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Chaos in differential equations driven by a nonautonomous force

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Abstract

Nonautonomous forces appear in many applications. They could be periodic, quasiperiodic and almost periodic in time; or they could take the form of a sample path of a random forcing driven by a stochastic process, which is without any periodicity in time. In this paper, we study the chaotic behaviour of differential equations driven by a general nonautonomous forcing without assuming any periodicity in time, aiming at applications to systems driven by a bounded random force. As a direct application, we prove that, for the Duffing equation driven by a bounded stationary stochastic process induced by a Brownian motion, chaotic dynamics exist almost surely. We also obtain various chaotic behaviour that are exclusively associated with equations driven by nonautonomous forcing without any periodicity in time. It has turned out that, unlike the systems driven by a periodic or almost periodic forcing, the transversal intersections of the stable and unstable manifolds are neither necessary nor sufficient for chaotic dynamics to exist. Finally, we apply all our results to the Duffing equation.

Mathematics Subject Classification: 37D45, 37C40

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Nonautonomous forces appear in many applications. They could be periodic, quasiperiodic and almost periodic in time; or they could take the form of a sample path of a random forcing driven by a stochastic process, which is without any periodicity in time. In this paper, we study the chaotic behaviour of differential equations driven by a general nonautonomous forcing without assuming any periodicity in time, aiming at applications to systems driven by a bounded random force. To simplify our presentation, we study nonautonomous ordinary differential equations in $\mathbb{R}^2$. The higher dimensional problem will be addressed in an upcoming paper.

Description of results. Let $(x, y) \in \mathbb{R}^2$ be the phase variables and t be the time. We start with an unforced system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x + f(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \beta y + g(x, y), \tag{1.1}$$

where $\alpha > \beta$ are positive constants, $f(x, y)$ and $g(x, y)$ are the higher order terms. We assume that equation (1.1) has a homoclinic orbit to the dissipative saddle $(x, y) = (0, 0)$. The precise conditions will be given in the next section.

Let $U \subset \mathbb{R}^2$ be an open neighbourhood of the unperturbed homoclinic loop of equation (1.1). To the right-hand side of equation (1.1) we add a time-dependent forcing to form a nonautonomous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x + f(x, y) + \mu P(x, y, t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \beta y + g(x, y) + \mu Q(x, y, t), \tag{1.2}$$

where μ is a small parameter representing the magnitude of the forcing, $P(x, y, t)$ and $Q(x, y, t)$ are higher order terms at $(x, y) = (0.0)$. In this paper we assume that, on $U \times \mathbb{R}$, $P(x, y, t)$ and $Q(x, y, t)$ are uniformly bounded and they are smooth in (x, y) but are only continuous in t. In applications, a nonautonomous forcing can be a sample path of a random forcing of the form

$$P(x, y, t) = F(x, y, \xi_t(\omega)), \qquad Q(x, y, t) = G(x, y, \xi_t(\omega)),$$

where $\xi_t(\omega)$ is a $\mathbb{R}^n$-valued stochastic process over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and F and G are uniformly bounded nonlinear functions on $U \times \mathbb{R}^n$.

We study the dynamical behaviour of equation (1.2) through the Poincaré return map $\mathcal{R}$ induced by equation (1.2) around the unforced homoclinic solution in the extended phase space. We introduce a characteristic function $\mathcal{W}(t)$ given by (2.4). This function is a natural extension of the classical Melnikov function. It is a function of time that measures the separations of the unstable manifold W^u and the stable manifold W^s of $(0, 0)$. We also extend the geometric approach of using vertical strips and horizontal strips, due to Smale, to describe chaotic dynamics of equation (1.2) (see section 2.4). Denote

$$m^{\pm} = \liminf_{t \to \pm\infty} \mathcal{W}(t), \qquad M^{\pm} = \limsup_{t \to \pm\infty} \mathcal{W}(t).$$

With the assumptions on the nonresonance of α and β and the uniform boundedness of the forcing functions, which we will introduce in detail in section 2, we prove the following.

