\omega s_1}\int_0^{+\infty} b^3(s_1+s_2)e^{\pm 3i\omega s_2} ds_2 ds_1 \\ B^{\pm} &= \int_0^{+\infty} H(s_1)b(s_1)e^{\pm i\omega s_1}\int_0^{+\infty} b^3(s_1+s_2)e^{\mp 2i\omega s_2} ds_2 ds_1. \end{aligned} \tag{3.8}$$

Step 2: Extending integral bounds. We have

$$
\begin{aligned}
A^+ &= \int_0^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_0^{+\infty} b^3(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1 \\
&= \int_0^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_{-\infty}^{+\infty} b^3(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1 \\
&\quad - \int_0^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_{-\infty}^{0} b^3(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1.
\end{aligned}
$$

Here, we first change the lower bound of the inner integral from 0 to $-\infty$, then nullify the effect of this replacement by subtracting the second integral. We do the same to the lower integral bound of the outer integral to obtain

$$A^+ = (I) + (II) + (III)$$

where

$$
\begin{aligned}
(I) &= \int_0^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_{-\infty}^{+\infty} b^3(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1 \\
(II) &= -\int_{-\infty}^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_{-\infty}^{0} b^3(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1 \\
(III) &= \int_{-\infty}^{0} H(s_1)b(s_1)e^{i\omega s_1} \int_{-\infty}^{0} b^3(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1.
\end{aligned}
$$

For (I), we let $\tau = s_1 + s_2$ to obtain

$$(I) = \left(\int_0^{+\infty} H(s_1)b(s_1)e^{-2i\omega s_1} ds_1 \right) \left(\int_{-\infty}^{+\infty} b^3(s_1+s_2)e^{3i\omega\tau} d\tau \right) = \mathcal{E}_1$$

where $\mathcal{E}_1$ is as in (3.7). We also note that $(III) = -A^-$. This is for us to conclude that

$$A^+ + A^- = \int_{-\infty}^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_0^{-\infty} b^3(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1 + \mathcal{E}_1. \tag{3.9}$$

In parallel, we have

$$B^+ + B^- = \int_{-\infty}^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_0^{-\infty} b^3(s_1+s_2)e^{-2i\omega s_2}ds_2ds_1 + \mathcal{E}_2 \tag{3.10}$$

where $\mathcal{E}_2$ is as in (3.7). It then follows from (3.9) and (3.10) that

$$I_{1,s} = -\frac{1}{2}\left(\tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 + \mathcal{E}_1 + \mathcal{E}_2\right)$$

where

$$\begin{aligned} \tilde{\mathcal{R}}_1 &= \int_{-\infty}^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_0^{-\infty} b^3(s_1+s_2)e^{3i\omega s_2}ds_2ds_1 \\ \tilde{\mathcal{R}}_2 &= \int_{-\infty}^{+\infty} H(s_1)b(s_1)e^{i\omega s_1} \int_0^{-\infty} b^3(s_1+s_2)e^{-2i\omega s_2}ds_2ds_1. \end{aligned} \tag{3.11}$$

Step 3: Final Reduction. Switching the order of integration from ds_2ds_1 to ds_1ds_2, we obtain

$$\begin{aligned} \tilde{\mathcal{R}} &= \int_0^{-\infty} e^{3i\omega s_2} \int_{-\infty}^{+\infty} H(s_1)b(s_1)e^{i\omega s_1}b^3(s_1+s_2)ds_1ds_2 \\ \tilde{\mathcal{R}}_2 &= \int_0^{-\infty} e^{-2i\omega s_2} \int_{-\infty}^{+\infty} H(s_1)b(s_1)e^{i\omega s_1}b^3(s_1+s_2)ds_1ds_2. \end{aligned}$$

Recall that, by definition,

$$G(t) = H(t)b(t) - \frac{i\pi}{16}b^3(t).$$

This is for us to have

$$\tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 = \mathcal{R}_1 + \mathcal{R}_2 + W_1 + W_2$$

where $\mathcal{R}_1, \mathcal{R}_2$ are as in (3.7), and

$$\begin{aligned} W_1 &= \frac{\pi i}{16} \int_0^{-\infty} e^{3i\omega s_2} \left(\int_{-\infty}^{+\infty} b^3(s_1)b^3(s_1+s_2)e^{i\omega s_1}ds_1 \right) ds_2 \\ W_2 &= \frac{\pi i}{16} \int_0^{-\infty} e^{-2i\omega s_2} \left(\int_{-\infty}^{+\infty} b^3(s_1)b^3(s_1+s_2)e^{i\omega s_1}ds_1 \right) ds_2. \end{aligned} \tag{3.12}$$

