CHAOTIC BEHAVIOR IN DIFFERENTIAL EQUATIONS DRIVEN BY A BROWNIAN MOTION

KENING LU AND QIUDONG WANG

Abstract. In this paper, we investigate the chaotic behavior of ordinary differential equations with a homoclinic orbit to a dissipative saddle point under an unbounded random forcing driven by a Brownian motion. We prove that, for almost all sample pathes of the Brownian motion in the classical Wiener space, the forced equation admits a topological horseshoe of infinitely many branches. This result is then applied to the randomly forced Duffing equation and the pendulum equation.

1. Introduction

In this paper, we study the chaotic dynamics of differential equations under an unbounded random force driven by a Brownian motion. We consider an ordinary differential equation with a homoclinic orbit to a dissipative saddle. We show that the randomly forced equation can be chaotic almost surely.

Let $(x, y) \in \mathbb{R}^2$ be the phase variables and t be the time. We start with an autonomous system

$$\frac{dx}{dt} = -\alpha x + f(x, y), \qquad \frac{dy}{dt} = \beta y + g(x, y) \tag{1.1}$$

where α and β are positive constants, $f(x, y)$ and $g(x, y)$ are the higher order terms.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classic Wiener probability space, where

$$\Omega = C_0(\mathbb{R}, \mathbb{R}) = \big\{\omega(t) \;:\; \omega(\cdot) : \mathbb{R} \to \mathbb{R} \text{ is continuous and } \omega(0) = 0\big\}$$

with the open compact topology under which it is a Polish space, $\mathcal{F}$ is its Borel σ-algebra, and $\mathbb{P}$ is the Wiener measure. The Brownian motion takes the form $B_t(\omega) = \omega(t)$. We consider the Wiener shift θ_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t). \tag{1.2}$$

It is well-known that $\mathbb{P}$ is an ergodic invariant measure of θ_t.

For small $\Delta > 0$, let $\mathcal{G} : \Omega \to \mathbb{R}$ be such that

$$\mathcal{G}(\omega) = \frac{1}{\Delta}\omega(\Delta). \tag{1.3}$$

Then, we have

$$\mathcal{G}(\theta_t \omega) = \frac{1}{\Delta}(\omega(t + \Delta) - \omega(t)). \tag{1.4}$$

From the properties of the Brownian motion, we have that $\mathcal{G}(\theta_t \omega)$ is an unbounded stationary stochastic process with a normal distribution, which in fact is unbounded almost surely. We can also view $\mathcal{G}(\theta_t \omega)$ as a discrete version of the white noise.

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To the right hand side of equation (1.1) we add a random forcing driven by the Brownian motion to have

$$\frac{dx}{dt} = -\alpha x + f(x, y) + \mu P(x, y)\mathcal{G}(\theta_t\omega), \qquad \frac{dy}{dt} = \beta y + g(x, y) + \mu Q(x, y)\mathcal{G}(\theta_t\omega) \tag{1.5}$$

where μ is a small parameter, $P(x, y)$ and $Q(x, y)$ are of higher order in (x, y).

The random forcing we have here is a multiplicative noise with a Brownian motion and is degenerated at 0.

We assume that equation (1.1) admits a homoclinic solution to the saddle $(x, y) = 0$, which we denote as

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$

We study the dynamical behavior of equation (1.5) through the random Poincaré return map $\mathcal{R}_\omega$ induced by equation (1.5) in extended phase space. Our result can be summarized as follows:

Main Theorem. *Assume that there exists a $t \in \mathbb{R}$ so that*

$$b'(t)P(a(t), b(t)) - a'(t)Q(a(t), b(t)) \neq 0.$$

Then there exist a $\mu_0 > 0$ and a θ_t-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for all $0 < \mu < \mu_0$ and all $\omega \in \tilde{\Omega}$, the return map $\mathcal{R}_\omega$ for equation (1.5) admits a topological horseshoe of infinitely many branches.

The condition $b'(t)P(a(t), b(t)) - a'(t)Q(a(t), b(t)) \neq 0$ simply says that the vector field (P, Q) is not tangent to the homoclinic orbit at some point of the orbit. This condition can be explicitly verified as we will see for the Duffing equation and the pendulum equation.

The existence of a topological horseshoe implies that for all $p_0 = (x_0, y_0)$ in the phase space that is located sufficiently close to the unperturbed homoclinic loop, we have that the dynamics of the solutions through p_0 depend sensitively on the time it is initiated at p_0. First, there exist infinitely many pairs t^0 and t^∞, arbitrarily close to each other, such that the solution for the initial time t^0 is attracted to the solution $(x, y) = (0, 0)$ so it never completes one loop around the homoclinic solution. The solution initiated from t^∞, on the other hand, does go around the homoclinic loop infinitely many times in a rather steady pace, taking roughly the same length of time to complete one round.[1] Second, in between t^0 and t^∞, the behavior of solutions are arbitrary in the sense that they assume *all* imaginable manners in going around the homoclinic loop in phase space. This is too say that, at any moment a solution could decide to slow down, taking, say, roughly twice, three times, or *any* number of times of whatever it took in the previous round in completing the next round. It could also decided to accelerate in similar fashion. In addition, arbitrarily close to each of these initial times, there are also solutions that decide to go away forever from the homoclinic loop at *any* imaginable moment. There exist infinitely many pairs p^0 and p^∞ around the homoclinic solution, arbitrarily close to each other, such that the dynamics of the solutions in between p^0 and p^∞ are the same as the ones described above with p^0 in the place of t^0 and p^∞ in the place of t^∞.

As applications, we first consider the Duffing equation driven by the Brownian motion

$$\frac{d^2q}{dt^2} - q + q^3 = \mu q^2 \mathcal{G}(\Theta_t\omega). \tag{1.6}$$

We have

Corollary A. *There exists a $\mu_0 > 0$ such that for almost all $\omega \in \Omega$ and all $0 < \mu < \mu_0$, the return map $\mathcal{R}_\omega$ for equation (1.6) admits a topological horseshoe.*

[1]This length of time is a multiple of the forcing period for periodically forced equations. It deviates from being a constant when the forcing is non-periodic.

Similarly, for the randomly perturbed pendulums, we have

Corollary B. *There exists a $\mu_0 > 0$ such that for almost all $\omega \in \Omega$ and all $0 < \mu < \mu_0$, the return map $\mathcal{R}_\omega$ for the equation for the randomly forced non-linear pendulum*

$$\frac{d^2q}{dt^2} + \sin q = \mu q^2 \mathcal{G}(\Theta_t\omega)$$

admits a topological horseshoe.

The results we obtained here are sample-wise. However, we cannot treat the problem simply as a nonautonomous problem by regarding ω as a parameter. The feature of the Brownian motion as a whole needs to be utilized.

There are two basic issues: (a) How to characterize the chaotic behavior of a differential equation driven by a stochastic process; (b) How to deal with the unboundedness of the forcing function. We study equation (1.5) for each sample path in the probability space in the extended phase space. Based the homoclinic orbit of the unforced equation, we introduce an extension of the classical Poincaré return map for the forced equation, which is partially defined on an infinite strip in the extended phase space. Under the forcing driven by a Brownian motion, this map has strong expansion in a direction. We extend the approach of using vertical strips and horizontal strips, due to Smale, to describe chaotic dynamics. The forcing term $\mathcal{G}(\theta_t\omega)$ is an unbounded stationary stochastic process with a normal distribution, which in fact is unbounded almost surely. The systems we study may not be dissipative. Consequently, for any given bounded domain U in the phase space, this unbounded forcing can immediately push a solution out of U and there is no global random attractor. Because of the degeneracy of the forcing, some solution will leave U and some solution will stay in U for all time, or at least in one of the direction of time. Those solutions staying in U may bear complicated dynamical structures. Contrasting to the bounded perturbation, instead of a whole piece of stable and unstable manifolds at the saddle point, we have infinite many slices of stable and unstable manifolds in the extended space. Nevertheless, there are infinitely many slices of stable manifolds intersecting with infinitely many slices of unstable manifolds under the forcing driven by the Brownian motion. The returned map is defined on infinitely many bounded sections of the bi-infinite strip. In order to construct a topological horseshoe, we also need to study the behavior of solutions of equation (1.5) in a tempered neighborhood of the saddle point, which may shrink to a point at a sub-exponential rate. To control the shrink, we use a semi-linearization result for random dynamical systems, which was presented in [15] for discrete-time systems. The almost sure property is due to the ergodicity of the Wiener shift and the Birkhoff ergodic theorem.

The study of complicated dynamics of ordinary differential equations under periodic perturbations has a long and rich history that dates back to Poincaré and Birkhoff. The complicated behavior induced by the presence of homoclinic intersections of the stable and the unstable manifold of a saddle fixed point was first observed by Poincaré [19], described by Birkhoff [5], proved by Smale [24, 25] in a geometry form, and was systematically studied by Alekseev [1] with applications to Sitnikov's three body problem [35]. There is a vast literature on the chaotic behavior induced by the presence of homoclinic intersections for differential equations driven by a periodic forcing, see for example, [13], [16], [27, 28, 29, 30, 31, 32], [12], [8],[11], [7], [10], [20], [4], [33, 34], [37], [38], and their references therein. There has also been a substantial literature on extending the Birkhoff-Smale theorem to quasi-periodically and almost periodically forced differential equations, see [26], [21], [23], [18], [39], [27], and [36]. The problem we deals with here is a system driven by a Brownian motion, an unbounded and irregular forcing.

This paper is organized as follows. In Section 2 we introduce in precise terms the equation of study and present the main theorem of this paper. We also present the applications of the main theorem to the randomly forced Duffing equation and the randomly forced pendulum. In Section 3 we study the local dynamics in a fixed neighborhood of the saddle fixed point. In Section 4 we study the properties of the Melnikov functions, and construct infinitely many vertical strips that will serve as the building blocks for the construction of the desired horseshoe map. In Section 5 we prove the main theorem by using the results obtained in Sections 3-4 and a semi-linearization result which is proved in the appendix.

2. Statement of results

In this section, we first introduce the basic setting, assumptions, and concepts. Then, we state our main result. The applications of the main theorem to the randomly forced Duffing equation and the randomly forced pendulum are presented at last.

2.1. **Equation of Study.**

Unforced equations. We consider an autonomous differential equation in $\mathbb{R}^2$,

$$\frac{dx}{dt} = -\alpha x + f(x, y), \qquad \frac{dy}{dt} = \beta y + g(x, y) \tag{2.1}$$

where $f(x, y)$ and $g(x, y)$ are high order terms at $(x, y) = (0, 0)$, C^r in a small neighborhood of $(0, 0)$ for an integer $r > 2$. We assume that equation (2.1) admits a homoclinic orbit to the saddle $(0, 0)$, which we denote as

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$

Let U be an open neighborhood in (x, y)-plane that contains the closure of ℓ. We also assume that $f(x, y), g(x, y)$ are C^r on U.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classic Wiener probability space, where

$$\Omega = C_0(\mathbb{R}, \mathbb{R}) = \big\{\omega(t) \ : \ \omega(\cdot) : \mathbb{R} \to \mathbb{R} \ \text{ is continuous and } \ \omega(0) = 0\big\}$$

with the open compact under which it is a Polish space, $\mathcal{F}$ is its Borel σ-algebra, and $\mathbb{P}$ is the Wiener measure. The Brownian motion takes the form $B_t(\omega) = \omega(t)$. We consider the Wiener shift θ_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is given by

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t). \tag{2.2}$$

It is well-known that $\mathbb{P}$ is an ergodic invariant measure for θ_t.

For small $\Delta > 0$, let $\mathcal{G} : \Omega \to \mathbb{R}$ be such that

$$\mathcal{G}(\omega) = \frac{1}{\Delta}\omega(\Delta). \tag{2.3}$$

Then, we have

$$\mathcal{G}(\theta_t\omega) = \frac{1}{\Delta}(\omega(t + \Delta) - \omega(t)). \tag{2.4}$$

From the properties of the Brownian motion, we know that $\mathcal{G}(\theta_t\omega)$ is an unbounded stationary stochastic process with a normal distribution, which in fact is unbounded almost surely. For each fixed ω, it is continuous in t. We can also view $\mathcal{G}(\theta_t\omega)$ as a discrete version of the white noise.

Let Δ be a small positive number that is fixed in the rest of this paper.

Equation driven by a Brownian motion. To the right hand side of equation (2.1) we add a random forcing driven by a Brownian motion, resulting in a new equation,

$$\begin{aligned} \frac{dx}{dt} &= -\alpha x + f(x, y) + \mu P(x, y)\mathcal{G}(\theta_t\omega) \\ \frac{dy}{dt} &= \beta y + g(x, y) + \mu Q(x, y)\mathcal{G}(\theta_t\omega) \end{aligned} \tag{2.5}$$

where μ is a small parameter. We also assume that $P(x, y)$ and $Q(x, y)$ are C^r-functions on U and they are high order terms at $(x, y) = (0, 0)$.

2.2. **Statement of result.** In this paper we prove that if the vector field (P, Q) is not tangent to the homoclinic orbit at some point of the orbit, then equation (2.5) is chaotic, namely has a topological horseshoe, almost surely.

A. The return map $\mathcal{R}$: Let $\omega = \tilde{\omega} \in \Omega$ be fixed in equation (2.5). We regard (2.5) as a non-autonomously forced equation. In the space of (x, y), we let U be the union of a small neighborhood B_ε (a ball at $(0, 0)$ with radius $\varepsilon > 0$) of $(0, 0)$ and a small neighborhood $D_{C_1(\varepsilon)}$ of size $C_1(\varepsilon)\mu$ around the part of ℓ that is outside of $B_{\frac{1}{4}\varepsilon}$ where $C_1(\varepsilon)$ is a constant independent of μ. We construct a subset of solutions in U with complicated dynamical structure for equation (2.5).

Re-write equation (2.5) in autonomous form as

$$\begin{aligned} \frac{dx}{dt} &= -\alpha x + f(x, y) + \mu P(x, y)\mathcal{G}(\theta_p(\tilde{\omega})) \\ \frac{dy}{dt} &= \beta y + g(x, y) + \mu Q(x, y)\mathcal{G}(\theta_p(\tilde{\omega})) \\ \frac{dp}{dt} &= 1. \end{aligned} \tag{2.6}$$

The space (x, y, p) is the *extended phase space.* Let $\mathcal{U} = U \times \mathbb{R}$. We study the dynamics of the solutions of equation (2.6) through the iterations of a return map we now introduce in $\mathcal{U}$. See Fig. 1. Let σ^- be of length 2μ and σ^+ be of length $2C_1(\varepsilon)\mu$, both are centered at their respective intersections to ℓ and are perpendicular to ℓ.

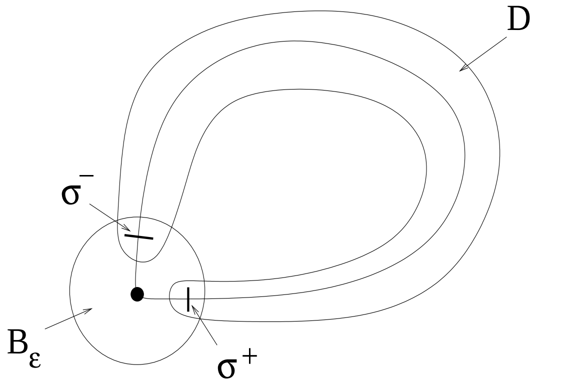

Fig. 1 B_ε, D_{K_1} and $\sigma^\pm$.

In the extended phase space (x, y, p) we denote

$$\mathcal{B}_\varepsilon = B_\varepsilon \times \mathbb{R}, \quad \mathcal{D}_{C_1(\varepsilon)} = D_{C_1(\varepsilon)} \times \mathbb{R}$$

and let

$$\Sigma^\pm = \sigma^\pm \times \mathbb{R}.$$

Let $\mathcal{N} : \Sigma^+ \to \Sigma^-$ be the map induced by the solutions on $\mathcal{B}_\varepsilon$ and $\mathcal{M} : \Sigma^- \to \Sigma^+$ be the map induced by the solutions on $\mathcal{D}_{K_1}$. We first consider $\mathcal{M}$ and $\mathcal{N}$ separately, then compose $\mathcal{N}$ and $\mathcal{M}$ to obtain a return map $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$.

B. Topological horseshoe. Even in the case of uniformly bounded forcing, $\mathcal{R}$ is only partially defined on Σ^-. Let $\mathcal{V}$ be the subset of Σ^- where $\mathcal{R}$ is defined. The following concept of a topological horseshoe for $\mathcal{R}$ was introduced previously in [**?**]. We first introduce some geometric terms. We call the direction of $p \in \mathbb{R}$ in Σ^- the *horizontal direction* and the direction of σ^- (transversal to the homoclinic solution ℓ in the original phase space) the *vertical direction*. In Σ^-, a *vertical curve* is a non-self-intersecting, continuous curve that connects the two horizontal boundaries of Σ^-. We call a region that is bounded by two non-intersecting vertical curves a *vertical strip*, which we denote as V. The two defining vertical curves for a given vertical strip V are the *vertical boundary* of V. We call a non-self-intersecting continuous curve connecting the two vertical boundaries of V a *fully extended* horizonal curve in V. Let V_1, V_2 be two non-intersecting vertical strips in $\mathcal{V}$. We say that $\mathcal{R}(V_1)$ *crosses* V_2 *horizontally* if for every fully extended horizontal curve h of V_1, there is a subsegment $\tilde{h}$ of h so that $\mathcal{R}(\tilde{h})$ is a fully extended horizontal curve in V_2.

Definition 2.1 (**Topological horseshoe**). *Let $\mathcal{R} : \mathcal{V} \to \Sigma^-$, $\mathcal{V} \subset \Sigma^-$, be continuous. We say that $\mathcal{R}$ admits a topological horseshoe of k-branches, $k \le \infty$, if there exists a bi-infinite sequence of non-intersecting vertical strips $\{V_n\}_{n=-\infty}^{\infty}$ lined up monotonically from $t = -\infty$ to $t = +\infty$ in Σ^-, $V_n \subset \mathcal{V}$ for all n, such that*

(1) For every n, there exists a $\hat{n}_1 > n$, such that $\mathcal{R}(V_n)$ crosses $V_{\hat{n}_1}, V_{\hat{n}_1+1}, \cdots V_{\hat{n}_2+k}$ horizontally.

(2) For every n, there exists a $\hat{n}_2 < n$, such that $\mathcal{R}(V_{\hat{n}_2-k}), \cdots, \mathcal{R}(V_{\hat{n}_2})$ crosses V_n horizontally.