Theorem A. *Assume that $m^{\pm} < 0 < M^{\pm}$. Then there exists a $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$, the solutions of equation (1.2) admit chaotic behaviour in the form of a full horseshoe of infinitely many branches for the return map $\mathcal{R}$.*

The full horseshoe here implies that for each positive integer $k \geqslant 2$, there exists an invariant set of solutions on which the dynamics are semi-conjugate to the full shift of k symbols. The chaotic behaviour we obtained for the solutions of equation (1.2) can also be described as follows. For any given $p_0 = (x_0, y_0)$ that is located sufficiently close to the unforced homoclinic loop, the dynamics of the solutions through p_0 depend sensitively on the time it is initiated at p_0. First, there exist infinitely many pairs t^0 and t^∞ of initial times, arbitrarily close to each other, such that the solution for the initial time t^0 is attracted to the solution $(x, y) = (0, 0)$ so it never completes one loop around the homoclinic solution. The solution initiated from t^∞, on the other hand, does go around the homoclinic loop infinitely many times in a rather steady pace. Second, in between t^0 and t^∞, the behaviour of solutions are arbitrary in the sense that they assume *all* imaginable manners in going around the homoclinic loop in phase space. This is to say that, at any moment a solution could decide to slow down, taking, say, roughly twice, three times or *any* number of times of the time it took for the previous round in completing the next round. It could also decide to accelerate in similar fashion. Third, arbitrarily close to each of these initial times, there are also solutions that decide, at *any* imaginable moment, to leave the neighbourhood of the unperturbed homoclinic loop by following the other unstable branch of (0, 0).

The dynamics of the solutions of equation (1.2) also depend sensitively on initial phase position. This is to say that, for a fixed initial time t_0, there exists infinitely many pairs p^0 and p^∞ in phase space (x, y) around the homoclinic loop, arbitrarily close to each other, such that the dynamics of the solutions in between p^0 and p^∞ initiated at time t_0 are similar to the ones described above with p^0 in the place of t^0 and p^∞ in the place of t^∞.

We apply theorem A to a Duffing equation driven by a random forcing. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classic Wiener probability space, where

$$\Omega = C_0(\mathbb{R}, \mathbb{R}) = \left\{\omega(t)\ :\ \omega(\cdot) : \mathbb{R} \to \mathbb{R}\ \text{ is continuous and }\ \omega(0) = 0\right\}$$

has the open compact topology so that Ω is a Polish space, $\mathcal{F}$ is its Borel σ-algebra and $\mathbb{P}$ is the Wiener measure. The Brownian motion takes the form $B_t(\omega) = \omega(t)$. We consider the Wiener shift θ_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is given by

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t). \tag{1.3}$$

It is well known that $\mathbb{P}$ is an ergodic invariant measure for θ_t. For a small $\Delta > 0$, let $G(\theta_t\omega)$ denote

$$G(\theta_t\omega) = \frac{1}{\Delta}(\omega(t + \Delta) - \omega(t))$$

which is a stationary stochastic process with a normal distribution. We can also view $G(\theta_t\omega)$ as a discrete version of the white noise. Note that $G(\theta_t\omega)$ is unbounded almost surely. In order to apply theorem A to the Duffing equation, we truncate the value of $|G(\theta_t\omega)|$ by a given $M_0 \gg \Delta^{-2}$. The resulted stochastic process, which we denote as $\mathcal{G}(\theta_t\omega)$, is a truncated discrete version of the white noise. We now study a Duffing equation driven by the bounded stationary stochastic process $\mathcal{G}(\theta_t\omega)$ in the form of

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + (\lambda - \gamma q^2)\frac{\mathrm{d}q}{\mathrm{d}t} - q + q^3 = \mu q^2\mathcal{G}(\theta_t\omega). \tag{1.4}$$

In the unforced case of $\mu = 0$, we know that for every small $\lambda > 0$ there exists a γ_λ for γ so that the unperturbed equation has a homoclinic solution, which we denote as ℓ_λ. Let $\gamma = \gamma_\lambda$ in equation (1.4). We have the following.