We work on $W_1 + W_2$. Letting $t_1 = s_1 + s_2$, we have

$$W_1 = \frac{\pi i}{16} \int_0^{-\infty} e^{2i\omega s_2} \left(\int_{-\infty}^{+\infty} b^3(t_1 - s_2) b^3(t_1) e^{i\omega t_1} dt_1 \right) ds_2$$
$$= -\frac{\pi i}{16} \int_0^{+\infty} e^{-2i\omega t_2} \left(\int_{-\infty}^{+\infty} b^3(t_1 + t_2) b^3(t_1) e^{i\omega t_1} dt_1 \right) dt_2$$

where the second equality is obtained by letting $t_2 = -s_2$. It then follows that $W_1 + W_2 = \mathcal{E}_3$ where $\mathcal{E}_3$ is as in (3.7). □

3.2.3. A generic setting

In this subsection, we introduce a generic setting to facilitate the computation of $\mathcal{R}_1$ and $\mathcal{R}_2$. Let

$$\mathcal{K}_{g,f}(z,t) = \frac{e^{i\omega z} g(z) f(t+z)}{(t+z)^3} \tag{3.13}$$

where $t \neq 0$ is a real parameter and z is a complex variable. We assume

(A1) the functions $f(z), g(z)$ are independent of the forcing frequency ω,
(A2) $f(z), g(z)$ are real analytic, non-constant functions and $f(0) \neq 0$,
(A3) $|f^{(k)}(t)|,\ |g^{(k)}(t)| \leq C_0 e^{-c_0|t|}$ for all $0 \leq k < 3$ where both C_0 and c_0 are positive constants.

Letting $t \neq 0$ be fixed, we regard $\mathcal{K}_{g,f}(z,t)$ as a function of z. The function $\mathcal{K}_{g,f}(z,t)$ has a pole of order three on the real z axis at $z = -t$ (with the possible exception of countably many values of t so that $g(t) = 0$). Denote the residue of this pole as $R(t)$. Our purpose is to calculate

$$I_{g,f}^{(m)}(\omega) = \int_0^{-\infty} e^{im\omega t} R(t) dt. \tag{3.14}$$

Lemma 3.5. *We have*

$$\begin{aligned} I_{g,f}^{(m)} = & -\frac{i\omega}{2!(m-1)} f(0)g(0) - \frac{1}{m-1}(f(0)g'(0) + g(0)f'(0)) \\ & + \frac{1}{2!(m-1)^2} f(0)g'(0) + O(\omega^{-1}). \end{aligned} \tag{3.15}$$

Proof. Denote

$$K(z) = e^{i\omega z} g(z) \tag{3.16}$$

to write $\mathcal{K}_{g,f}(z,t)$ as

$$\mathcal{K}_{g,f}(z,t) = \frac{K(z) f(t+z)}{(t+z)^3}. \tag{3.17}$$

We have

$$R(t) = \frac{1}{2!}\partial_{z^2}\ (f(z)K(z-t))|_{z=0}\,.$$

To compute $R(t)$, we start with Leibniz's formula (product rule)

$$\partial_{z^2}(h_1(z)h_2(z)) = \sum_{\alpha=0}^{2} \frac{2!}{\alpha!(2-\alpha)!}\partial_{z^\alpha}h_1(z)\cdot\partial_{z^{2-\alpha}}h_2(z). \tag{3.18}$$

We have, by using (3.18),

$$R(t) = \sum_{\alpha=0}^{2} \frac{1}{\alpha!(2-\alpha)!} f^{(\alpha)}(0)\partial_{\tau^{(2-\alpha)}}K(\tau)$$

where

$$\tau = -t.$$

We use $K(\tau) = e^{i\omega\tau}g(\tau)$ to obtain

$$K^{(2-\alpha)}(\tau) = e^{i\omega\tau}\sum_{\gamma=0}^{2-\alpha} \frac{(i\omega)^{2-\alpha-\gamma}(2-\alpha)!}{\gamma!(2-\alpha-\gamma)!} g^{(\gamma)}(\tau).$$