Observe that, if $\mathcal{R}$ admits a horseshoe of k-branches, then inside every vertical strip V_n, there exists a 2D Cantor set Λ_n formed by the intersections of a k-Cantor set of vertical curves and a k-Cantor set of fully extended horizontal curves, so that all orbit of Λ_n will be in $\mathcal{V}$ for all time. If $\mathcal{R} = \mathcal{R}_\omega$ is from equation (2.5) for a given $\omega \in \Omega$, then all solutions initiated at Λ_n will stay inside of $B_\varepsilon \cup D_{C_1}$ for all time. If $\mathcal{R}_\omega$ admits a topological horseshoe according to Definition 2.1, then the structure of solutions of equation (2.5) is *complicated* inside of $B_\varepsilon \cup D_{C_1}$ and we have *chaotic* dynamics for equation (2.5) around the unforced homoclinic loop ℓ. The horseshoe we have here can be regarded as an extension of Smale's horseshoe to non-periodic equations.

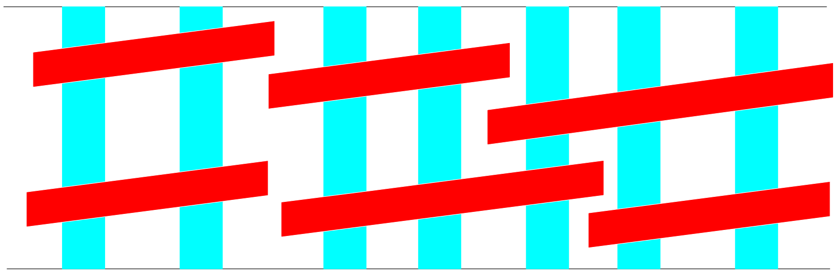

Fig. 2 A topological horseshoe of 2-branches

We are now ready to state our main theorem. Recall that $(a(s), b(s))$ is a homoclinic solution of equation (2.1) to the saddle $(0, 0)$.

Main Theorem. *Assume that there exist a $t \in \mathbb{R}$ so that*

$$b'(t)P(a(t), b(t)) - a'(t)Q(a(t), b(t)) \neq 0.$$

Then there exist a $\mu_0 > 0$ and a θ_t-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for all $0 < \mu < \mu_0$ and all $\omega \in \tilde{\Omega}$, the return map $\mathcal{R}_\omega$ for equation (2.6) admits a topological horseshoe of infinitely many branches.

2.3. **Examples.** We give two applications of the main theorem. We first apply it to the forced Duffing equation driven by a Brownian motion.

We start with the autonomous Duffing equation ([9])

$$\frac{d^2q}{dt^2} - q + q^3 = 0. \tag{2.7}$$

Equation (2.7) has a homoclinic solution to $(q,p) = (0,0)$, which we denote as $\ell_0(t) = (q_0(t), p_0(t))$, where

$$q_0(t) = \frac{2\sqrt{2}e^t}{(1+e^{2t})}, \quad p_0(t) = \frac{2\sqrt{2}(e^t - e^{3t})}{(1+e^{2t})^2}.$$

We add a Brownian motion driving force to the right of equation (2.7) to obtain

$$\frac{d^2q}{dt^2} - q + q^3 = \mu q^2 \mathcal{G}(\theta_t\omega), \tag{2.8}$$

where $\mathcal{G}(\theta_t\omega)$ is given by (2.4).

To apply the main theorem we re-write equation (2.8) as

$$\begin{aligned} \frac{dq}{dt} &= p, \\ \frac{dp}{dt} &= q - q^3 + \mu q^2 \mathcal{G}(\Theta_t). \end{aligned}$$

To write the linear part of equation in a canonical form, we introduce new variables (x,y) so that

$$q = x + y, \quad p = -x + y.$$

Then, the new equations in (x,y) are

$$\begin{aligned} \frac{dx}{dt} &= -x + f(x,y) + \mu P(x,y)\mathcal{G}(\Theta_t\omega) \\ \frac{dy}{dt} &= y + g(x,y) + \mu Q(x,y)\mathcal{G}(\Theta_t\omega), \end{aligned}$$

where

$$f(x,y) = \frac{1}{2}(x+y)^3, \quad g(x,y) = \frac{-1}{2}(x+y)^3;$$

$$P(x,y) = \frac{-1}{2}(x+y)^2, \quad Q(x,y) = \frac{1}{2}(x+y)^2.$$

Clearly, the condition for the Main theorem holds at any given $t \neq 0$. Therefore, we have

Corollary 2.1. *There exist a $\mu_0 > 0$ and a θ_t-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for all $0 < \mu < \mu_0$ and all $\omega \in \tilde{\Omega}$, the return map $\mathcal{R}_\omega$ for equation (2.8) admits a topological horseshoe.*

By a similar computation and applying the main theorem to the randomly perturbed non-linear pendulums, we have

Corollary 2.2. *There exist a $\mu_0 > 0$ and a θ_t-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that for all $0 < \mu < \mu_0$ and all $\omega \in \tilde{\Omega}$, the return map $\mathcal{R}_\omega$ for the equation for the randomly forced non-linear pendulum*

$$\frac{d^2q}{dt^2} + \sin q = \mu q^2 \mathcal{G}(\Theta_t\omega) \tag{2.9}$$

admits a topological horseshoe.

3. Local dynamics around the fixed point

Let B_ε be a neighborhood of $(x,y)=(0,0)$ of a fixed size in the (x,y)-space. In the extended phase space (x,y,p), let $\mathcal{B}_\varepsilon = B_\varepsilon \times \mathbb{R}$. If the magnitude of the forcing function $|\mathcal{G}(\theta_t\tilde{\omega})|$ were *uniformly bounded* for all time, then the forced equation would have a two-dimensional local unstable and a two dimensional local stable manifold in $\mathcal{B}_\varepsilon$ in the extended phase space. The intersection of the local unstable manifold with Σ^- and the intersection of the local stable manifold with Σ^+ would be a simple curve across Σ^- and a simple curve across Σ^+ respectively in the p-direction. Unfortunately, the forcing function $\mathcal{G}(\theta_t\tilde{\omega})$ is *unbounded* in time almost surely and for equation (2.6) there are no longer these nice local unstable and stable curves across Σ^- and Σ^+ all the way in the p-direction. We can, however, prove that on certain vertical strips in Σ^-, there remains a well-defined local unstable segment. We also have a similar local stable segment on Σ^+.

In the following, we will first introduce some preliminaries and conventions in notation that are used throughout of the rest of this paper. We will then prove the existence of local unstable solutions and study the properties of the local unstable curves. Finally, we will give the corresponding results for the local stable solutions.

3.1. **Preliminaries.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener space. From the law of logarithms, we have

$$\lim_{s\to\pm\infty} \frac{\omega(s)}{s} = 0$$

almost surely. Let $\hat{\Omega}$ be the set of full measure on which the above properties hold. Let

$$C_\omega = \sup_{s\in\mathbb{Q}} \frac{|\omega(s)|}{|s|+1}$$

where $\mathbb{Q}$ is the set of rational numbers. Since for each s, $\omega(s) : \Omega \to \mathbb{R}$ is measurable and the sup is finite, $C_\omega : \hat{\Omega} \to \mathbb{R}^+$ is a measurable function and

$$|\omega(s)| \le C_\omega(|s|+1)$$

for all $s \in \mathbb{R}$. Recall that

$$\theta_t\omega(s) = \omega(t+s) - \omega(t),$$

which is a measurable dynamical system and is continuous in t. It then follows that

$$C_{\theta_t\omega} < 2(C_\omega+1)(|t|+1). \tag{3.1}$$

For $K_0 > 0$, we denote

$$\Lambda_{K_0} = \{\omega \in \hat{\Omega} : \quad C_\omega \le K_0\}.$$

Since $C_\omega : \Omega \to \mathbb{R}^+$ is Wiener measurable, it follows that, for any given $\delta > 0$, there exists a $K_0 > 0$ sufficiently large, so that $\mathbb{P}(\Lambda_{K_0}) > 1-\delta$.

In this section we fix a $K_0 > 0$ such that $\mathbb{P}(\Lambda_{K_0}) > 0$. Let $\mathcal{G}$ be as in (2.3) and $\mathcal{X}_{\Lambda_{K_0}}(\cdot)$ be the characteristic function for Λ_{K_0} on $\hat{\Omega}$. Let $\tilde{\Omega} \subset \hat{\Omega}$ be the subset of $\hat{\Omega}$ of full measure on which the Birkhoff ergodic theory holds for $|\mathcal{G}(\cdot)|$ and $\mathcal{X}_{\Lambda_{K_0}}(\cdot)$ for θ_t. We note that both $\hat{\Omega}$ and $\tilde{\Omega}$ can be chosen to be θ_t-invariant. For the rest of this paper, $\tilde{\omega} \in \tilde{\Omega}$ is fixed once and for all. For $t \in \mathbb{R}$ we denote

$$\omega_t = \theta_t\tilde{\omega}.$$

It follows from the Birkhoff ergodic theory that there exists a bi-infinite sequence $\{q_n\}_{n=-\infty}^{\infty}$ such that

$$\omega_{q_n} := \theta_{q_n}\tilde{\omega} \in \Lambda_{K_0}$$

for all $n \in \mathbb{Z}$.

Parameters ε, L^-, L^+ and μ: The letter ε denotes the radius of B_ε. It depends on K_0 and a few other quantities that will be introduced in precise terms later in the process. For the purpose of this section it suffices to think that $\varepsilon << 1$.

Let $-L^-, L^+$ be the time that the homoclinic solution $\ell(t) = (a(t), b(t))$ reaches $B_{\frac{1}{2}\varepsilon}$ in the negative and the positive time directions respectively. L^-, L^+ are determined completely by ε and $\ell(t)$ and we have

$$L^-,\ L^+ \sim \ln \varepsilon^{-1}. \tag{3.2}$$

μ is the parameter presented in front of the forcing terms in equation (2.5). We assume that $\mu << \varepsilon$.

Notation: Constants are quantities that are independent of phase variables and time. When a constant is a dependent of $\omega \in \Omega$ or one of the parameters above, the dependency is usually made explicit. For example, C_ω is a dependent of ω, and $C_1(\varepsilon)$ depends on ε.

The letter K is reserved throughout to represent constants that are *independent* of ω and parameters $\varepsilon,\ \mu$. When standing alone, K represents a generic constant, the value of which is allowed to change from place to place, sometimes even in the same line. When K is with a subscript, such as $K_0, K_1, \cdots$, then it is non-generic, carrying a fixed value throughout. In particular, Δ in (2.3) is one specific value for K.

Let $\{q_n\}$ be such that $\omega_{q_n} \in \Lambda_{K_0}$, and

$$L = L^- + L^+.$$

Let

$$\tilde{\Sigma}^- = \{(x, y, p) \in \Sigma^-, \quad p \in \cup_n [q_n - 2L, q_n + 2L]\}.$$

We define (s, z) by letting

$$x = a(s) + v(s)z, \quad y = b(s) - u(s)z \tag{3.3}$$

where $(u(t), v(t))$ are the unit tangent vectors of the unforced homoclinic solution $\ell(t) = (a(t), b(t))$. We use new variables (s, z, p) interchangeable with (x, y, p) to represent points in the extended phase space. In these new variables, $s = -L^-$ for Σ^- and we denote

$$\tilde{\Sigma}^- = \{(z, p) : \quad |z| < \mu, \quad p \in \cup_n [q_n - 2L, q_n + 2L]\}.$$

3.2. **Local unstable solution.** In this subsection we prove the existence of local unstable solutions for equation (2.6) using the Lyapunov-Perron approach. Let $\omega = \tilde{\omega}$ be fixed in (2.5) and denote $\Theta_q \tilde{\omega} = \omega_q$. We study equation

$$\frac{dx}{dt} = -\alpha x + f(x, y) + \mu P(x, y)\mathcal{G}(\theta_t \omega_q)), \qquad \frac{dy}{dt} = \beta y + g(x, y) + \mu Q(x, y)\mathcal{G}(\theta_t \omega_q)) \tag{3.4}$$

regarding $q \in \mathbb{R}$ as a parameter.

To obtain local results we first use a cut-off function to modify the equation. Let $\sigma(s)$ be a C^∞ function from $(0, +\infty) \to [0, 1]$ with

$$\sigma(s) = 1 \text{ for } 0 \le s \le 1, \qquad \sigma(s) = 0 \text{ for } s \ge 2$$

$$\sup_{s \in [1,2]} |\sigma'(s)| < 2.$$

We multiply the cut off-function to the higher order terms of equation (3.4) to obtain

$$\frac{dx}{dt} = -\alpha x + \tilde{f}(x, y) + \mu \tilde{P}(x, y)\mathcal{G}(\theta_t \omega_q)), \qquad \frac{dy}{dt} = \beta y + \tilde{g}(x, y) + \mu \tilde{Q}(x, y)\mathcal{G}(\theta_t \omega_q)). \tag{3.5}$$

where $\omega_q = \theta_q \tilde{\omega}$,

$$\tilde{f}(x, y) = f(x, y) \cdot \sigma((2\varepsilon)^{-1}\sqrt{x^2 + y^2})$$

and $\tilde{g}(x,y), \tilde{P}(x,y)$, and $\tilde{Q}(x,y)$ are similarly defined.

In equation (3.5) we fix an $\tilde{\omega} \in \tilde{\Omega}$ and regard $q \in \mathbb{R}$ as a parameter, the value of which is allowed to vary over $\mathbb{R}$. Let $\gamma > 0$ be fixed such that $0 < 2\gamma < min\{\alpha, \beta\}$. We consider the Banach space

$$\mathcal{C}_\gamma^- = \left\{ \phi \mid \phi : \mathbb{R}^- \to \mathbb{R}^2 \text{ is continuous and } \sup_{t\leq 0} |e^{-\gamma t}\phi(t)| < \infty \right\}$$

with norm $\|\phi\| := \|\phi\|_\gamma^- = \sup_{t\leq 0} |e^{-\gamma t}\phi(t)|$. We represent the usual magnitude of vectors in $\mathbb{R}^n$ by using $|\cdot|$.

Let $w = (x,y)$ be the phase variable and let $w(t, w^0)$ denote the solution of equation (3.5) satisfying $w(0, w^0) = w^0$. We wish to prove that for any given $y^0 \in (-\varepsilon, \varepsilon)$, there exists a unique $x^0 \in (-\varepsilon, \varepsilon)$ so that $w(t, w^0) \in \mathcal{C}_\gamma^-$ where $w^0 = (x^0, y^0)$. We also wish to show that $|w(t)| < 2\varepsilon$ for all $t \in \mathbb{R}^-$ so $w(t) = w(t, w^0)$ is a solution of equation (3.4). This is, however, only true for some q. Recall that $\{q_n\}_{n=-\infty}^{\infty}$ is such that $\omega_{q_n} := \theta_{q_n}(\tilde{\omega}) \in \Lambda_{K_0}$.

Proposition 3.1. *There exists a positive $\varepsilon_0 << 1$ independent of $\tilde{\omega}$ such that for any given $0 < \varepsilon < \varepsilon_0$, there exists $\mu_0(\varepsilon) > 0$ independent of $\tilde{\omega}$, so that for all $0 < \mu < \mu_0(\varepsilon)$, and all $q \in \cup_n[q_n - 2L, q_n + 2L]$, the following hold for equation (3.5): for any given $y^0 \in (-\varepsilon, \varepsilon)$, there exists a unique $x^0 \in (-\varepsilon, \varepsilon)$, such that*

(a) the solution $w(t, w^0)$ satisfying $w(0, w^0) = w^0 = (x^0, y^0)$ is in $\mathcal{C}_\gamma^-$, and

(b) $|w(t, w^0)| < 2\varepsilon$ for all $t \in \mathbb{R}^-$.

Proof: This proof is based on a modified version of the Lyapunov-Perron approach.

We first note that if $w(t) = (x(t), y(t)) \in C_\gamma^-$, then for a given $y^0 \in \mathbb{R}$, $w(t)$ is a solution of (3.5) satisfying $y(0) = y^0$ if and only if

$$\begin{aligned} y(t) &= e^{\beta t}y^0 + \int_0^t e^{\beta(t-\tau)}\Big(\tilde{g}(w(\tau)) + \mu\tilde{Q}(w(\tau))\mathcal{G}(\theta_\tau\omega_q)\Big)d\tau \\ x(t) &= \int_{-\infty}^t e^{-\alpha(t-\tau)}\Big(\tilde{f}(w(\tau)) + \mu\tilde{P}(w(\tau))\mathcal{G}(\theta_\tau\omega_q)\Big)d\tau \end{aligned} \tag{3.6}$$

for all $t \leq 0$.

Next, we prove the existence and the uniqueness of the solution $w(t) = (x(t), y(t))$ of (3.6) in $\mathcal{C}_\gamma^-$. For $\phi \in \mathcal{C}_\gamma^-$, let $\mathcal{J}(\phi)(t)$ denote the right side of (3.6) with $\phi(\tau)$ in the place of $w(\tau)$. $\mathcal{J} : \mathcal{C}_\gamma^- \to \mathcal{C}_\gamma^-$ is the standard Lyapunov-Perron operator. Let $y_0 \in (-\varepsilon, \varepsilon)$ be fixed and let

$$\mathcal{C}_\gamma^-(1) = \{\phi \in \mathcal{C}_\gamma^- : \quad \|\phi\| \leq 1\}.$$

We first prove that $\mathcal{J}(\mathcal{C}_\gamma^-(1)) \subset \mathcal{C}_\gamma^-(1))$. We then prove that $\mathcal{J}$ is contracting on $\mathcal{C}_\gamma^-(1)$.