Theorem B. *There exists a $\mu_0 > 0$ sufficiently small and a θ_t-invariant subset $\tilde{\Omega} \subset \Omega$ of full Wiener measure such that for all $0 < \mu < \mu_0$ and all $\omega \in \tilde{\Omega}$, the random Poincaré return map $\mathcal{R}$ induced by equation (1.4) around the homoclinic loop ℓ_λ admits a horseshoe of infinitely many branches.*

The chaotic behaviour we have here is a sample-wise property. Theorem B simply states equation (1.4) driven by a bounded random force has a horseshoe almost surely. The almost sure property is due to the ergodicity of the Wiener shift and the Birkhoff ergodic theorem.

Furthermore, we obtain, through the Poincaré return map derived from the forced equation (1.2), much more than what is stated in theorem A. The quantities m^- and M^-

are intrinsic limit values of the Melnikov function $\mathcal{W}(t)$ in the negative t-direction, and m^+ and M^+ are their correspondences in the positive t-direction. We also denote

$$m = \inf_{t\in\mathbb{R}} \mathcal{W}(t), \qquad M = \sup_{t\in\mathbb{R}} \mathcal{W}(t).$$

In principle, any conceivable combination for $m^\pm, M^\pm, m$ and M satisfying $m \leqslant m^- \leqslant M^- \leqslant M$ and $m \leqslant m^+ \leqslant M^+ \leqslant M$ is possible; see the example presented in section 4. On the other hand, if the forcing of equation (1.2) is time periodic or almost periodic, then $m^\pm = m$, $M^\pm = M$. Therefore, the freedom in getting different combinations of $m, m^\pm, M$ and $M^\pm$ is from non-periodic forcing. Associated with this new freedom of choices is a set of new dynamical behaviour that are not permitted in the equations driven by a periodic or almost periodic forcing.

We will see that for a general nonautonomous forcing, unlike a periodic or an almost periodic forcing, the transversal intersections of the stable and unstable manifolds are neither necessary nor sufficient for chaotic dynamics to exist. In some cases, the stable and unstable manifolds intersect but there exists no complicated dynamical behaviour. In some other cases, a horseshoe exists even when the stable and the unstable manifolds are pulled apart completely by the non-periodic forcing. There are also cases in between, where we obtain only half of a horseshoe.

The rest of the results of this paper can be summarized as follows. For comparison, theorem A is included as item (i).

Theorem C. *For each of the items (i)–(v) below, there is a respective $\mu_0 > 0$ so that for all $0 < \mu < \mu_0$, we have*

- *(i)* (Intersection and full horseshoe). *If $m^\pm < 0 < M^\pm$, then $\mathcal{R}$ admits a full horseshoe of infinitely many branches.*
- *(ii)* (Intersection and half horseshoe). *If $M^\pm, m^- > 0$ and $m^+ < 0$ (or $M^\pm, m^+ > 0$ and $m^- < 0$), then $\mathcal{R}$ admits a half horseshoe of infinitely many branches.*
- *(iii)* (Trivial dynamics). *If either $M^+ < 0$ or $M^- < 0$, then $\mathcal{R}$ has trivial dynamics.*
- *(iv)* (Non-intersection and half horseshoe). *If $m^\pm, M^\pm > 0$ and there exist L^+, and sequences $a_n, b_n \to \infty$ with $0 < a_{n+1} - a_n, b_{n+1} - b_n < L^+$ for all $n \geqslant 0$ such that*

$$\lim_{n\to\infty} \mathcal{W}(a_n) = m^+, \qquad \lim_{n\to\infty} \mathcal{W}(b_n) = M^+$$

and

$$M^+ > 2m^+ e^{3\beta k L^+}$$

for an integer $k > 1$. Then $\mathcal{R}$ admits a half horseshoe of k-branches.
- *(v)* (Non-intersection and full horseshoe). *If the conditions in (iv) also hold for m^-, M^-, then $\mathcal{R}$ admits a full horseshoe of k-branches.*
- *(vi)* (Applications). *The above phenomena all appear in a forced Duffing's equation.*

Item (i) is theorem A. It assumes that the unstable manifold W^u and the stable manifold W^s of $(0, 0)$ intersect persistently in both directions of time. Item (ii) assumes persistent intersections of W^u and W^s in only one of the time directions. Item (iii) implies that the intersections of W^u and W^s are not sufficient for complicated dynamics.