This is then for us to have

$$\begin{aligned} R(t) &= \sum_{\alpha=0}^{2} \frac{1}{\alpha!(2-\alpha)!} f^{(\alpha)}(0)e^{i\omega\tau}\sum_{\gamma=0}^{2-\alpha} \frac{(i\omega)^{2-\alpha-\gamma}(2-\alpha)!}{\gamma!(2-\alpha-\gamma)!} g^{(\gamma)}(\tau) \\ &= e^{i\omega\tau}\sum_{\alpha=0}^{2}\sum_{\gamma=0}^{2-\alpha} \frac{(i\omega)^{2-\alpha-\gamma}}{\alpha!\gamma!(2-\alpha-\gamma)!} f^{(\alpha)}(0)g^{(\gamma)}(\tau) \\ &= e^{-i\omega t}\sum_{\alpha=0}^{3}\sum_{\gamma=0}^{2-\alpha} \frac{(i\omega)^{2-\alpha-\gamma}}{\alpha!\gamma!(2-\alpha-\gamma)!} f^{(\alpha)}(0)g^{(\gamma)}(-t). \end{aligned}$$

We have in conclusion

$$I_{g,f}^{(m)} = \sum_{\alpha=0}^{2}\sum_{\gamma=0}^{2-\alpha} \frac{(i\omega)^{2-\alpha-\gamma}}{\alpha!\gamma!(2-\alpha-\gamma)!} f^{(\alpha)}(0)\mathcal{I}_g^{\alpha,\gamma,(m)} \tag{3.19}$$

where

$$\mathcal{I}_g^{\alpha,\gamma,(m)} = \int_0^{-\infty} e^{i(m-1)\omega t} g^{(\gamma)}(-t)dt. \tag{3.20}$$

We have for $\mathcal{I}_g^{\alpha,\gamma,(m)}$,

$$\begin{aligned}\mathcal{I}_g^{\alpha,\gamma,(m)} &= -\int_0^{+\infty} e^{-i(m-1)\omega t} g^{(\gamma)}(t)dt = \frac{1}{i(m-1)\omega}\int_0^{+\infty} g^{(\gamma)}(t)de^{-i(m-1)\omega t} \\ &= \frac{-1}{i(m-1)\omega} g^{(\gamma)}(0) + \frac{1}{(i(m-1)\omega)^2}\int_0^{+\infty} g^{(\gamma+1)}(t)de^{-i(m-1)\omega t} \\ &= \frac{-1}{i(m-1)\omega} g^{(\gamma)}(0) - \frac{1}{(i(m-1)\omega)^2} g^{(\gamma+1)}(0) + O(\omega^{-3}).\end{aligned} \tag{3.21}$$

It then follows from (3.19) that

$$\begin{aligned}I_{g,f}^{(m)} =& -\frac{(i\omega)}{2!(m-1)} f(0)g(0) - \frac{1}{m-1}(f(0)g'(0) + g(0)f'(0)) \\ &+ \frac{1}{2!(m-1)^2} f(0)g'(0) + O(\omega^{-1}). \quad \square\end{aligned}$$

3.2.4. Evaluating $\mathcal{R}_1$ and $\mathcal{R}_2$

For a continuous curve ℓ in the complex z-plane, we let

$$I_\ell(t) = \int_\ell e^{i\omega z} G(z) b^3(t+z)dz$$

where $t \neq 0$ is a real parameter. Let

$$\begin{aligned}\ell_1(t) =& \{z = t_1 + is_1, \;\; t_1 \in (-\infty, +\infty), \;\; s_1 = 0\} \\ \ell_2(t) =& \{z = t_1 + is_1, \;\; t_1 \in (-\infty, +\infty), \;\; s_1 = i\rho\}.\end{aligned}$$

where $\rho = 3\pi/2 - \omega^{-1}$ for ℓ_2. First, we work on I_{ℓ_2}.