For $\mathcal{J}(\mathcal{C}_\gamma^-(1)) \subset \mathcal{C}_\gamma^-(1))$, we let $\phi \in \mathcal{C}_\gamma^-(1)$ and denote $(x(t), y(t)) = \mathcal{J}(\phi)$. We have for $t < 0$,

$$e^{-\gamma t}y(t) = e^{(\beta-\gamma)t}y^0 + e^{-\gamma t} \cdot \int_0^t e^{\beta(t-\tau)}\Big(\tilde{g}(u(\tau)) + \mu\tilde{Q}(u(\tau))\mathcal{G}(\theta_\tau\omega_q)\Big)d\tau.$$

Recall that f, g, P, Q are all second order terms at $(x,y) = (0,0)$. We have

$$\begin{aligned} |e^{-\gamma t}y(t)| &< \varepsilon + K\int_t^0 e^{-\gamma t} \cdot e^{\beta(t-\tau)}|\phi(\tau)|^2\sigma((2\varepsilon)^{-1}|\phi(\tau)|)(1 + \mu|\mathcal{G}(\theta_\tau\omega_q)|)d\tau \\ &< \varepsilon + 4\varepsilon K\|\phi\|\int_t^0 e^{(\beta-\gamma)(t-\tau)}d\tau + K\mu\|\phi\|^2\int_t^0 e^{(\beta-\gamma)(t-\tau)}|\mathcal{G}(\theta_\tau\omega_q)|e^{\gamma\tau}d\tau \\ &< \frac{1}{2}. \end{aligned} \tag{3.7}$$

Note that to obtain the second inequality, we used $|\phi(\tau)|\sigma((2\varepsilon)^{-1}|\phi|) < 4\varepsilon$ for the term without $\mathcal{G}$. For the term with $\mathcal{G}$ we obtain an extra copy of $e^{\gamma\tau}$ by converting $|\phi(\tau)|^2$ to $\|\phi\|^2$. For the third inequality we make ε small for the first integral. For the second integral we use

$$\begin{aligned}|\mathcal{G}(\theta_t\omega_q)| &= |\mathcal{G}(\theta_{t+(q-q_n)}\omega_{q_n}))| = \Delta^{-1}|\omega_{q_n}(t+q-q_n+\Delta) - \omega_{q_n}(t+q-q_n)|\\ &\le \Delta^{-1}|\omega_{q_n}(t+q-q_n+\Delta)| + \Delta^{-1}|\omega_{q_n}(t+q-q_n)|\\ &\le \Delta^{-1}(K_0+1)(|t+q-q_n+\Delta| + |t+q-q_n| + 2)\\ &< 2\Delta^{-1}(K_0+1)(|t| + 2L + 1)\end{aligned}$$

because $q_n \in \Lambda_{K_0}$. It then follows that

$$\left|\int_t^0 e^{(\beta-\gamma)(t-\tau)}|\mathcal{G}(\Theta_\tau\omega_q)|e^{\gamma\tau}d\tau\right| < K\left|\int_t^0 (|\tau| + 2L + 1)e^{\gamma\tau}d\tau\right| < KL.$$

Finally we make μ sufficiently small. The parameter μ is a dependent of L hence of ε but it is independent of $\tilde{\omega}$. Estimate for $|e^{-\gamma t}x(t)|$ is similar.

Next, we prove that $\mathcal{J}$ is a contraction on $\mathcal{C}_\gamma^-(1)$. Let $\phi, \bar{\phi} \in \mathcal{C}_\gamma^-(1)$ and denote $\mathcal{J}(\phi) = (x(t), y(t))$, $\mathcal{J}(\bar{\phi}) = (\bar{x}(t), \bar{y}(t))$. We have

$$\begin{aligned}|y(t) - \bar{y}(t)| \le K\Big\{ &\int_t^0 e^{\beta(t-\tau)}(|\phi| + |\bar{\phi}|)|\phi - \bar{\phi}|\sigma((2\varepsilon)^{-1}|\phi|)d\tau\\ &+ (2\varepsilon)^{-1}\int_t^0 e^{\beta(t-\tau)}|\bar{\phi}|^2|\phi-\bar{\phi}|\ |\sigma'((2\varepsilon)^{-1}\xi)|d\tau\\ &+ \mu\int_t^0 e^{(\beta-\gamma)(t-\tau)}(|\phi| + |\bar{\phi}|)|\phi - \bar{\phi}|\sigma((2\varepsilon)^{-1}|\phi|)|\mathcal{G}(\theta_\tau\omega_q)|d\tau\\ &+ \mu(2\varepsilon)^{-1}\int_t^0 e^{(\beta-\gamma)(t-\tau)}|\bar{\phi}|^2|\phi-\bar{\phi}|\ |\sigma'((2\varepsilon)^{-1}\xi)||\mathcal{G}(\theta_\tau\omega_q)|d\tau\Big\}\end{aligned}$$

where $\xi = \xi(\tau)$ is in between $|\bar{\phi}(\tau)|$ and $|\phi(\tau)|$.

Similarly, we have

$$\begin{aligned}|x(t) - \bar{x}(t)| \le K\Big\{ &\int_{-\infty}^t e^{-\alpha(t-\tau)}(|\phi| + |\bar{\phi}|)|\phi - \bar{\phi}|\sigma((2\varepsilon)^{-1}|\phi|)d\tau\\ &+ (2\varepsilon)^{-1}\int_{-\infty}^t e^{-\alpha(t-\tau)}|\bar{\phi}|^2|\phi-\bar{\phi}|\ |\sigma'((2\varepsilon)^{-1}\xi)|d\tau\\ &+ \mu\int_{-\infty}^t e^{-\alpha(t-\tau)}(|\phi| + |\bar{\phi}|)|\phi - \bar{\phi}|\sigma((2\varepsilon)^{-1}|\phi|)|\mathcal{G}(\theta_\tau\omega_q)|d\tau\\ &+ \mu(2\varepsilon)^{-1}\int_{-\infty}^t e^{-\alpha(t-\tau)}|\bar{\phi}|^2|\phi-\bar{\phi}|\ |\sigma'((2\varepsilon)^{-1}\xi)||\mathcal{G}(\theta_\tau\omega_q)|d\tau\Big\}.\end{aligned}$$

From these two it follows that

$$\begin{aligned}\|\mathcal{J}(\phi)-\mathcal{J}(\bar{\phi})\| &\leq K\sup_{t\in\mathbb{R}^-}\Big\{8\varepsilon\int_t^0 e^{(\beta-\gamma)(t-\tau)}d\tau+8\varepsilon\int_{-\infty}^t e^{-(\alpha+\gamma)(t-\tau)}d\tau\\
&\quad+4\mu\int_t^0 e^{(\beta-\gamma)(t-\tau)}|\mathcal{G}(\theta_\tau\omega_q)|e^{\gamma\tau}d\tau+4\mu\int_{-\infty}^t e^{-(\alpha+\gamma)(t-\tau)}|\mathcal{G}(\theta_\tau\omega_q)|e^{\gamma\tau}d\tau\Big\}\|\phi-\bar{\phi}\|\\
&\leq K\Big\{8\varepsilon(\frac{1}{\beta-\gamma}+\frac{1}{\alpha-\gamma})+8\mu\int_{-\infty}^0|\mathcal{G}(\theta_\tau\omega_q)|e^{\gamma\tau}d\tau\Big\}\|\phi-\bar{\phi}\|\\
&\leq K\Big(8\varepsilon(\frac{1}{\beta-\gamma}+\frac{1}{\alpha-\gamma})+16\mu\Delta^{-1}(K_0+1)\gamma^{-1}(\frac{1}{\gamma}+2L+1)\Big)\|\phi-\bar{\phi}\|\end{aligned}$$

where for the last inequality we again used

$$|\mathcal{G}(\theta_t\omega_q)|\leq 2\Delta^{-1}(K_0+1)(|t|+2L+1).$$

It follows that $\mathcal{J}$ is contracting on $\mathcal{C}_\gamma^-(1)$ provided that

$$\varepsilon << \min\{\alpha-\gamma,\beta-\gamma\};\qquad \mu << \gamma\Delta L^{-1}.$$

Thus, we have proved Proposition 3.1(a).

We now prove the part (b). Assume that $w(t)=(x(t),y(t))\in\mathcal{C}_\gamma^-$ is a solution of (3.6) with $|y^0|<\varepsilon$. We have

$$\begin{aligned}|y(t)| &< \varepsilon+K\int_t^0 e^{\beta(t-\tau)}\big(|w(\tau)|^2\sigma((2\varepsilon)^{-1}|w(\tau)|)+\mu|w(\tau)|^2|\mathcal{G}(\theta_\tau\omega_q)|\big)d\tau\\
&<\varepsilon+K\left\{4\varepsilon\int_t^0 e^{\beta(t-\tau)}d\tau+\mu\|w\|\int_t^0|\mathcal{G}(\theta_\tau\omega_q)|e^{\gamma\tau}d\tau\right\}\max_{\tau\in\mathbb{R}^-}|w(\tau)|\end{aligned}$$

Similarly,

$$\begin{aligned}|x(t)| &< K\int_{-\infty}^t e^{-\alpha(t-\tau)}\big(|w(\tau)|^2\sigma((2\varepsilon)^{-1}|w(\tau)|)+\mu|w(\tau)|^2|\mathcal{G}(\theta_\tau\omega_q)|\big)d\tau\\
&< K\left\{4\varepsilon\int_{-\infty}^t e^{-\alpha(t-\tau)}d\tau+\mu\|w\|\int_{-\infty}^t|\mathcal{G}(\theta_\tau\omega_q)|e^{\gamma\tau}d\tau\right\}\max_{\tau\in\mathbb{R}^-}|w(\tau)|\end{aligned}$$

These two together gives

$$\max_{t\in\mathbb{R}^-}|w(t)|<\varepsilon+K\big\{4\varepsilon\big(\alpha^{-1}+\beta^{-1}\big)+4\mu(K_0+1)\gamma^{-1}\Delta^{-1}(\gamma^{-1}+2L+1)\big\}\max_{\tau\in\mathbb{R}^-}|w(\tau)|$$

from which (b) follows directly provided that $\mu << \varepsilon << 1$. □

3.3. Local unstable manifold on $\tilde{\Sigma}^-$.

Recall that Σ^- is defined by $s=-L^-$ in the extended phase space (s,z,p) and is a two dimensional strip represented by $\{(z,q)\in(-\mu,\mu)\times\mathbb{R}\}$. It follows from Proposition 3.1 that, on $\tilde{\Sigma}^-\subset\Sigma^-$, we have a unique unstable curve, which we denote as

$$w^u=\{(w^u(q),q):\quad q\in\cup_n[q_n-2L,q_n+2L]\},$$

so that the solution of (3.4) initiated at $(z^0,q)\in\tilde{\Sigma}^-$ is in $\mathcal{C}_\gamma^-$ if and only if $z^0=w^u(q)$. In this subsection we prove

Proposition 3.2. *Let $w^u(q):\cup_n[q_n-2L,q_n+2L]\to[-\mu,\mu]$ be the unstable curve as in the above. Then,*
(a) $|w^u(q)|\leq K\mu\varepsilon\ln\varepsilon^{-1}$;
(b) $|w^u(q+\Delta q)-w^u(q)|\leq\mu|\Delta q|$.

Proof: We first recall the following three properties that will be used repeatedly in the proofs.

(P1) For $q \in [q_n - 2L, q_n + 2L]$,

$$|\mathcal{G}(\theta_t \omega_q)| \le K(|t| + 2L)$$

because of $q_n \in \Lambda_{K_0}$ and (3.1).

(P2) The randomly driven forcing is $\mathcal{G}(\theta_t \omega_q)$ multiplied by a high order term whose magnitude on B_ε is bounded by

$$K\varepsilon|w(t)| \le K\varepsilon\|w(\cdot)\|e^{-\gamma|t|}.$$

The exponential factor $e^{-\gamma|t|}$ is then used to balance the linear growth of $\mathcal{G}(\theta_t\omega_q)$ in (P1).

(P3) We have $L \sim \ln \varepsilon^{-1} << \varepsilon^{-1}$.

Proof of (a): Assume that for some $z^0 \in (0, \frac{1}{4}\varepsilon)$, we obtain a unique unstable solution $w(t) = (x(t), y(t))$ of (3.6) satisfying $w(0) = (x^0, y^0)$, where (x^0, y^0) is obtained through

$$x^0 = a(-L^-) + v(-L^-)z^0; \qquad y^0 = b(-L^-) - u(-L^-)z^0. \tag{3.8}$$

From (3.6) we also have

$$x^0 = \int_{-\infty}^0 e^{\alpha\tau}\big(\tilde{f}(w(\tau)) + \mu\tilde{P}(w(\tau))\mathcal{G}(\theta_\tau\omega_q)\big)d\tau. \tag{3.9}$$

Combining these two for x^0 in the above we have

$$a(-L^-) + v(-L^-)z^0 = \int_{-\infty}^0 e^{\alpha\tau}\big(\tilde{f}(w(\tau)) + \mu\tilde{P}(w(\tau))\mathcal{G}(\theta_\tau\omega_q)\big)d\tau. \tag{3.10}$$

For Proposition 3.2(a) on $\tilde{\Sigma}^-$, it suffices to prove that

$$|z^0| < K\mu\varepsilon\ln\varepsilon^{-1}. \tag{3.11}$$

To prove (3.11) we start with the equation for $\tilde{w}(t) := (a(t - L^-), b(t - L^-))$. Denote $\tilde{a}(t) = a(t - L^-)$, $\tilde{b}(t) = b(t - L^-)$. Then, $\tilde{w}(t)$ satisfies the equation

$$\begin{aligned} \tilde{b}(t) &= e^{\beta t}b(-L^-) + \int_0^t e^{\beta(t-\tau)}\tilde{g}(\tilde{w}(\tau))d\tau, \\ \tilde{a}(t) &= \int_{-\infty}^t e^{-\alpha(t-\tau)}\tilde{f}(\tilde{w}(\tau))d\tau, \end{aligned} \tag{3.12}$$

which together with (3.6) give

$$\begin{aligned} y(t) - \tilde{b}(t) &= -z^0u(-L^-)e^{\beta t} + \int_0^t e^{\beta(t-\tau)}\Big(\tilde{g}(w(\tau)) - \tilde{g}(\tilde{w}(\tau))\Big)d\tau \\ &\quad + \mu\int_0^t e^{\beta(t-\tau)}\tilde{Q}(w(\tau))\mathcal{G}(\theta_\tau\omega_q)d\tau \\ x(t) - \tilde{a}(t) &= \int_{-\infty}^t e^{-\alpha(t-\tau)}\Big(\tilde{f}(w(\tau)) - \tilde{f}(\tilde{w}(\tau))\Big)d\tau \\ &\quad + \mu\int_{-\infty}^t e^{-\alpha(t-\tau)}\tilde{P}(w(\tau))\mathcal{G}(\theta_\tau\omega_q)d\tau. \end{aligned} \tag{3.13}$$

For $t \in \mathbb{R}^-$, this leads to

$$\begin{aligned} |w(t) - \tilde{w}(t)| \le |z^0|K &+ K\varepsilon\max_{\tau\in\mathbb{R}^-}|w(t) - \tilde{w}(t)|\big(\int_t^0 e^{\beta(t-\tau)}d\tau + \int_{-\infty}^t e^{-\alpha(t-\tau)}d\tau\big) \\ &+ \mu K\varepsilon\|w\|\int_{-\infty}^0 |\mathcal{G}(\Theta_\tau\omega_q)|e^{\gamma\tau}d\tau. \end{aligned}$$

Recall that $\|w\| < 1$. We obtain, for $q \in [q_n - 2L, q_n + 2L]$ that

$$\max_{t\in\mathbb{R}^-} |w(t) - \tilde{w}(t)| \leq K|z^0| + K\varepsilon L\mu. \tag{3.14}$$

We now estimate z^0 through (3.10). We have

$$\begin{aligned}|z^0| &\leq \frac{1}{|v(-L^-)|}\left|\int_{-\infty}^0 e^{\alpha\tau}(\tilde{f}(w(\tau)) - \tilde{f}(\tilde{w}(\tau)))d\tau + \mu\int_{-\infty}^0 e^{\alpha\tau}|\tilde{P}(w(\tau))|\mathcal{G}(\theta_\tau\omega_q)|d\tau\right| \\ &\leq K\varepsilon\max_{t\in\mathbb{R}^-}|w(t) - \tilde{w}(t)| + K\varepsilon^2 L\mu.\end{aligned}$$

By using (3.14), we finally conclude that

$$|z^0| \leq K\mu\varepsilon^2 \ln\varepsilon^{-1}.$$

Proof of (b): Proposition 3.2(b) claims that $\mu^{-1}w^u(q)$, the rescaled unstable curve, is a Lipschitz function on $\cup_n[q_n - 2L, q_n + 2L]$. Let $z^0(q)$ be the same as in the above for $q \in [q_n - 2L, q_n + 2L]$. We need to prove

$$|z^0(q + \Delta q) - z^0(q)| \leq \mu|\Delta q|. \tag{3.15}$$

Let $w_q(t) = (x_q(t), y_q(t))$ be the unstable solution for $q \in [q_n - 2L, q_n + 2L]$ with $(q, z^0(q)) \in \Sigma^-$. We claim

(A) $\max_{t\in\mathbb{R}^-} |w_q(t + \Delta q) - w_q(t)| \leq \varepsilon|\Delta q|$;

(B) $\max_{t\in\mathbb{R}^-} |w_{q+\Delta q}(t) - w_q(t)| \leq K\varepsilon^2|(z^0(q) - z^0(q + \Delta q)| + K\mu\varepsilon\ln\varepsilon^{-1}|\Delta q|$.

Observe that (3.15) follows directly from (B).

We first show claim (A). Without loss of generality, we may assume that $\Delta q \leq 0$. Using (3.13), we have

$$\begin{aligned}&|w_q(t + \Delta q) - w_q(t)| \\ &\quad\leq K\mu\varepsilon^2|\Delta q| + \int_{t+\Delta q}^t |e^{\beta(t-\tau)}\tilde{g}(w_q(\tau)) + e^{-\alpha(t-\tau)}\tilde{f}(w_q(\tau))|d\tau \\ &\qquad + \mu\int_{t+\Delta q}^t |e^{\beta(t-\tau)}\tilde{Q}(w_q(\tau)) + e^{-\alpha(t-\tau)}\tilde{P}(w_q(\tau))||\mathcal{G}(\theta_\tau\omega_q)|d\tau \\ &\qquad + \int_{t+\Delta q}^0 |e^{\beta\Delta q} - I||e^{\beta(t-\tau)}\tilde{g}(w_q(\tau))|\,d\tau + \int_{-\infty}^{t+\Delta q} |e^{-\alpha\Delta q} - I||e^{-\alpha(t-\tau)}\tilde{f}(w_q(\tau))|\,d\tau \\ &\qquad + \mu\int_0^{t+\Delta q} |e^{\beta\Delta q} - I||e^{\beta(t-\tau)}\tilde{Q}(w_q(\tau))\mathcal{G}(\theta_\tau\omega_q)|\,d\tau \\ &\qquad + \mu\int_{-\infty}^{t+\Delta q} |e^{-\alpha\Delta q} - I||e^{-\alpha(t-\tau)}\tilde{P}(w_q(\tau))\mathcal{G}(\theta_\tau\omega_q)|d\tau \\ &\quad\leq \varepsilon|\Delta q|.\end{aligned}$$

Here the part (a) in Proposition 3.2 is used.