Not only the non-empty intersections of W^u and W^s are not sufficient, but also they are not necessary for chaotic dynamics to emerge. This is stated in (iv). The condition $M^+ > 2m^+ e^{3\beta k L^+}$ in (iv) can be achieved in two ways. The first is by making m^+ small. This implies that, even though W^u and W^s are eventually separated as $t \to +\infty$, their persistently getting close would generate enough expansion to create complicated dynamics. The second way is to make L^+ small, namely to have high frequency oscillation of the Melnikov function between m^+ and M^+ as $t \to +\infty$.

Relation to existing literature. The study of complicated dynamics of ordinary differential equations under a periodic forcing has long and rich history that dates back to Poincaré and Birkhoff. The complicated behaviour induced by the presence of homoclinic intersections of the stable and unstable manifolds of a saddle fixed point was first observed by Poincaré [P], described by Birkhoff [B], proved by Smale [Sm1, Sm2] in a geometry form, and was systematically studied by Alekseev [A] with applications to Sitnikov's three body problem [Sit]. Since then, the study of chaotic behaviour has flourished and the literature on this subject is vast, see for example, [Le, M, Shi3, Shi5, AS, Ho, CHM-P, HM, CH, GH, Pa1, BP, SSTC1, SSTC2, BL, G, WO, WOk] and the references therein. There has also been substantial literature on extending the Birkhoff–Smale theorem to quasiperiodically and almost periodically forced differential equations, see [Sch, Pa2, St1, St2, PS, MS, Ya, Shi1, Wi]. For works on chaos on non-smooth ordinary differential equations, see [ARHO, BF1, BF2, F].

There are two basic approaches to obtain a horseshoe for the periodically forced differential equations. The first is a combination of the Melnikov function and the Birkhoff–Smale theorem, see, for example, [GH]. In this case, the Melnikov function is used to detect a homoclinic point in the time-period solution map. The second approach was introduced by Palmer [Pa1, Pa2] and it is based on analytic shadowing. This approach does not rely on Smale's geometric construction. It consists of three steps: (a) to prove, using analytic shadowing, that if there is a countable collection of homoclinic solutions for the perturbed equation satisfying exponential dichotomy, then there is a collection of solutions corresponding to full Bernoulli shift using these homoclinic solutions as generating symbols; (b) to prove that for the perturbed equation, a homoclinic solution satisfies exponential dichotomy if and only if it corresponds to a transversal homoclinic point of the time-period map; (c) to use the Melnikov function to verify the existence of transversal homoclinic points.

The extension of the first approach to quasiperiodically forced equations can be found in [Wi], which is based on an observation that goes back to Shilnikov [Shi1]: if the saddle point is replaced by a normally hyperbolic tori, and the stable and the unstable manifolds of this tori intersect transversally, then Smale's geometric construction of a horseshoe remains valid but each of the point of the horseshoe is replaced by a tori. Palmer's approach is also extended to almost periodically forced equations [Pa2, PS, Sch, MS]. This extension again focuses on using the homoclinic solutions satisfying exponential dichotomy as generating symbols to create solutions of full Bernoulli shift by analytic shadowing.

Lerman and Shilnikov [LS] studied the complicated dynamics for differential equations under time-dependent perturbations. Their approach is similar to Palmer's. They assume that there is a sequence of homoclinic solutions satisfying the exponential dichotomy and a certain uniform properties hold. Then, they constructed solutions corresponding to full Bernoulli shift using these solutions as generating symbols. The result they obtained is a version corresponding to the part (a) in Palmer's approach. Palmer's approach was also extended by Gundlach [Gun] to the random difference equations. Based on this approach, he established a random version of the Birkhoff–Smale theorem. However, it is very difficult to verify the assumptions of [LS, Gun] when considering a given equation, such as the examples presented in theorem B and in section 4 of this paper.

The theory developed in this paper follows the first approach, which is more geometric in flavour. Instead of zooming into the neighbourhood of a collection of homoclinic solutions, as did Palmer's approach, we zoom out, deriving a return map in the extended phase space that catches dynamical activities of all solutions that stay in the neighbourhood of the unforced homoclinic loop. For non-periodic equations, the return maps are no longer defined on compact surfaces and there is no recurrent orbit. Nevertheless, chaotic behaviour can been established for these return maps.