Lemma 3.6. *There exists a constant C independent of ω such that*

$$\left|I_{\ell_2}\right| < C\omega^3 e^{-3\pi\omega/2} e^{-|t|}.$$

Proof. Recall that $\rho = 3\pi/2 - \omega^{-1}$. First, by the asymptotic behavior of $G(t), b(t)$ on ℓ_2, we have

$$|G(t+i\rho)| \leq Ce^{-|t|}, \quad |b(t+i\rho)a^2(t+i\rho)| \leq C\omega^3 e^{-3|t|}. \tag{3.22}$$

We have a factor ω^3 in the second estimate because the order of the pole of the function $b^3(t)$ at $t = 3i\pi/2$ is three. Note that the distance from ℓ_2 to the pole located at $-t + i3\pi/2$ is $\geq \omega^{-1}$. We have

$$
\begin{aligned}
I_{\ell_2} &= e^{-3\pi\omega/2+1} \int_{-\infty}^{+\infty} e^{i\omega t_1} G(t_1 + i\rho) b^3(t + t_1 + i\rho) dt_1 \\
&= e^{-3\pi\omega/2+1} e^{-i\omega t} \int_{-\infty}^{+\infty} e^{i\omega s_1} G(s_1 - t + i\rho) b^3(s_1 + i\rho) ds_1.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\left|I_{\ell_2}\right| &\le e^{-3\pi\omega/2+1} \int_{-\infty}^{+\infty} \left|G(s_1 - t + i\rho) b^3(s_1 + i\rho)\right| ds_1 \\
&\le C\omega^3 e^{-3\pi\omega/2} \int_{-\infty}^{+\infty} e^{-|s_1-t|} e^{-3|s_1|} dt_1 \\
&\le C\omega^3 e^{-3\pi\omega/2} \int_{-\infty}^{+\infty} e^{-(|t|-|s_1|)} e^{-3|s_1|} dt_1 \\
&\le C\omega^3 e^{-3\pi\omega/2} e^{-|t|}.
\end{aligned}
$$

For the second inequality here, we use (3.22); for the third, we use $|s_1 - t| \ge |t| - |s_1|$. □

Lemma 3.7. *We have*

$$
\mathcal{R}_1 + \mathcal{R}_2 = 2\pi\omega e^{-\pi\omega/2} \left(-\frac{4}{9} + O(\omega^{-1}) \right) + O(\omega^3 e^{-3\pi\omega/2}).
$$

Proof. We have, by definition,

$$
\mathcal{R}_1 = \int_0^{-\infty} e^{3i\omega t} I_{\ell_1} dt, \qquad \mathcal{R}_2 = \int_0^{-\infty} e^{-2i\omega t} I_{\ell_1} dt.
$$

By the residue theorem,

$$
I_{\ell_1} = I_{\ell_2} + 2\pi i \; Res(e^{i\omega z} G(z) b^3(t+z))\Big|_{z=-t+\pi i/2} = I_{\ell_2} + 2\pi i e^{-\pi\omega/2} R(t)
$$

where

$$
R(t) = Res(e^{i\omega z} G(z) b^3(t+z))\Big|_{z=-t+\pi i/2} = Res\left(\frac{e^{i\omega z_1} g(z_1) f(z_1 + t)}{(t+z_1)^3} \right)\Bigg|_{z_1=-t}.
$$

We note that z_1 in the second equality is such that $z_1 = z - \pi i/2$ and the functions f, g are as in (3.6). We then have by Lemma 3.6,

$$\mathcal{R}_1 = \int_0^{-\infty} e^{3i\omega t} I_{\ell_1}(t)dt = \int_0^{-\infty} e^{3i\omega t} \left(I_{\ell_2} + 2\pi i e^{-\pi\omega/2} R(t) \right) dt$$
$$= 2\pi i e^{-\pi\omega/2} I_{g,f}^{(3)} + O(\omega^3 e^{-3\pi\omega/2}).$$

In parallel, we have

$$\mathcal{R}_2 = 2\pi i e^{-\pi\omega/2} I_{g,f}^{(-2)} + O(\omega^3 e^{-3\pi\omega/2}).$$

By using Lemmas 3.5 and 3.3,

$$I_{g,f}^{(3)} = -\frac{(i\omega)}{4}\frac{16}{3} + O(\omega^{-1}) \quad I_{g,f}^{(-2)} = \frac{(i\omega)}{6}\frac{16}{3} + O(\omega^{-1}).$$

We finally conclude

$$\mathcal{R}_1 + \mathcal{R}_2 = 2\pi\omega e^{-\pi\omega/2} \left(-\frac{4}{9} + O(\omega^{-1}) \right) + O(\omega^3 e^{-3\pi\omega/2}). \quad \square$$