We now prove claim (B). We assume that $q+\Delta q\in[q_n-2L,q_n+2L]$. From (3.13), we have

$$\begin{aligned}y_{q+\Delta q}(t)-y_q(t)&=(z^0(q)-z^0(q+\Delta q))u(-L^-)e^{\beta t}+\int_0^t e^{\beta(t-\tau)}\Big(\tilde g(w_{q+\Delta q}(\tau))-\tilde g(w_q(\tau))\Big)d\tau\\&\quad+\mu\int_0^t e^{\beta(t-\tau)}\left(\tilde Q(w_{q+\Delta q}(\tau))\mathcal G(\Theta_\tau\omega_{q+\Delta q})-\tilde Q(w_q(\tau))\mathcal G(\Theta_\tau\omega_q)\right)d\tau\\x_{q+\Delta q}(t)-x_q(t)&=\int_{-\infty}^t e^{-\alpha(t-\tau)}\Big(\tilde f(w_{q+\Delta q}(\tau))-\tilde f(w_q(\tau))\Big)d\tau\\&\quad+\mu\int_{-\infty}^t e^{-\alpha(t-\tau)}\left(\tilde P(w_{q+\Delta q}(\tau))\mathcal G(\Theta_\tau\omega_{q+\Delta q})-\tilde P(w_q(\tau))\mathcal G(\Theta_\tau\omega_q)\right)d\tau,\end{aligned}$$

from which it follows that

$$\begin{aligned}|w_{q+\Delta q}(t)-w_q(t)|&\le K|z^0(q)-z^0(q+\Delta q)|\varepsilon^2+K\varepsilon\max_{\tau\in\mathbb R^-}|w_{q+\Delta q}(t)-w_q(t)|\\&\quad+\mu\left(|(I)|+|(II)|\right)\end{aligned}\tag{3.16}$$

where

$$\begin{aligned}(I)&=\int_t^0 e^{\beta(t-\tau)}[\tilde Q(w_{q+\Delta q}(\tau))\mathcal G(\theta_\tau\omega_{q+\Delta q})-\tilde Q(w_q(\tau))\mathcal G(\theta_\tau\omega_q)]d\tau,\\(II)&=\int_{-\infty}^t e^{-\alpha(t-\tau)}[\tilde P(w_{q+\Delta q}(\tau))\mathcal G(\theta_\tau\omega_{q+\Delta q})-\tilde P(w_q(\tau))\mathcal G(\theta_\tau\omega_q)]d\tau.\end{aligned}$$

We re-write (I) as

$$\begin{aligned}(I)&=\int_{t+\Delta q}^{\Delta q} e^{\beta(t-\tau+\Delta q)}\tilde Q(w_{q+\Delta q}(\tau-\Delta q))\mathcal G(\theta_\tau\omega_q)d\tau-\int_t^0 e^{\beta(t-\tau)}\tilde Q(w_q(\tau))\mathcal G(\theta_\tau\omega_q)d\tau\\&=\left(\int_{t+\Delta q}^t+\int_0^{\Delta q}\right)e^{\beta(t-\tau+\Delta q)}\tilde Q(w_{q+\Delta q}(\tau-\Delta q))\mathcal G(\theta_\tau\omega_q)d\tau\\&+\int_t^0\left(e^{\beta(t-\tau+\Delta q)}\tilde Q(w_{q+\Delta q}(\tau-\Delta q))-e^{\beta(t-\tau)}\tilde Q(w_q(\tau))\right)\mathcal G(\theta_\tau\omega_q)d\tau.\end{aligned}$$

Here we have changed $\tau\to\tau+\Delta q$ for the first integral to avoid the potential trouble of estimating $|\mathcal G(\theta_\tau\omega_{q+\Delta q})-\mathcal G(\theta_\tau\omega_q)|$. We have

$$\begin{aligned}|(I)|&\le K\varepsilon L|\Delta q|+K\varepsilon\int_t^0|w_{q+\Delta q}(\tau-\Delta q)-w_q(\tau)|(|\tau|+2L)e^{-\gamma|\tau|}d\tau\\&\le K\varepsilon L|\Delta q|+K\varepsilon L\max_{t\in\mathbb R^-}|w_{q+\Delta q}(t)-w_q(t)|+K\varepsilon\int_t^0|w_q(\tau-\Delta q)-w_q(\tau)|(|\tau|+2L)e^{-\gamma|\tau|}d\tau\\&\le K\varepsilon L|\Delta q|+K\varepsilon L\max_{t\in\mathbb R^-}|w_{q+\Delta q}(t)-w_q(t)|.\end{aligned}$$

Here we use (A) for the last inequality. Estimates for $|(II)|$ is similar. (B) follows then directly from combining (3.16) and the estimates for $|(I)|$ and $|(II)|$. This completes the proof of the proposition □

3.4. **Local stable manifold on $\tilde\Sigma^+$.** Recall that Σ^+ is defined by $s=L^+$ in the extended phase space (s,z,p) and it is represented by $\{(z,q)\in(-C_1(\varepsilon)\mu,C_1(\varepsilon)\mu)\times\mathbb R\}$. In this section we let

$$\tilde\Sigma^+=\{(z,p)\in\Sigma^+:\quad p\in\cup_n[q_n-L-1,q_n+3L+1]\}.$$

Notice that the non-symmetric definition on the intervals for p is designed for $\tilde{\Sigma}^+$ to cover the images of $\tilde{\Sigma}^-$. Again, let $2\gamma < \min\{\alpha, \beta\}$. We define local stable solutions by first letting

$$\mathcal{C}_\gamma^+ = \left\{\phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^2 \text{ is continuous and } \sup_{t \geq 0} |e^{\gamma t}\phi(t)| < \infty\right\}$$

with norm $\|\phi\| := \|\phi\|_\gamma^+ = \sup_{t\geq 0} |e^{\gamma t}\phi(t)|$. We have

Proposition 3.3. *There exists a positive $\varepsilon_0 << 1$ independent of $\tilde{\omega}$ such that for any given $0 < \varepsilon < \varepsilon_0$, there exists $\mu_0(\varepsilon) > 0$ independent of $\tilde{\omega}$, so that for all $0 < \mu < \mu_0(\varepsilon)$, and all $q \in \cup_n[q_n - L - 1, q_n + 3L + 1]$, the following holds: that for any give $x^0 \in (-\varepsilon, \varepsilon)$, there is a unique $y^0 \in (-\varepsilon, \varepsilon)$, such that*
(a) the solution $w(t, w^0)$ satisfying $w(0, w^0) = w^0 = (x^0, y^0)$ is in $\mathcal{C}_\gamma^+$, and
(b) $|w(t, w^0)| < 2\varepsilon$ for all $t \in \mathbb{R}^+$.

From this proposition we have on $\tilde{\Sigma}^+$ a unique unstable curve, which we denote as

$$w^s = \{(w^s(q), q) : \quad q \in \cup_n[q_n - L - 1, q_n + 3L + 1]\},$$

so that the solution of (3.4) initiated at $(z^0, q) \in \tilde{\Sigma}^+$ is in $\mathcal{C}_\gamma^+$ if and only if $z^0 = w^s(q)$. We have, in addition,

Proposition 3.4. *Let $w^s(q)$ on $\cup_n[q_n - L - 1, q_n + 3L + 1]$ be the stable curve as in the above. Then*
(a) $|w^s(q)| \leq K\mu\varepsilon \ln \varepsilon^{-1}$;
(b) $|w^s(q + \Delta q) - w^s(q)| \leq \mu|\Delta q|$.

4. Intersection of the stable and unstable manifold

In this section we use the signs of a random Melnikov function as a guidance to find the intersections of the stable and the unstable solutions obtained in the previous section in the extended phase space. Recall that $\ell(t) = (a(t), b(t))$ is the homoclinic solution ℓ of the unperturbed equation (2.1) and

$$(u(t), v(t)) = \left|\frac{d}{dt}\ell(t)\right|^{-1} \frac{d}{dt}\ell(t)$$

is the unit tangent vector of ℓ at $\ell(t)$. Let

$$\begin{aligned} E(t) = {} & v^2(t)(-\alpha + \partial_x f(a(t), b(t))) + u^2(t)(\beta + \partial_y g(a(t), b(t))) \\ & - u(t)v(t)(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))). \end{aligned} \tag{4.1}$$

The quantity $E(t)$ measures the rate of expansion of the solutions of equation (2.1) in the direction normal to ℓ at $\ell(t)$. We introduce a random characteristic function which we call the **Random Melnikov function** $\mathcal{W}(\omega)$ for equation (2.5) defined as

$$\mathcal{W}(\omega) = \int_{-\infty}^{\infty} F(s)\mathcal{G}(\theta_s\omega)e^{-\int_0^s E(\tau)d\tau} ds, \tag{4.2}$$

where

$$F(s) = v(s)P(a(s), b(s)) - u(s)Q(a(s), b(s)). \tag{4.3}$$

Observe that $E(t) \to \beta$ as $t \to +\infty$, $E(t) \to -\alpha$ as $t \to -\infty$, and

$$|\mathcal{G}(\theta_t\omega)| < \Delta^{-1}(|\omega(t + \Delta)| + |\omega(t)|) \leq 2\Delta^{-1}C_\omega(|t| + \Delta + 1).$$

$\mathcal{W}(\omega)$ is well-defined for almost all $\omega \in \Omega$. For a fixed $\tilde{\omega} \in \Omega$, denote

$$\mathcal{W}(t) = \mathcal{W}(\theta_t\tilde{\omega}).$$

Note that $\mathcal{W}(t) : \mathbb{R} \to \mathbb{R}$ is a continuous function. This follows from

$$\begin{aligned}
|\mathcal{W}(t+\delta) - \mathcal{W}(t)| &= \left| \int_{-\infty}^{\infty} F(s-\delta) e^{-\int_0^{s-\delta} E(\tau)d\tau} \mathcal{G}(\theta_{s+t}\tilde{\omega}) ds - \int_{-\infty}^{\infty} F(s) e^{-\int_0^{s} E(\tau)d\tau} \mathcal{G}(\theta_{s+t}\tilde{\omega}) ds \right| \\
&\le \int_{-\infty}^{\infty} \left| F(s-\delta) e^{-\int_s^{s-\delta} E(\tau)d\tau} - F(s) \right| e^{-\int_0^{s} E(\tau)d\tau} |\mathcal{G}(\theta_{s+t}\tilde{\omega})| ds \\
&\le K C_{\tilde{\omega}} \delta \int_{-\infty}^{\infty} |t+s+1| e^{-\int_0^{s} E(\tau)d\tau} ds \\
&\le K C_{\tilde{\omega}} |t| \delta.
\end{aligned}$$

Here for the second inequality we use the estimate

$$\left| F(s-\delta) e^{-\int_s^{s-\delta} E(\tau)d\tau} - F(s) \right| < K\delta,$$

which follows directly from

$$\left| \frac{d}{d\delta} F(s-\delta) e^{-\int_s^{s-\delta} E(\tau)d\tau} \right| < K.$$

4.1. **Qualified intervals of initial times.** In this subsection we prove the existence of a bi-infinite sequence of intervals $\{I_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}$, monotonically lined up from $-\infty$ to $+\infty$, such that the values of $\mathcal{W}(t) = \mathcal{W}(\theta_t\tilde{\omega})$ on I_n range from $> K^{-1}$ to $< -K^{-1}$ for some $K > 0$ for all n. In order to use the previous results on the stable and unstable curves obtained in Section 3, we will also prove that on every I_n there is a point t_n so that $C_{\theta_{t_n}\tilde{\omega}} < K_0$, and the lengths of all I_n are uniformly bounded by a constant that is independent of ε where ε is the same as in Section 3.

Assumptions for this subsection: In this subsection we assume that there exists $s \in \mathbb{R}$ such that $F(s) \neq 0$ where $F(s)$ is as in (4.3). We also assume that $\Delta > 0$ is sufficiently small where Δ is as in (2.4). Note that these are the assumptions of the Main Theorem.

Proposition 4.1. *There exists a bi-infinite sequence of intervals $\{I_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}$, lined up monotonically in $\mathbb{R}$ from $-\infty$ to $+\infty$, such that there exist K_0, K_1, K_2 sufficiently large and $q_n^+, q_n^- \in I_n$ for all n, so that*

(a) $\omega_{q_n^+} \in \Lambda_{K_0}$;
(b) $\mathcal{W}(\theta_{q_n^+}\tilde{\omega}) > K_1^{-1}$, $\quad \mathcal{W}(\theta_{q_n^-}\tilde{\omega}) < -K_1^{-1}$; and
(c) $|I_n| < K_2$.

To prove this proposition, we need to the following lemmas.

Lemma 4.1. *There exist positive constants K_1 and K_3 and subsets Ω^+ and Ω^- in Ω, both with Wiener measure $> K_3^{-1}$, such that $\mathcal{W}(\omega) > 2K_1^{-1}$ for all $\omega \in \Omega^+$ and $\mathcal{W}(\omega) < -2K_1^{-1}$ for all $\omega \in \Omega^-$.*

Proof: Let $\mathbb{E}(\mathcal{W})$ be the expectation and $\mathbb{V}(\mathcal{W})$ be the variance of the random variable $\mathcal{W}(\omega)$, respectively. To prove Lemma 4.1, it suffices to prove that

$$\mathbb{E}(\mathcal{W}) = 0, \quad \mathbb{V}(\mathcal{W}) \neq 0.$$

First, we have

$$\begin{aligned}
\mathbb{E}(\mathcal{W}) &= \int_\Omega \int_{-\infty}^{+\infty} F(s)\mathcal{G}(\theta_s\omega) ds d\mathbb{P} = \int_{-\infty}^{+\infty} F(s) \left\{ \int_\Omega \mathcal{G}(\theta_s\omega) d\mathbb{P} \right\} ds \\
&= 0
\end{aligned}$$

since $\mathcal{G}(\theta_s\omega)$ is given by the Brownian motion.

Next, we compute $\mathbb{V}(\mathcal{W})$. Let $(s,t)\in\mathbb{R}^2$ be fixed and denote

$$X_1=\omega(s+\Delta)-\omega(s),\qquad X_2=\omega(t+\Delta)-\omega(t),$$

We have

$$\begin{aligned}\mathbb{V}(\mathcal{W}) &= \int_\Omega\int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\mathcal{G}(\theta_t\omega)\mathcal{G}(\theta_s\omega)dsdtd\mathbb{P}\\ &= \int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_\Omega\mathcal{G}(\theta_t\omega)\mathcal{G}(\theta_s\omega)d\mathbb{P}\right\}dsdt\\ &= \frac{1}{\Delta^2}\int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_\Omega X_1X_2d\mathbb{P}\right\}dsdt\end{aligned}$$

We write

$$\omega(s)\omega(t)=-\frac{1}{2}(\omega(s)-\omega(t))^2+\frac{1}{2}\omega^2(s)+\frac{1}{2}\omega^2(t),$$

and do the same for $\omega(s+\Delta)\omega(t+\Delta)$, $\omega(s)\omega(t+\Delta)$ and $\omega(t)\omega(s+\Delta)$. We have

$$\begin{aligned}\mathbb{V}(\mathcal{W}) &= -\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_\Omega(\omega(s+\Delta)-\omega(t+\Delta))^2)d\mathbb{P}\right\}dsdt\\ &\quad -\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_\Omega(\omega(s)-\omega(t))^2)d\mathbb{P}\right\}dsdt\\ &\quad +\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_\Omega(\omega(t+\Delta)-\omega(s))^2)d\mathbb{P}\right\}dsdt\\ &\quad +\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_\Omega(\omega(t)-\omega(s+\Delta))^2)d\mathbb{P}\right\}dsdt\\ &= (I)+(II)+(III)+(IV).\end{aligned}$$

By the properties of the Brownian motion, we have

$$\begin{aligned}(I) &= -\frac{1}{2\Delta^2}\int_{(t,s)\in\mathbb{R}^2}|t-s|F(s)F(t)dsdt\\ (II) &= -\frac{1}{2\Delta^2}\int_{(t,s)\in\mathbb{R}^2}|t-s|F(s)F(t)dsdt\\ (III) &= \frac{1}{2\Delta^2}\int_{(t,s)\in\mathbb{R}^2}|t-s+\Delta|F(s)F(t)dsdt\\ (IV) &= \frac{1}{2\Delta^2}\int_{(t,s)\in\mathbb{R}^2}|t-s-\Delta|F(s)F(t)dsdt\end{aligned}$$

Let

$$\mathbb{D}=\{(s,t)\in\mathbb{R}^2: s-\Delta<t<s+\Delta\}.$$

We have

$$\begin{aligned}\mathbb{V}(\mathcal{W}) &= \frac{1}{\Delta^2}\int_{\mathbb{D}}(\Delta-|t-s|)F(s)F(t)dsdt\\ &= \frac{1}{\Delta^2}\int_{-\infty}^{+\infty}F(s)\left\{\int_{s-\Delta}^{s+\Delta}(\Delta-|t-s|)F(t)dt\right\}ds\\ &= \int_{-\infty}^{+\infty}F^2(s)ds+\mathcal{O}(\Delta).\end{aligned}$$

By the assumption that $F(s) \neq 0$ for some s, we have

$$\int_{-\infty}^{+\infty} F^2(s)ds > 0.$$

We also let $\Delta > 0$ be sufficiently small so that the $\mathcal{O}(\Delta)$ terms in the above $< \frac{1}{2}\int_{-\infty}^{+\infty} F^2(s)ds$. We conclude that

$$\mathbb{V}(\mathcal{W}) > \frac{1}{2}\int_{-\infty}^{+\infty} F^2(s)ds > 0.$$

This completes the proof of the lemma. □

This lemma is the only place where a restriction is placed on Δ. For the rest of this paper we fix a $\Delta > 0$ sufficiently small, and treat it as one specific value generically represented by the letter K.

Combining Lemma 4.1 with the fact that $C_\omega : \Omega \to \mathbb{R}^+$ is a measurable function, we conclude that

$$\Lambda(K_0, -) := \Lambda_{K_0} \cap \Omega^-$$

is a set of positive Wiener measure provided that K_0 is sufficiently large.

Let $T, \delta > 0$ be a given pair of positive numbers, and $\hat{\omega} \in \Omega$. Denote

$$B_{T,\delta}(\hat{\omega}) = \{\omega \in \Omega : \ |\omega(s) - \hat{\omega}(s)| < \delta, \quad |s| < T\}.$$

Note that the Wiener space Ω is a Polish space and $B_{T,\delta}(\hat{\omega})$ is an open neighborhood of $\hat{\omega}$.

We also need the following assertion on the Wiener measure.