The study of the Poincaré return map around a homoclinic orbit for autonomous differential equations goes back to Shilnikov. In [Shi1, Shi2, Shi3, Shi4, Shi5], Shilnikov developed an approach to estimate the return map through which the chaotic behaviour was obtained. This method was extensively used and extended by others to study the homoclinic bifurcation of autonomous equations, see [Deng1, Deng2, DS]. In [AS], Afraimovich and Shilnikov showed that this return map can also be used to study periodically perturbed equations if the form of the return map is known. A long standing problem is to estimate the Ponicaré return map for equations driven by a time dependent forcing. Recently, the rigorous estimation of the return maps for the periodic forced equation was carried out in [WO, WOk], where rich and complicated dynamics were also obtained.

The rest of this paper is organized as follows. In section 2 we introduce in detail the assumptions, the definitions and the dynamical objects needed, leading to a precise presentation of theorems A and C. In section 3 we study equations driven by a random forcing using theorem A. Theorem B is proved in section 3. In section 4 we apply all theorems stated in section 2 to a Duffing equation driven by non-periodic forces. In section 5 we estimate the Poincaré return map $\mathcal{R}$ and get its leading terms. In section 6 we prove the theorems stated in sections 2 using the estimates of $\mathcal{R}$ obtained in section 5. Two technical propositions used in section 5 are proved in the appendices.

2. Statement of theorems

2.1. Equations of study

Let $(x, y) \in \mathbb{R}^2$ be the phase variables and t be the time. We start with an autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x + f(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \beta y + g(x, y), \tag{2.1}$$

where $f(x, y)$ and $g(x, y)$ are C^N functions for sufficient large N satisfying $f(0, 0) = g(0, 0) = \partial_x f(0, 0) = \partial_y f(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0$. First, we assume the following.

(H1) *α and β satisfy*

(i) the nonresonant conditions up to order N: there are no nonnegative integers m and n with $2 \leqslant m + n \leqslant N$ such that

$$-\alpha = -m\alpha + n\beta \qquad \text{or} \qquad \beta = -m\alpha + n\beta;$$

(ii) $0 < \beta < \alpha$.

A sufficient condition for (H1)(i) is that α and β are rationally independent. (H1)(ii) assumes that the saddle point $(0, 0)$ is dissipative. We also assume that the positive x-side of the local stable manifold and the positive y-side of the local unstable manifolds of $(0, 0)$ are included as a part of a homoclinic solution, which we denote as

$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2,\ t \in \mathbb{R}\}.$$

To the right-hand side of equation (2.1) we add a time-dependent forcing to form a nonautonomous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x + f(x, y) + \mu P(x, y, t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \beta y + g(x, y) + \mu Q(x, y, t), \tag{2.2}$$

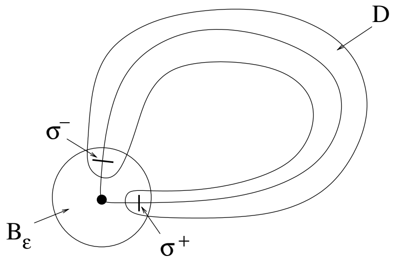

Figure 1. B_ε, D and $\sigma^\pm$.

where μ is a small parameter representing the magnitude of the forcing. Let U be an open neighbourhood in the (x, y)-plane that contains the closure of ℓ, and $\mathcal{U} = U \times \mathbb{R}$.

We also assume the following.

(H2) *the nonautonomous forcing satisfies*

(i) the forcing $P(x, y, t)$, $Q(x, y, t)$ are C^N in the phase variables (x, y) and the C^N-norm of P, Q in (x, y) are uniformly bounded by a constant that is independent of μ on $\mathcal{U}$ for all t;

(ii) $P(x, y, t)$ and $Q(x, y, t)$ are high order terms at $(x, y) = (0, 0)$ for all t. P, Q are C^0 in t.