Proof of Proposition 3.1. Let

$$I_{m,n} = \int_{-\infty}^{+\infty} e^{in\omega t} b^m(t)dt.$$

Recall that $b(t)$ are meromorphic function in the complex t-plane and both functions take $(2k + 1)\pi i/2$ for all k as poles. By a direct application of the residue theorem, we obtain

$$|I_{m,n}| \le K\omega^{m-1} e^{-n\omega\pi/2}. \tag{3.23}$$

We leave the details of this estimate to the reader. Applying (3.23) to $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, we have

$$|\mathcal{E}_1| \le K\omega^2 e^{-3\pi\omega/2}, \quad |\mathcal{E}_2| \le K\omega^2 e^{-\pi\omega}, \quad |\mathcal{E}_3| \le K\omega^4 e^{-5\pi\omega/2}.$$

Proposition 3.1 then follows by combining these estimates with the conclusion of Lemma 3.7. $\square$

3.3. Proof of Theorem 2

First, we prove

Lemma 3.8. $|I_{4,s}| < K\omega^2 e^{-\pi\omega}, \quad |I_{5,s}| < K\omega^2 e^{-5\pi\omega/4}, \quad |I_{6,s}| < K\omega^2 e^{-3\pi\omega/2}.$

Proof. We start with $I_{4,s}$. We follow the process of Sect. 3.2 closely to decompose $I_{4,s}$ into a collection of integrals to obtain

$$I_{4,s} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

where

$$\mathcal{R}_1 = -\int_{-\infty}^{0} e^{2i\omega s_2}\left(\int_{-\infty}^{+\infty} e^{4i\omega s_1} G(s_1)b^3(s_1+s_2)ds_1\right)ds_2$$

$$\mathcal{R}_2 = -\int_{-\infty}^{0} e^{-2i\omega s_2}\left(\int_{-\infty}^{+\infty} e^{-4i\omega s_1} G(s_1)b^3(s_1+s_2)ds_1\right)ds_2$$

$$\mathcal{E}_1 = \left(\int_{0}^{+\infty} H(\tau)b(\tau)e^{2i\omega\tau}d\tau\right)\left(\int_{-\infty}^{+\infty} b^3(\tau)e^{2i\omega\tau}d\tau\right)$$

$$\mathcal{E}_2 = \left(\int_{0}^{+\infty} b(\tau)H(\tau)e^{-2i\omega\tau}d\tau\right)\left(\int_{-\infty}^{+\infty} b^3(\tau)e^{-2i\omega\tau}d\tau\right)$$

$$\mathcal{E}_3 = -\frac{\pi i}{16}\left(\int_{-\infty}^{+\infty} b^3(\tau)e^{2i\omega\tau}d\tau\right)\left(\int_{-\infty}^{+\infty} b^3(\tau)e^{2i\omega\tau}d\tau\right).$$

In contrast to $\mathcal{R}_1, \mathcal{R}_2$ for $I_{1,s}$, the triangular part of the inner integrals for $\mathcal{R}_1, \mathcal{R}_2$ here is respectively $e^{4i\omega s_1}$ and $e^{-4i\omega s_1}$. Consequently, applying the residue theorem to the inner integral would induce a factor $e^{-2\pi\omega}$ for $I_{4,s}$ instead of $e^{-\pi\omega/2}$ for $I_{1,s}$. It then follows directly that

$$|\mathcal{R}_1|,\ |\mathcal{R}_2| < K\omega^2 e^{-2\pi\omega}.$$

We further have, for $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$,

$$|\mathcal{E}_1|,\ |\mathcal{E}_2|,\ |\mathcal{E}_3| < K\omega^2 e^{-\pi\omega}.$$

In conclusion, we have

$$|I_{4,s}| < K\omega^2 e^{-\pi\omega}.$$

The proofs for $I_{5,s}$ and $I_{6,s}$ are similar. □

Proof of Theorem 2. By (3.4),

$$E_1(t_0,\omega) = -\frac{\omega}{16}\sum_{n=1}^{6} I_{n,s}\sin n\omega t_0. \tag{3.24}$$

The acclaimed estimate then follows directly from Proposition 3.1 for $I_{1,s}$, Lemma 3.8 for $I_{4,s}$, $I_{5,s}$ and $I_{6,s}$. □

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