Lemma 4.2. *Let $\Lambda \subset \Omega$ be a set of positive Wiener measure. Then there exists an $\hat{\omega} \in \Lambda$, such that, for all $T, \delta > 0$, we have*

$$\mathbb{P}(B_{T,\delta}(\hat{\omega}) \cap \Lambda) > 0.$$

Proof: Suppose that the statement is not true. Then, for each $\omega \in \Lambda$, there exists a $B_{T(\omega),\delta(\omega)}(\omega)$ such that

$$\mathbb{P}(B_{T(\omega),\delta(\omega)}(\omega) \cap \Lambda) = 0.$$

Since Ω is a Polish space which is separable, then Λ has a countable covering

$$\{B_{T(\omega_i),\delta(\omega_i)}(\omega_i) \mid \omega_i \in \Lambda, \ i = 1, 2, \cdots\},$$

which yields that

$$\mathbb{P}(\Lambda) \le \sum_{i=1}^{\infty} \mathbb{P}\big(B_{T(\omega_i),\delta(\omega_i)}(\omega_i) \cap \Lambda\big) = 0.$$

□

Now, let $\Lambda = \Lambda(K_0, -)$ in Lemma 4.2. Then, it follows that there exists an $\omega^- \in \Lambda(K_0, -)$ such that $\mathbb{P}(B_{T,\delta,K_0,-}(\omega^-)) > 0$ for all $T, \delta > 0$ where

$$B_{T,\delta,K_0,-}(\omega^-) := B_{T,\delta}(\omega^-) \cap \Lambda(K_0, -).$$

We have

Lemma 4.3. *Let $\omega^- \in \Lambda(K_0, -)$ be as in the above. Then there exists $T_0 > 0$ sufficiently large, so that for any given $T > T_0$, there exists $\delta_0(T) > 0$ such that for any given $\delta < \delta_0(T)$ and for all $\omega \in B_{T,\delta,K_0,-}(\omega^-)$, we have*

$$\mathcal{W}(\omega) < -\frac{3}{2}K_1^{-1}.$$

Proof: Let $\gamma = \frac{1}{2}\min\{\alpha, \beta\}$. We have by definition,

$$\begin{aligned}|\mathcal{W}(\omega) - \mathcal{W}(\omega^-)| &\le \hat{K}\delta T + \hat{K}\int_{\mathbb{R}\setminus[-T,T]} (|\mathcal{G}(\theta_s\omega)| + |\mathcal{G}(\theta_s\omega^-)|)e^{-\gamma|s|}ds \\ &< \hat{K}\delta T + 4\hat{K}\int_T^{+\infty} K_0(s+1)e^{-\gamma s}ds \\ &< \hat{K}\delta T + 4\hat{K}K_0\gamma^{-1}(\gamma^{-1}+T+1)e^{-\gamma T} \\ &< \frac{1}{2}K_1^{-1}\end{aligned}$$

where for the last inequality, we first make T_0 sufficiently large so that $(\gamma^{-1} + T_0 + 1)e^{-\gamma T_0} < (16\gamma\hat{K}K_0)^{-1}K_1^{-1}$. We then let $\delta_0(T) = (4\hat{K}T)^{-1}K_1^{-1}$. □

Since $B_{T_0,\delta_0(T_0),K_0,-}(\omega^-)$ is a set of positive Wiener measure, the subset of Ω that is θ-typical with respect to $B_{T_0,\delta_0(T_0),K_0,-}(\omega^-)$ is a set of full Wiener measure, which we denote as $\hat{\Omega}$. We now let

$$\Lambda(K_0,+) = \Lambda_{K_0} \cap \Omega^+ \cap \hat{\Omega}$$

and use Lemma 4.2 to obtain an $\omega^+ \in \Lambda(K_0,+)$ such that

$$\mathbb{P}(B_{T,\delta,K_0,+}(\omega^+)) > 0$$

for all $T, \delta > 0$ where

$$B_{T,\delta,K_0,+}(\omega^+) := B_{T,\delta}(\omega^+) \cap \Lambda(K_0,+).$$

Similar to Lemma 4.3, there exists a $T_0 > 0$ sufficiently large, and for every $T > T_0$, there exists a $\delta_0(T) > 0$ so that for every $\delta < \delta_0(T)$, and for all $\omega \in B_{T,\delta,K_0,+}(\omega^+)$, we have

$$\mathcal{W}(\omega) > \frac{3}{2}K_1^{-1}. \tag{4.4}$$

The proof for (4.4) is identical to the proof of Lemma 4.3 with the same T_0 and $\delta_0(T)$.

Let $\hat{q}$ be such that $\theta_{\hat{q}}\omega^+ \in B_{T_0,\delta_0(T_0),K_0,-}(\omega^-)$. This $\hat{q}$ exists because $\omega^+ \in \hat{\Omega}$ is θ-typical with respect to $B_{T_0,\delta_0(T_0),K_0,-}(\omega^-)$. Without loss of generality, we can also assume that $\hat{q}$ is sufficiently large such that

$$\hat{q}^2 e^{-\gamma\hat{q}} < \frac{1}{8}. \tag{4.5}$$

We have

Lemma 4.4. *Assume that $q^+ \in \mathbb{R}$ is such that $\omega_{q^+} := \theta_{q^+}\tilde{\omega} \in B_{\hat{T},\hat{\delta},K_0,+}(\omega^+)$ where*

$$\hat{T} = 3(T_0 + \hat{q}), \quad \hat{\delta} = \frac{\delta_0(T_0)}{4(1+\hat{q}T_0^{-1})}.$$

Then we have

$$\mathcal{W}(\omega_{q^-}) < -K_1^{-1}$$

where $q^- = q^+ + \hat{q}$, $\omega_{q^-} = \theta_{q^-}\tilde{\omega}$.

Proof: First we observe that ω_{q^-} is such that $C_{\omega_{q^-}} < 2(K_0+1)(\hat{q}+1)$. This follows from (3.1) and the fact that $\omega_{q^+} \in \Lambda_{K_0}$. We also have

$$\mathcal{W}(\theta_{\hat{q}}\omega^+) < -\frac{3}{2}K_1^{-1} \tag{4.6}$$

since $\theta_{\hat{q}}\omega^+ \in B_{T_0,\delta_0(T_0),K_0,-}(\omega^-)$ and Lemma 4.3.

From $\omega_{q^+} \in B_{\hat{T},\hat{\delta},K_0,+}(\omega^+)$, it follows that

$$\omega_{q^-} = \theta_{\hat{q}}(\theta_{q^+}\tilde{\omega}) \in B_{\frac{2}{3}\hat{T},2\hat{\delta}}(\theta_{\hat{q}}\omega^+). \tag{4.7}$$

In fact, for $|t| < \frac{2}{3}\hat{T}$, we have

$$\begin{aligned} |\omega_{q^-}(t) - \theta_{\hat{q}}\omega^+(t)| &= |\theta_{\hat{q}}\omega_{q^+}(t) - \theta_{\hat{q}}\omega^+(t)| \\ &\le |\omega_{q^+}(t+\hat{q}) - \omega^+(t+\hat{q})| + |\omega_{q^+}(\hat{q}) - \omega^+(\hat{q})| \\ &\le 2\hat{\delta}. \end{aligned}$$

The last inequality holds because $t + \hat{q} < \frac{2}{3}\hat{T} + \hat{q} < \hat{T}$ and $\omega_{q^+} \in B_{\hat{T},\hat{\delta},K_0,+}$.

We now have

$$\begin{aligned} |\mathcal{W}(\theta_{\hat{q}}\omega^+) - \mathcal{W}(\theta_{q^-}\tilde{\omega})| &\le 2\hat{K}\hat{\delta}(T_0+\hat{q}) + \hat{K}\int_{\mathbb{R}\setminus[-(T_0+\hat{q}),T_0+\hat{q}]} (|\mathcal{G}(\theta_s(\theta_{\hat{q}}\omega^+))| \\ &\quad + |\mathcal{G}(\theta_s(\theta_{\hat{q}}\Theta_{q^+}\tilde{\omega}^+))|)e^{-\gamma|s|}ds \\ &< \hat{K}\frac{1}{2(1+\hat{q}T_0^{-1})}\delta_0(T_0)(T_0+\hat{q}) + 4\hat{K}\int_{T_0+\hat{q}}^{+\infty} K_0|\hat{q}|(s+1)e^{-\gamma s}ds \\ &< \hat{K}\delta_0(T_0)T_0 + 4\hat{K}K_0\gamma^{-1}\hat{q}(T_0+\hat{q}+1)e^{-\gamma(T_0+\hat{q})} \\ &< \frac{1}{2}K_1^{-1} \end{aligned}$$

where (4.5) is also used to estimate the second term for the last inequality. Lemma 4.4 is then proved by combining the last inequality with (4.6). Note that in the estimate above, $\hat{K}$ is the same as in the proof of Lemma 4.3. □

Proof of Proposition 4.1: By the fact that $\theta_t : \Omega \to \Omega$ is ergodic, we can let $\tilde{\omega} \in \Omega$ and be θ-typical to $B_{\hat{T},\hat{\delta},K_0,+}(\omega^+)$. Then there exists a bi-infinite sequence $q_n \to \pm\infty$ as $n \to \pm\infty$ so that, regarding q_n as q^+, Lemma 4.4 applies to all q_n. We now let $I_n = [q_n, q_n + \hat{q}]$ and let $K_2 = \hat{q} + 1$. □

4.2. **Melnikov function and the intersections of the stable and unstable manifold.** Let us recall the following from Sections 2 and 3. The equation of study is (2.6) and the variables for the extended phase space are (x, y, p). We also use (s, z) to replace the original phase variables (x, y) in a small neighborhood of the part of the unforced homoclinic solution ℓ out of $B_{\frac{1}{2}\varepsilon^2}$, where (s, z) are defined by letting

$$x = a(s) + v(s)z, \quad y = b(s) - u(s)z. \tag{4.8}$$

In (4.8), $\ell(t) = (a(t), b(t))$ is the unforced homoclinic solution ℓ and $(u(t), v(t))$ are the unit tangent vector of ℓ at $\ell(t) = (a(t), b(t))$. In (s, z, p)-space, we have

$$\Sigma^- = \{(-L^-, z, p), \quad |z| < \mu, \ p \in \mathbb{R}\}.$$

The surface $\Sigma^- = (-\mu, \mu) \times \mathbb{R}$ is a bi-infinite 2D strip. Similarly,

$$\Sigma^+ = \{(L^+, z, p), \quad |z| < C_1(\varepsilon)\mu, \ p \in \mathbb{R}\}$$

where $C_1(\varepsilon)$ is a constant, the value of which we will give in precise terms momentarily. The surface $\Sigma^+ = (-C_1(\varepsilon)\mu,\ C_1(\varepsilon)\mu) \times \mathbb{R}$ is again a bi-infinite 2D strip. In what follows, points on Σ^- and Σ^+ are both denoted by (z, p) where $|z| < \mu$ for Σ^- and $|z| < C_1(\varepsilon)\mu$ for Σ^+. We also denote

$$\mathcal{D}_{C_1} = \{(s, z, p) : \quad s \in [-2L^-, 2L^+], \quad |z| < C_1(\varepsilon)\mu, \quad p \in \mathbb{R}\}.$$

Let $\tilde{\omega}$ be the same as before and q_n^+ be the left end point of the interval I_n of Proposition 4.1. By Proposition 4.1(a), $\omega_{q_n^+} \in \Lambda_{K_0}$. Let $\varepsilon > 0$ be sufficiently small so that

$$L := L^- + L^+ > K_2 \tag{4.9}$$

where K_2 is such that $|I_n| < K_2$ for all n (see Proposition 4.1(c)). We define $\tilde{\Sigma}^-$ by letting

$$\tilde{\Sigma}^- = \cup_{n\in\mathbb{Z}}\tilde{\Sigma}^-(n)$$

where

$$\tilde{\Sigma}^-(n) = \{(z,p) \in \Sigma^-, \quad p \in \cup_n[q_n^+ - 2L, q_n^+ + 2L]\}.$$

By (4.9) we have

$$I_n \subset [q_n^+ - 2L, q_n^+ + 2L].$$

We also let

$$\tilde{\Sigma}^+ = \cup_{n\in\mathbb{Z}}\tilde{\Sigma}^+(n)$$

where

$$\tilde{\Sigma}^+(n) = \{(z,p) \in \Sigma^+, \quad p \in \cup_n[q_n^+ - L - 1, q_n^+ + 3L + 1]\}.$$

From Propositions 3.2 and 3.4, there exists an unstable curve defined on $\tilde{\Sigma}^-$ that can be written as

$$z = w^u(q), \quad q \in \cup_n[q_n^+ - 2L, q_n^+ + 2L].$$

There is also a stable curve defined on $\tilde{\Sigma}^+$ that can be written as

$$z = w^s(q), \quad q \in \cup_n[q_n^+ - L - 1, q_n^+ + 3L + 1].$$

In addition, we have

$$\begin{aligned} &|w^u(q)|, \quad |w^s(q)| < K\varepsilon \ln \varepsilon^{-1}\mu; \\ &|w^u(q + \Delta q) - w^u(q)|, \ \ |w^s(q + \Delta q) - w^s(q)| < \mu|\Delta q|. \end{aligned} \tag{4.10}$$

We first prove

Proposition 4.2. *Let $\mathcal{M} : \Sigma^- \to \Sigma^+$ be the map induced by the solutions of equation (2.6). Then $\mathcal{M}$ is well-defined on $\tilde{\Sigma}^-$, and $\mathcal{M}(\tilde{\Sigma}^-(n)) \subset \tilde{\Sigma}^+(n)$ for all n. In addition, for any given $n \in \mathbb{Z}$, there exists a continuous, non-self intersecting 1D curve $\xi_n : [0,1] \to \tilde{\Sigma}^-(n)$ such that*

(a) $\mathcal{M}(\xi_n)$ is a continuous segment of the stable curve in $\tilde{\Sigma}^+(n)$; and

(b) ξ_n connects the unstable curve $w^u(q)$ and the horizontal curve defined by $z = \mu$.

Proof: We divide the proof of this proposition into two steps.

Step 1: Equations in (s, z, p). We derive the equations for the new variables (s, z) through (4.8). Differentiating (4.8) we obtain

$$\begin{aligned} \frac{dx}{dt} &= (-\alpha a(s) + f(a(s), b(s)) + v'(s)z)\frac{ds}{dt} + v(s)\frac{dz}{dt}, \\ \frac{dy}{dt} &= (\beta b(s) + g(a(s), b(s)) - u'(s)z)\frac{ds}{dt} - u(s)\frac{dz}{dt}, \end{aligned} \tag{4.11}$$

where $u'(s) = \frac{du(s)}{ds}$, $v'(s) = \frac{dv(s)}{ds}$. Let us denote

$$\begin{aligned} F(s,z) &= -\alpha(a(s) + zv(s)) + f(a(s) + zv(s), b(s) - zu(s)), \\ G(s,z) &= \beta(b(s) - zu(s)) + g(a(s) + zv(s), b(s) - zu(s)), \\ P(s,z) &= P(a(s) + zv(s), b(s) - zu(s)), \\ Q(s,z) &= Q(a(s) + zv(s), b(s) - zu(s)). \end{aligned}$$

By using equation (2.6), we obtain from (4.11) the new equations for s, z as

$$\begin{aligned}
\frac{ds}{dt} &= \frac{u(s)F(s,z) + v(s)G(s,z) + \mu(u(s)P(s,z) + v(s)Q(s,z))\mathcal{G}(\omega_p)}{\sqrt{F(s,0)^2 + G(s,0)^2} + z(u(s)v'(s) - v(s)u'(s))},\\
\frac{dz}{dt} &= v(s)F(s,z) - u(s)G(s,z) + \mu(v(s)P(s,z) - u(s)Q(s,z))\mathcal{G}(\omega_p),\\
\frac{dp}{dt} &= 1.
\end{aligned}$$

We re-write these equations as

$$\begin{aligned}
\frac{ds}{dt} &= 1 + zw_1(s,z) + \mu W(s,z)\mathcal{G}(\omega_p),\\
\frac{dz}{dt} &= E(s)z + z^2 w_2(s,z) + \mu(v(s)P(s,z) - u(s)Q(s,z))\mathcal{G}(\omega_p),\\
\frac{dp}{dt} &= 1
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
E(s) = v^2(s)(-\alpha + \partial_x f(a(s),b(s))) &+ u^2(s)(\beta + \partial_y g(a(s),b(s))),\\
&- u(s)v(s)(\partial_y f(a(s),b(s)) + \partial_x g(a(s),b(s)))
\end{aligned}$$

is the same function as in (4.1);

$$W(s,z) = \frac{u(s)P(s,z) + v(s)Q(s,z)}{\sqrt{F(s,0)^2 + G(s,0)^2 + z(u(s)v'(s) - v(s)u'(s))}};$$

and the C^r-norms of $w_1(s,z)$, $w_2(s,z)$ and $W(s,z)$ are bounded from above by a constant K.

Finally we re-scale z by letting

$$Z = \mu^{-1} z. \tag{4.13}$$

We arrive at the following equations

$$\begin{aligned}
\frac{ds}{dt} &= 1 + \mu Z w_1(s,\mu Z) + \mu W(s,\mu Z)\mathcal{G}(\omega_p),\\
\frac{dZ}{dt} &= E(s)Z + \mu Z^2 w_2(s,\mu Z) + (v(s)P(s,\mu Z) - u(s)Q(s,\mu Z))\mathcal{G}(\omega_p),\\
\frac{dp}{dt} &= 1.
\end{aligned} \tag{4.14}$$

Step 2: Intersection to local stable manifold Assume that $u^0 = (-L^-, \mu Z_0, q) \in \tilde{\Sigma}^-(n)$. We denote the solution of equation (4.14) initiated at u^0 as $u(t) = (s(t), Z(t), p(t))$. First we have

$$p(t) = q + t,$$

and from the first equation in (4.14),

$$s(t) = t - L^- + \mu \int_0^t Zw_1(s(\tau), \mu Z(\tau))d\tau + \mu \int_0^t W(s(\tau), \mu Z(\tau))\mathcal{G}(\theta_{\tau+q}\tilde{\omega})d\tau.$$

Since $|q| < q_n^+ + 2L$, we have for $|t| < 2L$,

$$s(t) = t - L^- + D \tag{4.15}$$

where $D = \mathcal{O}(\mu)$ is a term of magnitude $< \mu C(\varepsilon)$ where $C(\varepsilon)$ is dependent of ε. To obtain (4.15), we observe that

$$\left| \int_0^t W(s(\tau), \mu Z(\tau))\mathcal{G}(\omega_{\tau+q})d\tau \right| < K \int_0^t (C_{\omega_q} + 1)(|\tau| + 1)d\tau < K(K_0 + 1)L^3 := C(\varepsilon)$$

where for the last inequality we use

$$C_{\omega_q} < 2(C_{\omega_{q_n^+}} + 1)(|2L| + 1), \quad \text{and} \quad \mathrm{C}_{\omega_{\mathrm{q_n^+}}} < \mathrm{K}_0.$$

We also obtain from Equation (4.14) that for $|t| < 2L$,

$$\begin{aligned} Z(t) &= e^{\int_0^t E(\tau - L^- + D)d\tau} \left(Z_0 + \int_0^t (\mathbb{F}(\tau - L^-) + \mathcal{O}(\mu))\mathcal{G}(\omega_{\tau+q})e^{-\int_0^\tau E(\hat{\tau} - L^- + D)d\hat{\tau}} d\tau \right) + \mathcal{O}(\mu) \\ &= e^{\int_0^t E(\tau - L^-)d\tau} \left(Z_0 + \int_0^t \mathbb{F}(\tau - L^-)\mathcal{G}(\omega_{\tau+q})e^{-\int_0^\tau E(\hat{\tau} - L^-)d\hat{\tau}} d\tau \right) + \mathcal{O}(\mu) \end{aligned}$$

where $\mathbb{F}(\tau) = v(\tau)P(\tau, 0) - u(\tau)Q(\tau, 0)$, and for the $\mathcal{O}(\mu)$ term in the second equality we also use $D = \mathcal{O}(\mu)$,

$$C_{\omega_{\tau+q}} < 2(C_{\omega_{q_n^+}} + 1)(|\tau| + 2L + 1), \tag{4.16}$$

and $|t| < 2L$.