2.2. Return maps in the extended phase space

We study equation (2.2) on $\mathcal{U} = U \times \mathbb{R}$ in the extended phase space (x, y, t) through the iterations of a return map we now introduce. A small neighbourhood U in the space of (x, y) is constructed by taking the union of a small neighbourhood B_ε (a ball at $(0, 0)$ with radius ε) of $(0, 0)$ and a small neighbourhood D around ℓ outside of $B_{\frac{1}{4}\varepsilon}$. See figure 1. In the extended phase space (x, y, t) we denote

$$\mathcal{B}_\varepsilon = B_\varepsilon \times \mathbb{R}, \qquad \mathcal{D} = D \times \mathbb{R}$$

We also denote

$$\Sigma^\pm = \sigma^\pm \times \mathbb{R},$$

where $\sigma^\pm \in B_\varepsilon \cap D$ are the two segments depicted in figure 1, both perpendicular to the homoclinic solution ℓ. Let $\mathcal{N} : \Sigma^+ \to \Sigma^-$ be the map induced by the solutions on $\mathcal{B}_\varepsilon$ and $\mathcal{M} : \Sigma^- \to \Sigma^+$ be the map induced by the solutions on $\mathcal{D}$. The return map $\mathcal{R} : \Sigma^- \to \Sigma^-$ is obtained by composing $\mathcal{N}$ and $\mathcal{M}$. This is to say that $\mathcal{R} = \mathcal{N} \circ \mathcal{M}$.

2.3. Objects of study

We introduce a characteristic function, which we call the Melnikov function for equation (2.2), as follows: let $(u(t), v(t))$ be the unit tangent vector of ℓ at $\ell(t)$, and let

$$\begin{aligned} E(t) = {} & v^2(t)(-\alpha + \partial_x f(a(t), b(t))) + u^2(t)(\beta + \partial_y g(a(t), b(t))) \\ & - u(t)v(t)(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))). \end{aligned} \tag{2.3}$$

The quantity $E(t)$ measures the rate of expansion of the solutions of equation (2.1) in the direction normal to ℓ at $\ell(t)$. The *Melnikov function* $\mathcal{W}(t)$ for equation (2.2) is defined as

$$\mathcal{W}(t) = \int_{-\infty}^{\infty} (v(s)P(a(s), b(s), s + t) - u(s)Q(a(s), b(s), s + t))\mathrm{e}^{-\int_0^s E(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s. \tag{2.4}$$

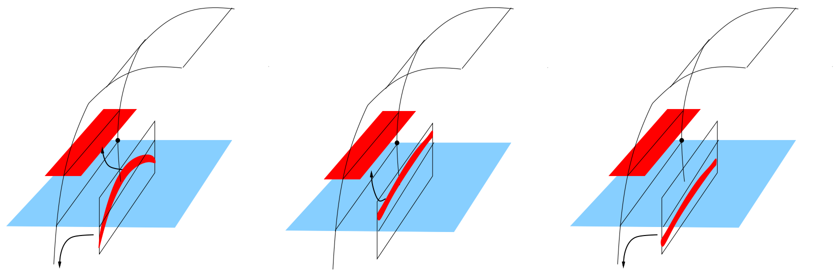

Figure 2. Three cases for the return map $\mathcal{R}$.

Observe that $E(t) \to \beta$ as $t \to +\infty$ and $E(t) \to -\alpha$ as $t \to -\infty$. With the assumption that P and Q are uniformly bounded on $\mathcal{U}$, $\mathcal{W}(t)$ is uniformly bounded for all t. Also observe that, as a normally hyperbolic set, the line $(x, y) = (0, 0)$ in the extended phase space has a two-dimensional unstable manifold which we denote by W^u and a two-dimensional stable manifold which we denote by W^s. Let

$$m = \inf_{t\in\mathbb{R}} \mathcal{W}(t), \qquad M = \sup_{t\in\mathbb{R}} \mathcal{W}(t).$$

We start with homoclinic intersections.

Theorem 2.1.

(a) Assume that $m < 0 < M$. *Then there exists* $\mu_0 > 0$ *sufficiently small so that for all* $0 < \mu < \mu_0$, $W^s \cap W^u \neq \emptyset$.

(b) Assume $m > 0$. *Then there exists* $\mu_0 > 0$ *sufficiently small so that for all* $0 < \mu < \mu_0$, $W^s \cap W^u = \emptyset$.

(c) Assume $M < 0$. *Then there exists* $\mu_0 > 0$ *sufficiently small so that for all* $0 < \mu < \mu_0$, $W^s \cap W^u = \emptyset$.