Change τ to $\tau - L^-$, we have

$$Z(t) = e^{\int_{-L^-}^{t-L^-} E(\tau)d\tau} \left(Z_0 + \int_{-L^-}^{t-L^-} F(\tau)\mathcal{G}(\omega_{\tau+q+L^-})e^{-\int_{-L^-}^{\tau} E(\hat{\tau})d\hat{\tau}} d\tau \right) + \mathcal{O}(\mu). \tag{4.17}$$

We then have for $|t| < 2L$,

$$|Z(t)| < Ke^{KL}K_0L^3 := C_1(\varepsilon).$$

This defined $C_1(\varepsilon)$ and it follows that $\mathcal{M}$ is well-defined on $\tilde{\Sigma}^-(n)$. That $\mathcal{M}(\tilde{\Sigma}^-(n)) \subset \tilde{\Sigma}^+(n)$ follows from (4.15).

To prove the existence of the 1D curve ξ_n as stated in Proposition 4.2, we start with the initial data $(Z_0, q) \in \tilde{\Sigma}^-(n)$ where $q \in [q_n^+ - L^-, q_n^- + L^+]$ and let t_h be the first time at which the solution reaches to $\tilde{\Sigma}^+(n)$. Then, we have

$$t_h = L^+ + L^- + \mathcal{O}(\mu), \quad p(t_h) = q + L^+ + L^- + \mathcal{O}(\mu). \tag{4.18}$$

Using (4.17) and (4.18), we have

$$Z(t_h) = A(\varepsilon) \left(B(\varepsilon)Z_0 + \int_{-L^-}^{L^+} F(\tau)\mathcal{G}(\omega_{\tau+L^-+q})e^{-\int_0^\tau E(\hat{\tau})d\hat{\tau}} d\tau \right) + \mathcal{O}(\mu), \tag{4.19}$$

where

$$A(\varepsilon) = e^{\int_0^{L^+} E(\tau)d\tau} \sim \varepsilon^{-\frac{\beta}{\alpha}}; \quad B(\varepsilon) = e^{\int_{-L^-}^0 E(\tau)d\tau} \sim \varepsilon^{\frac{\alpha}{\beta}}. \tag{4.20}$$

Note that (4.20) follows from

$$\varepsilon \sim e^{-\alpha L^+} \sim e^{-\beta L^-}$$

and

$$\lim_{t\to+\infty} E(t) = \beta, \qquad \lim_{t\to-\infty} E(t) = -\alpha.$$

Suppose that $(\mu Z(t_h), p(t_h))$ is on the stable manifold in $\tilde{\Sigma}^+$. Then, we have

$$\mu Z(t_h) = w^s(p(t_h)),$$

which means

$$A(\varepsilon) \left(B(\varepsilon)Z_0 + \int_{-L^-}^{L^+} F(\tau)\mathcal{G}(\omega_{\tau+L^-+q})e^{-\int_0^\tau E(\hat{\tau})d\hat{\tau}} d\tau \right) + \mathcal{O}(\mu) = \mu^{-1}w^s(q + L^+ + L^- + \mathcal{O}(\mu)),$$

namely,

$$F(Z_0, q) \equiv B(\varepsilon)Z_0 + \mathcal{W}(L^- + q) - \mathbb{E} - (A(\varepsilon))^{-1}\mu^{-1}w^s(q + L^+ + L^- + \mathcal{O}(\mu)) + \mathcal{O}(\mu) = 0 \tag{4.21}$$

where

$$\mathbb{E} = \int_{\mathbb{R}\setminus[-L^-,L^+]} F(\tau)\mathcal{G}(\omega_{\tau+L^-+q})e^{-\int_0^\tau E(\hat{\tau})d\hat{\tau}}d\tau.$$

Let $\gamma = \frac{1}{2}\min\{\alpha,\beta\}$. Observe that

$$\begin{aligned}|\mathbb{E}| &\le K\int_{-\infty}^{-L^-} |\mathcal{G}(\omega_{\tau+L^-+q})|e^{-\gamma|\tau|}d\tau + K\int_{L^+}^{+\infty} |\mathcal{G}(\omega_{\tau+L^-+q})|e^{-\gamma|\tau|}d\tau \\ &\le K\int_{-\infty}^{-L^-} K_0(2L+1)(|\tau|+1)e^{-\gamma|\tau|}d\tau + K\int_{L^+}^{+\infty} K_0(2L+1)(|\tau|+1)e^{-\gamma|\tau|}d\tau \\ &\le K(\varepsilon^{\gamma\alpha^{-1}} + \varepsilon^{\gamma\beta^{-1}})|\ln\varepsilon|^3.\end{aligned}$$

Here (4.16) is again used to control $|\mathcal{G}(\omega_{\tau+L^-+q})|$.

To prove Proposition 4.2(a)(b), we first show that there exists a $q(n) \in [q_n^+ - L^-, q_n^- + L^+]$ such that $\mathcal{M}(w^u(q(n)), q(n))$ is on the local stable manifold in $\tilde{\Sigma}^+(n)$. Let

$$F(q) = \frac{1}{\mu}B(\varepsilon)w^u(q) + \mathcal{W}(L^- + q) - \mathbb{E} - (A(\varepsilon))^{-1}\left(\mu^{-1}w^s(q + L^+ + L^- + \mathcal{O}(\mu)) + \mathcal{O}(\mu)\right)$$

for $q \in [q_n^+ - L^-, q_n^- + L^+]$, which is obtained by letting $Z_0 = \mu^{-1}w^u(q)$ on the left hand side of (4.21). We have

$$F(q_n^- - L^-) < 0, \quad F(q_n^+ - L^-) > 0$$

provided that ε is sufficiently small, for we have from Proposition 4.1(b)

$$\mathcal{W}(\omega_{q_n^-}) < -K_1^{-1}, \quad \mathcal{W}(\omega_{q_n^+}) > K_1^{-1}$$

but everything else in $F(q_n^- - L^-)$ and $F(q_n^+ - L^-)$ approach to zero as $\varepsilon \to 0$. Therefore, there is $q(n) \in [q_n^+ - L^-, q_n^- - L^-]$ such that $F(q(n)) = 0$.

Let $D_n = \{(Z,q) \in \tilde{\Sigma}^-(n),\ q_n^+ - L^- \le q \le q_n^- - L^-\}$.

Now, we claim that there exists a non-self intersecting, continuous curve ξ_n in $\tilde{\Sigma}^-(n)$, connecting $Z = -1$ and $Z = 1$ and satisfying

$$F(Z,q) = 0.$$

This claim holds because

(a) the set of points inside of D_n satisfying $F = 0$ is the intersection of D_n with the pre-image of $\mathcal{M}$ of the stable manifold in $\tilde{\Sigma}^+(n)$ Σ^+. This set consists of at least one and at most finitely many non-self intersecting continuous curves;

(b) these curve segments can only end at either $Z = -1$ or $Z = 1$ because $\mathbb{F}(Z, q_n^+ - L^-) > 0$, $F(Z, q_n^- - L^-) < 0$ for $|Z| \le 1$; and

(c) if none of these continuous segments connecting $Z = -1$ and $Z = 1$, then we could find a continuous path in D_n connecting $q = q_n^+ - L^-$ and $t = q_n^- - L^-$, on which $F \neq 0$, but this is not possible because the values of F at the end of this path has opposite sign.

Therefore, there is a non-self intersecting, continuous curve ξ_n connecting the unstable curve and the boundary $Z = 1$ in $\tilde{\Sigma}^-(n)$. This completes the proof of the proposition

□

5. Construction of topological horseshoe

In this section, we prove the main theorem by combining the results of Sections 3 and 4 with a result on straightening the local state and unstable manifolds.

5.1. **Straightening stable and unstable manifolds.** We consider Equation (2.5). We have the following proposition.

Proposition 5.1. *There is a transformation*

$$x = X + H^+(X, Y, \omega, \mu) \text{ and } y = Y + H^-(X, Y, \omega, \mu), \tag{5.1}$$

defined on a tempered ball $B(0, R(\omega)) = \{(X, Y) \mid |(X, Y)| < R(\omega)\}$ with $R(\omega) > (K(C_\omega + 1))^{-1}$, where $H^+(X, Y, \theta_t\omega, \mu)$ and $H^-(X, Y, \theta_t\omega, \mu)$ are C^r in (X, Y) and continuous in (t, μ) and satisfies

$$|H^+(X, Y, \omega, \mu)|, |H^(X, Y, \omega, \mu)| \leq K(1 + C_\omega)(X^2 + Y^2)$$

which maps solutions of

$$\frac{dX}{dt} = (-\alpha + \tilde{\alpha}(X, Y, \Theta_t\omega))X, \qquad \frac{dY}{dt} = (\beta + \tilde{\beta}(X, Y, \Theta_t\omega))Y$$

to solutions of Equation (2.5, where $\tilde{\alpha}(X, Y, \omega), \tilde{\beta}(X, Y, \omega)$ be C^r in (X, Y) satisfying

$$|\tilde{\alpha}(X, Y, \omega)|, \qquad |\tilde{\beta}(X, Y, \omega)| < K(1 + C_\omega)|(X, Y)|. \tag{5.2}$$

The proof of this proposition is given in the appendix.

5.2. **The map $\mathcal{N}$.** Recall that $\mathcal{M} : \Sigma^- \to \Sigma^+$ and $\mathcal{N} : \Sigma^+ \to \Sigma^-$ are maps induced by the solutions of equation (2.6) in the extended phase space, and $\mathcal{R} := \mathcal{N} \circ \mathcal{M}$ is the return map. We start with the infinitely long strip Σ in Σ^- defined by $z = \mu$ and the curve made up by the unstable segments in $\tilde{\Sigma}^-$ connected by straight lines. We call the direction of $p \in \mathbb{R}$ in Σ the *horizontal direction* and the direction of z the *vertical direction.* In Σ, a *vertical curve* is a non-self-intersecting, continuous curve that connects the two horizontal boundaries of Σ. We call a region that is bounded by two non-intersecting vertical curves a *vertical strip*, which we denote as V. The two defining vertical curves for a given vertical strip V are the *vertical boundary* of V. We call a non-self-intersecting continuous curve connecting the two vertical boundaries of V a *fully extended* horizonal curve in V. Let V_1, V_2 be two non-intersecting vertical strips in $\mathcal{V}$. We say that $\mathcal{R}(V_1)$ *crosses* V_2 *horizontally* if for every fully extended horizontal curve h of V_1, there is a subsegment $\tilde{h}$ of h so that $\mathcal{R}(\tilde{h})$ is a fully extended horizontal curve in V_2.

Let $\xi_n \in \tilde{\Sigma}^-(n)$ be as in Proposition 4.2. From the proof of Proposition 4.2, we note the vertical curve ξ_n can be chosen such that the function $F(Z, q)$ changes its sign as crossing the curve. We define the vertical strip V_n by using ξ_n as one vertical boundary; and a fully extended vertical curve η_n locating entirely on the positive side of ξ_n as the other vertical boundary. We also assume that η_n is sufficiently close to ξ_n. The choice of η_n is fairly arbitrary. In this subsection we prove the Main Theorem through the following proposition.

Proposition 5.2. *Let V_n be the vertical strip above and γ_0 be a fully extended horizontal curve in V_n. Let $\gamma_{\mathcal{M}} = \mathcal{M}(\gamma_0)$, and $\gamma_{\mathcal{N}} = \mathcal{N}(\gamma_{\mathcal{M}})$. There then exists an $m_0 >> n$ so that for all $m > m_0$, $\gamma_{\mathcal{N}}$ horizontally crosses V_m.*

The way Proposition 5.1 is used in proving Proposition 5.2 is as follows. For $q \in [q_n^+ - 2L, q_n^+ + 2L]$, let $(z_0, q) \in \tilde{\Sigma}^-(n)$ be an initial point in the extended phase space, which we also write in (x, y, p)-coordinate as $u^0 = (x_0, y_0, q)$. For notation we again write $\omega_q = \theta_q\tilde{\omega}$. The coordinate transformation (5.1) apply at $t = 0$: Observe that $C_{\omega_q} < 2(K_0 + 1)(2L + 1)$ so

$$R(\omega_q) > \big(K(2(K_0 + 1)(2L + 1) + 1)\big)^{-1} > |(x_0, y_0)|$$

where the last inequality is from $L \approx \ln \varepsilon^{-1}$ and $(x_0, y_0) \approx \frac{1}{2}\varepsilon$. Write u^0 in new variables (X, Y, p) as $u^0 = (X_0, Y_0, q)$.

Denote $u^{\mathcal{M}} = \mathcal{M}(u^0) = (x_{\mathcal{M}}, y_{\mathcal{M}}, q_{\mathcal{M}}) \in \tilde{\Sigma}^+(n)$. We have

$$C_{\omega_{q_{\mathcal{M}}}} < 2(K_0 + 1)(3L + 2)$$

so the coordinate transformation (5.1) again applies to $u^{\mathcal{M}}$ since

$$R(\omega_{q^{\mathcal{M}}}) > \big(K(2(K_0+1)(3L+2)+1)\big)^{-1} > |(x_{\mathcal{M}}, y_{\mathcal{M}})|$$

where the last inequality is from $L \approx \ln \varepsilon^{-1}$ and $|(x_{\mathcal{M}}, y_{\mathcal{M}})| \approx \frac{1}{2}\varepsilon$. For $u^0 \in V_n$, we also have

$$X_{\mathcal{M}} \approx \frac{1}{2}\varepsilon, \qquad Y_{\mathcal{M}} = \delta_{\mathcal{M}}, \qquad q_{\mathcal{M}} = q + L^- + L^+ + \mathcal{O}(\mu) \tag{5.3}$$

where $\delta_{\mathcal{M}} \in [0, \mu C_1(\varepsilon)]$. This implies that started from u^0 at $t = 0$, the solution reaches $u^{\mathcal{M}} \in \tilde{\Sigma}^+(n)$ at

$$t_{\mathcal{M}} = L^- + L^+ + \mathcal{O}(\mu)$$

where

$$u^{\mathcal{M}} \approx (\frac{1}{2}\varepsilon,\ \delta_{\mathcal{M}},\ q + t_{\mathcal{M}}).$$

Using Proposition 5.1, we have

$$X(t + t_{\mathcal{M}}) \approx \frac{1}{2}\varepsilon e^{(-\alpha+\mathcal{O}(\varepsilon)))t}, \qquad Y(t + t_{\mathcal{M}}) \approx \mu\delta_{\mathcal{M}} e^{(\beta+\mathcal{O}(\varepsilon))t} \tag{5.4}$$

for $t < T_{\delta_{\mathcal{M}}}$ where

$$T_{\delta_{\mathcal{M}}} := \frac{2}{3\beta}\ln(K\delta_{\mathcal{M}} t_{\mathcal{M}})^{-1}.$$

In fact, we have

Lemma 5.1. *For $0 \le t < T_{\delta_{\mathcal{M}}}$,*
(a) $R(\omega_{t+q_{\mathcal{M}}}) > |(X(t + t_{\mathcal{M}}), Y(t + t_{\mathcal{M}}))|$; and
(b) $X(T_{\delta_{\mathcal{M}}} + t_{\mathcal{M}}) < \varepsilon(K\delta_{\mathcal{M}} \ln \varepsilon^{-1})^{\frac{2\alpha}{3\beta}}$; $Y(T_{\delta_{\mathcal{M}}} + t_{\mathcal{M}}) > (\delta_{\mathcal{M}})^{\frac{1}{3}}(K \ln \varepsilon^{-1})^{-\frac{2}{3}}$.

Proof: Both (a) and (b) follows directly from

$$C_{\omega_{t+q_{\mathcal{M}}}} = C_{\theta_{t+t_{\mathcal{M}}}\omega_q} < 2\big(2(K_0+1)(2L+1)+1\big)(t+L+1)$$

and (5.4). □

We now turn to the backward solutions initiated on the vertical strip $V_m \subset \tilde{\Sigma}^-(m)$ for $m > m_0 >> n$. The initial point, which we denote as $u^{\mathcal{N}} = (x_{\mathcal{N}}, y_{\mathcal{N}}, q_{\mathcal{N}}) \in \tilde{\Sigma}^-(m)$, is such that $q_{\mathcal{N}} \in [q_m^+ - 2L, q_m^- + 2L]$, and in (X, Y, p)-variables we have $u^{\mathcal{N}} = (X_{\mathcal{N}}, Y_{\mathcal{N}}, q_{\mathcal{N}})$ where

$$X_{\mathcal{N}} \approx \delta_{\mathcal{N}}, \qquad Y_{\mathcal{N}} \approx \frac{1}{2}\varepsilon, \qquad q_{\mathcal{N}} \in [q_m^+ - 2L, q_m^- + 2L] \tag{5.5}$$

where $\delta_{\mathcal{N}} \in [0, \mu]$. We go backward in time, applying Proposition 5.1. Denote

$$T_{\delta_{\mathcal{N}}} = \frac{2}{3\alpha}\ln(K\delta_{\mathcal{N}} L)^{-1}.$$

Lemma 5.2. *For $-T_{\delta_{\mathcal{N}}} < t < 0$, we have*
(a) $R(\omega_{t+q_{\mathcal{N}}}) > |(X(t), Y(t))|$; and
(b) $X(-T_{\delta_{\mathcal{N}}}) > (\delta_{\mathcal{N}})^{\frac{1}{3}}(K \ln \varepsilon^{-1})^{-\frac{2}{3}}$; $Y(-T_{\delta_{\mathcal{N}}}) < \varepsilon(K\delta_{\mathcal{N}} \ln \varepsilon^{-1})^{\frac{2\beta}{3\alpha}}$.

The proof is the same as that of Lemma 5.1.

Proof of Proposition 5.2: Let γ_0 be a fully extended horizontal curve in V_n, then $\gamma_{\mathcal{M}}$ is a curve segment in $\tilde{\Sigma}^+(n)$ with one end located at the local stable manifold. Let us denote $\gamma_{\mathcal{M}}$ as

$$\gamma_{\mathcal{M}}(\tau) = (X_{\mathcal{M}}(\tau), Y_{\mathcal{M}}(\tau), q_{\mathcal{M}}(\tau)) :\ [0,1] \to \tilde{\Sigma}^+(n).$$

We have

$$X_{\mathcal{M}}(\tau) \approx \frac{1}{2}\varepsilon, \quad Y_{\mathcal{M}}(\tau) = \delta_{\mathcal{M}}(\tau), \quad q_{\mathcal{M}}(\tau) \in [q_n^+ - L - 1, q_n^- + 3L + 1]$$

where $\delta_{\mathcal{M}}(0) = 0$, and $\delta_{\mathcal{M}}(1) > 0$ depend on V_n.

Let $\tau_0 \leq 1$ be fixed. We apply Lemmas 5.1 to all points on $\gamma_{\mathcal{M}}(\tau), \tau \in [0, \tau_0]$; running the time forward up to $T_{\delta_{\mathcal{M}}(\tau_0)}$. Without loss of generality, let us assume that $\delta_{\mathcal{M}}(\tau) < \delta_{\mathcal{M}}(\tau_0)$ for all $\tau \in [0, \tau_0]$.

We now fix a $m >> n$, and apply Lemma 5.2 to a vertical segment in $V_m \subset \gamma_{\mathcal{N}}$ in $\tilde{\Sigma}^-(m)$. Let us denote $\gamma_{\mathcal{N}}$ as

$$\gamma_{\mathcal{N}}(\tau) = (X_{\mathcal{N}}(\tau), Y_{\mathcal{N}}(\tau), q_{\mathcal{N}}(\tau)) : [0, 1] \to \tilde{\Sigma}^-(m).$$

We have

$$X_{\mathcal{N}}(\tau) = \delta_{\mathcal{N}}(\tau), \quad Y_{\mathcal{N}}(\tau) \approx \frac{1}{2}\varepsilon, \quad q_{\mathcal{N}}(\tau) \in [q_m^+ - 2L, q_m^- + 2L]$$

where $\delta_{\mathcal{N}}(0) = 0$ and $\delta_{\mathcal{N}}(1) > 0$. Since $\delta_{\mathcal{N}}(0) = 0$ and $\delta_{\mathcal{M}}(0) = 0$, one can choose $\hat{\tau}_0,\ \tau_0 < 1$ such that $\delta_{\mathcal{N}}(\hat{\tau}_0),\ \delta_{\mathcal{M}}(\tau_0) < 1$ and

$$\delta_{\mathcal{N}}(\hat{\tau}_0) = (\delta_{\mathcal{M}}(\tau_0))^{\frac{\alpha}{\beta}}.$$

Without loss of generality, we assume $\delta_{\mathcal{N}}(\tau) < \delta_{\mathcal{N}}(\hat{\tau}_0)$ for all $\tau \in [0, \hat{\tau}_0]$. We apply Lemma 5.2 to all points on $\gamma_{\mathcal{N}}(\tau),\ \tau \in [0, \hat{\tau}_0]$, running the time backward to $-T_{\delta_{\mathcal{N}}(\hat{\tau}_0)}$. Denote the solution starting from $\gamma_{\mathcal{M}}(\tau)$ as $(X_{\mathcal{M},\tau}(t), Y_{\mathcal{M},\tau}(t), q_{\mathcal{M},\tau}(t))$ and the solution starting from $\gamma_{\mathcal{N}}(\tau)$ as $(X_{\mathcal{N},\tau}(t), Y_{\mathcal{N},\tau}(t), q_{\mathcal{N},\tau}(t))$. We have from Lemmas 5.1 and 5.2 that

$$X_{\mathcal{N},\hat{\tau}_0}(-T_{\delta_{\mathcal{N}}(\hat{\tau}_0)}) > X_{\mathcal{M},\tau_0}(T_{\delta_{\mathcal{M}}(\tau_0)}); \qquad Y_{\mathcal{N},\hat{\tau}_0}(-T_{\delta_{\mathcal{N}}(\hat{\tau}_0)}) < Y_{\mathcal{M},\tau_0}(T_{\delta_{\mathcal{M}}(\tau_0)}).$$

This implies that the two images, this is, the time-$T_{\delta_{\mathcal{M}}(\tau_0)}$ image of $\gamma_{\mathcal{M}}(\tau), \tau \in [0, \tau_0)$ and the time-$-T_{\delta_{\mathcal{N}}(\hat{\tau}_0)}$ image of $\gamma_{\mathcal{N}}(\tau), \tau \in [0, \hat{\tau}_0)$ do intersect. In another word, there exists $\tau \in [0, \tau_0),\ \hat{\tau} \in [0, \hat{\tau}_0)$, such that

$$(X_{\mathcal{M},\tau}(T_{\delta_{\mathcal{M}}(\tau_0)}), Y_{\mathcal{M},\tau}(T_{\delta_{\mathcal{M}}(\tau_0)})) = (X_{\mathcal{N},\hat{\tau}}(-T_{\delta_{\mathcal{N}}(\hat{\tau}_0)}), Y_{\mathcal{N},\hat{\tau}}(T_{\delta_{\mathcal{N}}(\hat{\tau}_0)})).$$

We caution that, in order for these two intersecting segments of solutions to be part of one solution, we need to have

$$q_{\mathcal{N}}(\hat{\tau}) = T_{\delta_{\mathcal{M}}(\tau_0)} + T_{\delta_{\mathcal{N}}(\hat{\tau}_0)} + q_{\mathcal{M}}(\tau). \tag{5.6}$$

This equality is achieved by first assume that m is such that

$$q_m^+ - 2L > T_{\delta_{\mathcal{M}}(\tau_0)} + T_{\delta_{\mathcal{N}}(\hat{\tau}_0)} + q_n^+ + 3L + 1,$$

for some $\tau_0 > 0$. For this specific choice of τ_0, and the ensuing values of $\hat{\tau}_0$, $\tau, \hat{\tau}$, we have

$$q_{\mathcal{N}}(\hat{\tau}) > T_{\delta_{\mathcal{M}}(\tau_0)} + T_{\delta_{\mathcal{N}}(\hat{\tau}_0)} + q_{\mathcal{M}}(\tau).$$

We then observe that as $\tau_0 \to 0$,

$$T_{\delta_{\mathcal{M}}(\tau_0)} + T_{\delta_{\mathcal{N}}(\hat{\tau}_0)} \to \infty.$$

This implies that there exist appropriate τ_0, and $\hat{\tau}_0$, τ nd $\hat{\tau}$, such that equality (5.6) holds. We have proved the existence of a solution starting from $\gamma_{\mathcal{M}}$ reaching $\gamma_{\mathcal{N}}$. □

The main theorem follows directly from Proposition 5.2.

Appendix A. Proof of Proposition 5.1

In this appendix, we first give the results on the random stable and unstable manifolds, then we use them to prove Proposition 5.1.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a measurable $\mathbb{P}$-preserving flow θ_t in the probability space:

$$\theta_t \circ \theta_\tau \omega = \theta_{t+\tau}\omega \quad \text{ for } t,\ \tau \in \mathbb{R},\ \omega \in \Omega, \qquad \theta_0 = \mathrm{id}_\Omega.$$

The quadruplet $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is the so-called metric dynamical system (see [2]). This metric dynamical system models the evolution of noise. Throughout of this paper, we assume the probability measure $\mathbb{P}$ is ergodic with respect to the flow θ_t.

Consider a random differential equation in $\mathbb{R}^d$ driven by θ_t.

$$\frac{du}{dt} = Au + f(u) + \mu g(\theta_t\omega, u), \tag{A.1}$$

where A is a $d \times d$ real matrix, f is a nonlinear term, g is a random forcing, and $\mu \in [-\mu_0, \mu_0]$ is a parameter, $0 < \mu_0 < 1$.

For the matrix A, we assume that

Hypothesis A: A is hyperbolic, that is, A has no eigenvalues on the imaginary axis.

This condition implies that there exist an invariant splitting of the phase space $\mathbb{R}^d = E_+ \oplus E_-$ with the associated projections Π_+ and Π_- and positive constants α, β, and K such that

$$\begin{aligned} ||e^{At}\Pi_-|| &\le Ke^{-\beta t} \quad \text{for } t \ge 0, \\ ||e^{At}\Pi_+|| &\le Ke^{\beta t} \quad \text{for } t \le 0. \end{aligned} \tag{A.2}$$

We assume that for the nonlinear term $f(u)$ and $g(\theta_t\omega, u)$

Hypothesis B: There are an open neighborhood U of 0 in $\mathbb{R}^d$ and such that

(i) $f : U \to \mathbb{R}^d$ is C^N for $N \ge 2$ and there is a deterministic ball $B_{r_0}(0) \subset U$ with constant radius r_0 such that
$$\sup_{u \in B_{r_0}(0)} ||D^k f(u)|| \le C_0, \quad \text{for all } 0 \le k \le N;$$

(ii) $g : \Omega \times U \to \mathbb{R}^d$ is measurable and is C^N with respect to u and $\partial_u^i g(\theta_t\omega, u)$ is continuous in t for $0 \le i \le N$. There is a ball, $U(\omega) = B(0, \rho_0(\omega)) = \{u \in \mathbb{R}^d \mid |u| < \rho_0(\omega)\}$, where $\rho_0 : \Omega \to (0, \infty)$ is tempered from below and $\rho_0(\theta_t\omega)$ is continuous in t, such that
$$\sup_{u \in U(\omega)} ||D^k g(\omega, u)|| \le B_k(\omega), \quad \text{for all } 0 \le k \le N, \quad \omega \in \Omega,$$
where $B_k(\omega)$ is tempered from above and $B_k(\theta_t\omega)$ is continuous in t.

(iii) $f(0) = 0$, $g(\omega, 0) = 0$, $D_u f(0) = 0$ and $D_u g(\omega, 0) = 0$.

In order to construct stable and unstable manifolds, we use the standard cut-off function to modify the nonlinearities f and g.

Let $\sigma(s)$ be a C^∞ function from $(-\infty, \infty)$ to $[0, 1]$ with
$$\sigma(s) = 1 \ \text{ for } \ |s| \le 1, \quad \sigma(s) = 0 \ \text{ for } \ |s| \ge 2,$$
$$\sup_{s \in \mathbb{R}} |\sigma'(s)| \le 2.$$

We first choose r such that $0 < r < \min\{r_0, \beta/120K^2C_0\}$. Let $\rho : \Omega \to (0, \infty)$ be a tempered random variable such that $2\rho(\omega) \le \min\{r, \rho_0(\omega)\}$ and $\rho(\theta_t\omega)$ is continuous in t. We consider a modification of $f(u)$ and $g(\omega, u)$. Let

$$F(u) = \sigma_r(|u|)f(u), \quad \text{where} \quad \sigma_r(|u|) = \sigma\left(\frac{|u|}{\rho(\omega)}\right),$$
$$G(\omega, u) = \sigma_{\rho(\omega)}(|u|)g(\omega, u), \quad \text{where} \quad \sigma_{\rho(\omega)}(|u|) = \sigma\left(\frac{|u|}{\rho(\omega)}\right),$$

with trivial extensions on the outside of $B(0, 2r)$ and $B(0, 2\rho)$ respectively. An elementary calculation gives

Lemma A.1. (i) *$F(u) = f(u)$, for $|u| \le r$ and $G(\omega, u) = f(\omega, u)$, for $|u| \le \rho(\omega)$;*

(ii) *$||DF(u)|| \le 10C_0 r$ and $||DG(\omega, u)|| \le 10B_2(\omega)\rho(\omega)$ for all $\omega \in \Omega$ and $u \in \mathbb{R}^d$;*

(iii) $\sup_{x\in\mathbb{R}^d} ||D^k F(u)|| \leq \tilde{C}_k(\omega)$ *and* $\sup_{x\in\mathbb{R}^d} ||D^k G(\omega, u)|| \leq \tilde{B}_k(\omega)$ *for* $2 \leq k \leq N$ *and all* $\omega \in \Omega$. *Here* $\tilde{C}_k(\omega)$ *and* $\tilde{B}_k(\omega)$ *are random variables tempered from above and* $\tilde{C}_k(\theta_t\omega)$ *and* $\tilde{B}_k(\theta_t\omega)$ *are continuous in* t.

We also choose $\rho(\omega)$ such that

$$\rho(\omega) = \frac{\beta}{120K^2(C_0 + B_2(\omega))}. \tag{A.3}$$

Clearly, $\rho(\omega)$ is tempered from below. Furthermore, we have

$$||D_u F(u)|| \leq \frac{\beta}{12K^2} \text{ and } ||D_u G(\omega, u)|| \leq \frac{\beta}{12K^2}. \tag{A.4}$$

We consider the modified random differential equation

$$\frac{du}{dt} = Au + F(u) + \mu G(\theta_t\omega, u). \tag{A.5}$$

We first recall that a multifunction $M = \{M(\omega)\}_{\omega\in\Omega}$ of nonempty closed sets $M(\omega)$, $\omega \in \Omega$, contained in $\mathbb{R}^d$ is called a *random set* if

$$\omega \to \inf_{y\in M(\omega)} |x - y|$$

is a random variable for any $x \in \mathbb{R}^d$.

Definition A.1. *A random set* $M(\omega)$ *is called an invariant set for equation (A.5) if we have*

$$u(t, \omega, M(\omega), \mu) \subset M(\theta^t\omega) \quad \textit{for} \quad t \geq 0.$$

We define for equation (A.5) the stable set

$$W^-(\omega) = \{u_0 \in \mathbb{R}^d \mid u(t, \omega, u_0, \mu) \to 0, \text{ as } t \to +\infty\}$$

and the unstable set

$$W^+(\omega) = \{u_0 \in \mathbb{R}^d \mid u(t, \omega, u_0, \mu) \to 0, \text{ as } n \to -\infty\}.$$

Clearly, both $W^s(\omega)$ and $W^u(\omega)$ are invariant for equation (A.5).

Let $\gamma \in (0, \beta/5)$. We define the following Banach spaces

$$C_\gamma^- = \left\{\phi \mid \phi : (-\infty, 0] \to \mathbb{R}^d \text{ is continuous and } \sup_{t\leq 0} |\phi(t)|e^{-\gamma t} < \infty\right\}$$

with the norm

$$|\phi|_{C_\gamma^-} = \sup_{t\leq 0} |\phi(t)|e^{-\gamma t}$$

and

$$C_\gamma^+ = \left\{\phi \mid \phi : [0, \infty) \to \mathbb{R}^d \text{ is continuous and } \sup_{t\geq 0} |\phi(t)|e^{\gamma t} < \infty\right\}$$

with the norm

$$|\phi|_{C_\gamma^+} = \sup_{t\geq 0} |\phi(t)|e^{\gamma t}$$

Proposition A.1. (**Unstable manifold**) *Assume that Hypotheses A and B hold and choose the tempered radius $\rho(\omega)$ such that (A.3) holds. Then there exists a C^N unstable manifold for equation (A.5) which is given by*

$$W^+(\omega) = \{\xi + h^+(\omega, \xi, \mu) \mid \xi \in E_+(\omega)\}$$

where $h^+(\cdot,\cdot,\cdot) : \Omega \times E_+ \times [-\mu_0, \mu_0] \to E_-(\omega)$ is measurable, C^N in ξ, continuous in μ, and satisfies $h^+(\omega, 0, \mu) = 0$ and $D_\xi h^+(\omega, 0, \mu) = 0$. Furthermore, $h^+(\theta_t\omega, \xi, \mu)$ is differentiable in t.

Remark A.1. *One can show that the unstable manifold is C^N in μ. However, for our application, we do not need it.*

Proof. The proof of this proposition follows from the standard Lyapunov and Perron approach. We gives an outline here. Note that $W^+(\omega)$ is nonempty since $u = 0 \in W^u(\omega)$, and invariant for the random dynamical system generated by (A.5). We will prove that $W^+(\omega)$ is given by the graph of a C^N function over $E_+(\omega)$.

We first claim that for $u(\cdot) \in C_\gamma^-(\omega)$, $u(0) \in M^u(\omega)$ if and only if $u(t)$ satisfies

$$\begin{aligned} u(t) = e^{At}\xi &+ \int_0^t e^{A(t-\tau)}\Pi^+\big(F(u) + \mu G(\theta_\tau\omega, u)\big)d\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)}\Pi^-\big(F(u) + \mu G(\theta_\tau\omega, u)\big)d\tau, \end{aligned} \tag{A.6}$$

where $\xi = \Pi^+ u(0)$. The proof of this claim follows from the variation of constants formula using (A.2).

Let $\mathcal{J}^+(u, \xi, \omega, \mu)$ be the right hand side of equality (A.6). We first have that $\mathcal{J}^u(\cdot, \xi, \omega, \mu)$ maps C_γ^- into C_γ^-. In fact, for $u \in C_\gamma^-$, using (A.2) and (A.4), we have

$$|\mathcal{J}^+(u, \xi, \omega, \mu))|_\gamma^- \le K|\xi| + \Big(\frac{\beta}{6(\beta-\gamma)} + \frac{\beta}{6(\beta+\gamma)}\Big)|u|_\gamma^- < \infty.$$

Next we show that $\mathcal{J}^+(u, \xi, \omega, \mu)$ is a uniform contraction in u with respect to ξ, ω, and μ. Using (A.2) and (A.4), we have for $u, \bar{u} \in C_\gamma^-$

$$\begin{aligned} &|\mathcal{J}^u(u, \xi, \omega, \mu) - \mathcal{J}^u(\bar{u}, \xi, \omega, \mu)|_\gamma^- \\ &\le \sup_{t\le 0}\Big\{\int_t^0 e^{(\beta-\gamma)(t-\tau)}\frac{\beta}{6}d\tau + \int_{-\infty}^t e^{-(\beta+\gamma)(t-\tau)}\frac{\beta}{6}d\tau\Big\}|u - \bar{u}|_\gamma^- \\ &\le \Big(\frac{\beta}{6(\beta-\gamma)} + \frac{\beta}{6(\beta+\gamma)}\Big)|u - \bar{u}|_\gamma^- \le \frac{9}{24}|u - \bar{u}|_\gamma^-. \end{aligned}$$

Hence, $\mathcal{J}^+(u, \xi, \omega, \mu)$ is a contraction in u uniformly in (ξ, ω, μ). By the contraction principle, for each $\xi \in E_+$, $\mathcal{J}^+(\cdot, \xi, \omega, \mu)$ has a unique fixed point $u(\cdot; \xi, \omega, \mu) \in C_\gamma^-$, which satisfies equation (A.6). Clearly, $u(t; 0, \omega, \mu) = 0$. Since $C_{\gamma_2}^- \subset C_{\gamma_1}^-$ for any $0 \le \gamma_1 \le \gamma_2$, a fixed point in $C_{\gamma_2}^-$ must be in $C_{\gamma_1}^-$. By the uniqueness, $u(t; \xi, \omega, \mu)$ is independent of $\gamma \in [0, \beta/5]$. Furthermore, we have for $\xi, \xi_0 \in E_+$ that

$$|u(\cdot, \xi, \omega, \mu) - u(\cdot; \xi_0, \omega, \mu)|_{C_{\beta/5}^-} \le \frac{24}{15}K|\xi - \xi_0|. \tag{A.7}$$

Since $u(t; \eta, \omega; \mu)$ is the ω-wise limit of iteration of contraction mapping $\mathcal{J}^+$ starting at 0 and $\mathcal{J}^+$ maps a $\mathcal{F}$-measurable function to a measurable function, $u(t; \xi, \cdot, \mu)$ is $\mathcal{F}$-measurable. On the other hand, since $u(t; \cdot, \omega, \mu)$ is Lipschitz continuous, by Castaing and Valadier [6], Lemma III.14, $u(t; \xi, \omega, \mu)$ is measurable with respect to (ξ, ω, μ). Since $g(\theta_t\omega, u)$ is continuous in t, we also have that $u(t; \xi, \theta_s\omega, \mu)$ is continuous in s.

In order to show that $u(t;\xi,\omega,\mu)$ is C^N in ξ, we show by induction that $u(\cdot;\xi,\omega,\mu)$ is C^i in ξ for any $1\le i\le N$ from E_+ to $C^-_{\beta/(5+2i)}$. The first step is to show that $u(\cdot;\cdot,\omega,\cdot)$ is C^1 from E_+ to $C^-_{\beta/7}$. Using the same arguments we used in [15], we first have that $u(\cdot;\cdot,\omega,\mu)$ is differentiable from E_+ to $C^-_{\beta/6}$. Then, we have that $Du(\cdot;\cdot,\omega,\mu): E_+\to L(E_+\times\mathbb{R},C^-_{\beta/7})$ is continuous, where $L(E_+,C^-_{\beta/7})$ is the space of all bounded linear operators from E_+ to $C^-_{\beta/7}$. We note that $D_\xi u(t;\xi,\omega,\mu)$ satisfies

$$
\begin{aligned}
D_\xi u(t) = e^{At} &+ \int_0^t e^{A(t-\tau)}\Pi^+\big(DF(u)+\mu D_uG(\theta_\tau\omega,u)\big)D_\xi u\,d\tau \\
&+ \int_{-\infty}^t e^{A(t-\tau)}\Pi^-\big(DF(u)+\mu D_uG(\theta_\tau\omega,u)\big)D_\xi u\,d\tau
\end{aligned}
\tag{A.8}
$$

and

$$
||D_\xi u(\cdot;\xi,\omega,\mu)||_{L(E_+,C^-_{\beta/7})}\le \frac{36}{23}K
\tag{A.9}
$$

Letting $2\le m\le N$, by the induction hypothesis, we have that $u(\cdot;\cdot,\omega,\mu)$ is C^j from E_+ to $C^-_{\beta/(5+2j)}$ for all $1\le j\le m-1$ and there are random variables $K_j(\omega)$ tempered from above such that

$$
||D^j u(\cdot;\xi,\omega,\mu)||_{L^j(E_+,C^-_{\beta/(5+2j)})}\le K_j(\omega),
\tag{A.10}
$$

$$
||D^j u(\cdot,\xi,\omega,\mu)-D^j u(\cdot;\xi_0,\omega,\mu)||_{L^j(E_+,C^-_{\beta/(5+2j)})}\le K_j(\omega)|\xi-\xi_0|.
\tag{A.11}
$$

Here $L^j(E_+,C^-_{\beta/(5+2j)})$ is the usual space of bounded j-linear forms. In fact for $m=2$ we have

$$
||D^2_\xi u(\cdot;\xi,\omega,\mu)||_{L^2(E_+,C^-_{\beta/9})}\le \frac{17\cdot 36^2}{31\cdot 23^2}K(C_0+\mu B_\epsilon(\omega)),
\tag{A.12}
$$

where $B_\epsilon(\omega)$ is a tempered random variable from above with $\epsilon\in(0,\beta/9)$ such that

$$
B_2(\theta_t\omega)\le B_\epsilon(\omega)e^{\epsilon|t|},\quad t\in\mathbb{R}
$$

which follows from the properties of a tempered random variable.

Then, in the same fashion as showing than u is C^1, we have that $u(\cdot;\cdot,\omega,\mu)$ is C^m from E_+ to $C^-_{\beta/(5+2m)}$ and $D^m u(\cdot;\cdot,\omega,\mu)$ is Lipschitz continuous from E_+ to $L^m(E_+,C^-_{\beta/(5+2m)})$ if $m<N$. We also have that $D^m u(\cdot;\cdot,\omega,\mu)$ is continuous in μ.

Let

$$
h^+(\omega,\xi,\mu)=\Pi^-u(0;\xi,\omega,\mu)=\int_{-\infty}^0 e^{-A\tau}\Pi^-\big(F(u(\tau;\xi,\omega,\mu))+\mu G(\theta_\tau\omega,u(\tau;\xi,\omega,\mu))\big)d\tau.
$$

Then $h^+(\omega,0,\mu)=0$, $D_\xi h^+(\omega,0,\mu)$, and $h^+(\omega,\xi,\mu)$ is C^N in ξ and $h^+(\theta_t\omega,\xi,\mu)$ is continuous in t.

Furthermore, we have

$$
\begin{aligned}
||\partial_\xi h^+(\omega,\xi,\mu)|| &= ||\int_{-\infty}^0 e^{-A\tau}\Pi^-\big(\partial_u F(u(\tau;\xi,\omega,\mu))+\mu\partial_u G(\theta_\tau\omega,u(\tau;\xi,\omega,\mu))\big)\partial_\xi u(\tau;\xi,\omega,\mu)\,d\tau|| \\
&\le \frac{13}{23}.
\end{aligned}
\tag{A.13}
$$

(A.14)
$$\begin{aligned}||\partial_\xi^2 h^+(\omega,\xi,\mu)|| = || \int_{-\infty}^0 & e^{-A\tau}\Pi^-\Big(\big(\partial_u F(u(\tau;\xi,\omega,\mu)) + \mu\partial_u G(\theta_\tau\omega, u(\tau;\xi,\omega,\mu))\big)\partial_\xi^2 u(\tau;\xi,\omega,\mu) \\ & + \big(\partial_u^2 F(u(\tau;\xi,\omega,\mu)) + \mu\partial_u^2 G(\theta_\tau\omega, u(\tau;\xi,\omega,\mu))\big)\big(\partial_\xi u(\tau;\xi,\omega,\mu)\big)^2\Big)\, d\tau|| \\ & \le \frac{17\cdot 36^2}{48\cdot 23^2}\left(\frac{17}{31}+K\right)(C_0+\mu B_\epsilon).\end{aligned}$$

By the definition of h^+ and the fact that $u^0 \in W^+(\omega)$ if and only if (A.6) has a unique solution $u(\cdot)$ in C_γ^- with $u(0) = u^0 = \xi + h^+(\omega,\xi,\mu)$ for some $\xi \in E_+(\omega)$, it follows that

$$W^+(\omega) = \{\xi + h^+(\omega,\xi,\mu) \mid \xi \in E_+(\omega)\}.$$

Clearly, $W^+(\omega)$ is a random set. Finally, we now show that $h^+(\theta_t\omega,\xi,\mu)$ is continuous in t. Let $u_0 \in W^+(\omega)$, by the invariance, we have $u(t,u_0) \in W^+(\theta_t\omega)$. Thus, we can write $u(t,u_0)$ as

$$u(t,u_0) = \xi(t) + h^+(\theta_t\omega,\xi(t)).$$

Since $u(t,u_0)$ is differentiable in t and $\xi(t) = \Pi_+ u(t,u_0)$, $\xi(t)$ is differentiable in t. Hence, using the above identity and the fact h^+ is differentiable in ξ, we have that for each $\xi \in E_+$, $h^+(\theta_t\omega,\xi)$ is differentiable in t. This completes the proof of the proposition.

□

Reversing the time in (A.5), we have the following

Proposition A.2. (**Stable manifold**) *Assume that Hypotheses A and B hold and choose the tempered radius $\rho(\omega)$ such that (A.3) holds. Then there exists a C^N stable manifold for equation (A.5) which is given by*

$$W^-(\omega) = \{\eta + h^-(\omega,\eta,\mu) \mid \eta \in E_-(\omega)\}$$

where $h^-(\cdot,\cdot,\cdot) : \Omega \times E_- \times [-\mu_0,\mu_0] \to E_+$ is measurable in all variables and C^N in η and continuous in μ and satisfies $h^-(\omega,0,\mu) = 0$ and $D_\eta h^-(\omega,0,\mu) = 0$. Furthermore, $h^-(\theta_t\omega,\eta,\mu)$ is continuous in t.

Proof of Proposition 5.1: Since $f(x,y), g(x,y), P(x,y)$, and $Q(x,y)$ are C^r with $r > 2$ and are high order terms at $(x,y) = (0,0)$, there are a ball $B_{\rho_0}(0) \subset \mathbb{R}^2$ and a constant $B_0 > 0$ such that they and their derivatives up to order r are bounded by B_0 for $(x,y) \in B_{\rho_0}(0)$. Let $\rho(\omega)$ be a tempered random variable such that $\rho(\omega) \le 2\rho_0$ and $\rho(\theta_t\omega)$ is continuous in t. We consider the modification of $f(x,y), g(x,y), P(x,y)$, and $Q(x,y)$ by using the cut-off function $\sigma(s)$.

$$\tilde f(x,y) = \sigma\left(\frac{|(x,y)|}{\rho(\omega)}\right) f(x,y), \quad \tilde g(x,y) = \sigma\left(\frac{|(x,y)|}{\rho(\omega)}\right) g(x,y),$$
$$\tilde P(x,y) = \sigma\left(\frac{|(x,y)|}{\rho(\omega)}\right) P(x,y), \quad \tilde Q(x,y) = \sigma\left(\frac{|(x,y)|}{\rho(\omega)}\right) Q(x,y).$$

Then, there is a tempered random variable $\tilde B_0(\omega)$ depending only on σ, f, g, P, Q, and B_0 such that $\tilde f(x,y), \tilde g(x,y), \tilde P(x,y)$, and $\tilde Q(x,y)$ and their derivatives up to order r are bounded by $\tilde B_0$. We note that

$$|G(\theta_t\omega)| \le \frac{1}{\Delta}C_{\theta_t\omega} \ \text{ and } \ C_{\theta_t\omega} < 2(C_\omega+1)(|t|+1).$$

We choose

$$\rho(\omega) = \frac{\Delta \min\{\alpha,\beta\}}{120 B_0(1+C_\omega)}$$

which implies that

$$||D_u F(u)|| \le \frac{\min\{\alpha,\beta\}}{12} \quad \text{and} \quad ||D_u G(\omega,u)|| \le \frac{\min\{\alpha,\beta\}}{12}.$$

By using Proposition A.1, Proposition A.1, and the estimates (A.9) and (A.12), equation (2.5) has a local stable manifold

$$W^s_{loc}(\omega) = \Big\{ \big(x, h^-(x,\omega,\mu)\big) \;\big|\; |x| < \frac{23}{36}\rho(\omega) \Big\}$$

and local unstable manifold

$$W^u_{loc}(\omega) = \Big\{ \big(h^+(y,\omega,\mu), y\big) \;\big|\; |y| < \frac{23}{36}\rho(\omega) \Big\}$$

where $h^-(x,\omega,\mu)$ and $h^+(y,\omega,\mu)$ satisfy the following

(i) $h^-(0,\omega,\mu)=0, \partial_x h^-(0,\omega,\mu)=0$ and $h^+(0,\omega,\mu)=0, \partial_y h^+(0,\omega,\mu)=0$

(ii) The following estimates hold

$$||\partial_x h^-(x,\omega,\mu)||,\ ||\partial_x h^-(x,\omega,\mu)|| \le \frac{13}{23}. \tag{A.15}$$

$$||\partial_x^2 h^-(x,\omega,\mu)||,\ ||\partial_x^2 h^+(y,\omega,\mu)|| \tag{A.16}$$

$$\le \frac{17\cdot 36^2}{31\cdot 23^2}\Big(\tilde{C}_0 + \frac{2\mu}{\Delta}\tilde{B}_0 \max\{1, \frac{9}{\min\{\alpha,\beta\}} e^{\min\{\alpha,\beta\}/9}\}(C_\omega+1)\Big). \tag{A.17}$$

(iii) The following equalities hold

$$\begin{aligned}
&-\alpha h^+ + f(h^+,y) + \mu P(h^+,y)\mathcal{G}(\theta_t\omega) \\
&\qquad = \partial_y h^+\Big(\beta y + g(h^+,y) + \mu Q(h^+,y)\mathcal{G}(\theta_t\omega)\Big) + \partial_t h^+(y,\theta_t\omega,\mu) \\
&\beta h^- + g(x,h^-) + \mu P(x,h^-)\mathcal{G}(\theta_t\omega) \\
&\qquad = \partial_x h^-\Big(-\alpha x + g(x,h^-) + \mu Q(x,h^-)\mathcal{G}(\theta_t\omega)\Big) + \partial_t h^-(x,\theta_t\omega,\mu)
\end{aligned} \tag{A.18}$$

Now we want to straighten the stable and unstable manifolds by using the transformation:

$$X = x - h^+(y,\omega,\mu) \quad \text{and} \quad Y = y - h^-(x,\omega,\mu).$$

Using (A.15), (A.16) and the uniform contraction mapping principle, we have that the above transformation has an inverse from $B_{\frac{10}{36}\rho(\omega)}$ into $B_{\frac{23}{36}\rho(\omega)}$ which is given by

$$x = X + H^+(X,Y,\omega,\mu) \quad \text{and} \quad y = Y + H^-(X,Y,\omega,\mu).$$

Furthermore, we have that $H^+(X,Y,\theta_t\omega,\mu)$ and $H^-(X,Y,\theta_t\omega,\mu)$ are C^r in (X,Y) and continuous in (t,μ) and satisfy

$$||\partial_{(X,Y)} H^-(X,Y,\omega,\mu)||,\ ||\partial_{(X,Y)} H^+(x,\omega,\mu)|| \le \frac{13}{10}. \tag{A.19}$$

$$\begin{aligned}
&||\partial^2_{(X,Y)} H^-(X,Y,\omega,\mu)||,\ ||\partial^2_{(X,Y)} H^+(X,Y,\omega,\mu)|| \\
&\le \frac{23\cdot 17\cdot 9^2\cdot 2}{31\cdot 5^3}\Big(\tilde{C}_0 + \frac{2\mu}{\Delta}\tilde{B}_0 \max\{1, \frac{9}{\min\{\alpha,\beta\}} e^{\min\{\alpha,\beta\}/9}\}(C_\omega+1)\Big).
\end{aligned} \tag{A.20}$$

Let $(x(t), y(t))$ be a solution of equation (2.5). If $(X(t), Y(t)) \in B_{\frac{10}{36}\rho(\theta_t\omega)}$, then it follows from (A.18) that

$$\frac{dX}{dt} = (-\alpha + \tilde{\alpha}(X, Y, \theta_t\omega))X, \qquad \frac{dY}{dt} = (\beta + \tilde{\beta}(X, Y, \theta_t\omega))Y \tag{A.21}$$

where

$$\begin{aligned}
\tilde{\alpha}(X, Y, \theta_t\omega)) = & \frac{1}{X}\Big(f(x, y) - f(h^+, y) + \mu\big(P(x, y) - P(h^+, y)\big)\mathcal{G}(\theta_t\omega) \\
& - \partial_y h^+\big(g(x, y) - g(h^+, y) + \mu\big(Q(x, y) - Q(h^+, y)\big)\mathcal{G}(\theta_t\omega)\Big) \\
\tilde{\beta}(X, Y, \theta_t\omega)) = & \frac{1}{Y}\Big(g(x, y) - g(x, h^-) + \mu\big(Q(x, y) - Q(x, h^-)\big)\mathcal{G}(\theta_t\omega) \\
& - \partial_y h^-\big(f(x, y) - f(x, h^-) + \mu\big(P(x, y) - P(x, h^-)\big)\mathcal{G}(\theta_t\omega)\Big)
\end{aligned}$$

with replacing x and y by $X + H^+(X, Y, \theta_t\omega, \mu)$ and $Y + H^-(X, Y, \theta_t\omega, \mu)$, respectively. Clearly, if $(X(t), Y(t))$ is a solution of (A.21) and $(X(t), Y(t)) \in B_{\frac{10}{36}\rho(\theta_t\omega)}$ for $T_1 \le t \le T_2$, then

$$(x(t), y(t)) = (X(t) + H^+(X(t), Y(t), \theta_t\omega, \mu), Y(t) + H^-(X(t), Y(t), \theta_t\omega, \mu))$$

is a solution of equation (2.5) and $(x(t), y(t)) \in B_{\frac{23}{36}\rho(\theta_t\omega)}(0)$ for $T_1 \le t \le T_2$.

Using estimates (A.19) and (A.20) there exist constant K^* such that for $(X, Y) \in B_{\frac{10}{36}\rho(\omega)}$

$$|\tilde{\alpha}(X, Y, \omega)|, \qquad |\tilde{\beta}(X, Y, \omega)| < K^*\big(1 + C_\omega\big)|(X, Y)|.$$

This completes the proof of the proposition.

$\square$

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(Kening Lu) Department of Mathematics, Brigham Young University, Provo, Utah 84602
E-mail address, k. Lu: `klu@math.byu.edu`

(Qiudong Wang) Department of Mathematics, University of Arizona, Tuscon, AZ 85721
E-mail address, D. Wang: `dwang@math.arizona.edu`