For case (a), the return map $\mathcal{R} : \Sigma^- \to \Sigma^-$ is only partially defined on Σ^- (see figure 2(a)); for case (b), $\mathcal{R}$ is well defined on Σ^- (figure 2(b)); and for case (c), solutions initiated from Σ^- all leave and there is no direct return to Σ^- (figure 2(c)).

We study the geometrical and dynamical structure of solutions that stay inside of U in phase space (x, y) for all time. Among the three scenarios of theorem 2.1, there is obviously nothing interesting about (c). So we will only consider scenarios (a) and (b). Let W be the subsect of Σ^- on which the return map $\mathcal{R}$ is defined (for scenarios (b), $W = \Sigma^-$). Let

$$\Omega = \{p \in W : \mathcal{R}^n(p) \in W, \text{ for all } n \geqslant 1\}; \qquad \Lambda = \cap_{n\geqslant 1}\mathcal{R}^n(\Omega).$$

The solutions initiated in Ω are all solutions that stay forever in U in forward times and the solutions initiated in Λ are all solutions that stays forever in U in both forward and backward times. Our *ultimate objective* is to understand the geometrical and dynamical structure of Λ for the return map $\mathcal{R}$ derived from equation (2.2).

2.4. Description of chaotic behaviour

To study chaos dynamics for equations driven by a nonautonomous forcing, we start with the periodically perturbed case, for which the return map $\mathcal{R}$ is reduced to an annulus map that resembles the Hénon maps and the dissipative standard maps, to which many established theories can be applied. For the corresponding equations, Λ assumes a variety of complicated structures ranging from horseshoes to strange attractors with SRB measures, see [WO, WOk]. However, if the forcing functions are not periodic in time, then the return maps $\mathcal{R}$ do not admit similar reductions and they are not those studied in the standard dynamical systems theory: these return maps are defined on non-compact surfaces and their orbits are all wandering. We start from defining horseshoes for these maps.

First some geometric terms. We call the direction of t in Σ^- the horizontal direction and the direction of σ^- (transversal to the homoclinic solution ℓ in the original phase space) the vertical direction. In Σ^-, a vertical curve is a non-self-intersecting, continuous curve that connects the two horizontal boundaries of Σ^-. We call a region that is bounded by two non-intersecting vertical curves a *vertical strip*, which we denote as V. The two defining vertical curves for a given vertical strip V are the vertical boundaries of V. We call a non-self-intersecting continuous curve connecting the two vertical boundaries of V a *fully extended* horizontal curve in V. Let V_1, V_2 be two non-intersecting vertical strips. We say that $\mathcal{R}(V_1)$ *crosses* V_2 *horizontally* if for every fully extended horizontal curve h of V_1, there is a subsegment $\tilde{h}$ of h so that $\mathcal{R}(\tilde{h})$ is a fully extended horizontal curve in V_2.

Definition 2.1 (Topological horseshoe). *Let* $\mathcal{R} : W \to \Sigma^-$, $W \subset \Sigma^-$, *be continuous. We say that* $\mathcal{R}$ *admits a topological horseshoe of* k*-branches,* $k < \infty$, *if there exists a bi-infinite sequence of non-intersecting vertical strips* $\{V_n\}_{n=-\infty}^{\infty}$ *lined up monotonically from* $t = -\infty$ *to* $t = +\infty$ *in* Σ^-, $V_n \subset W$ *for all* n, *such that*

(1) for every n, *there exists a* $\hat{n}_1 > n$, *such that* $\mathcal{R}(V_n)$ *crosses* $V_{\hat{n}_1}, V_{\hat{n}_1+1}, \ldots V_{\hat{n}_2+k}$ *horizontally;*

(2) for every n, *there exists a* $\hat{n}_2 < n$, *such that* $\mathcal{R}(V_{\hat{n}_2-k}), \ldots, \mathcal{R}(V_{\hat{n}_2})$ *crosses* V_n *horizontally.*

This definition is a natural extension of the topological horseshoe introduced previously [Mo, MM, BW, S, CM, CKM, KY] (see figure 3). The existence of a topological horseshoe of k-branches of definition 2.1 implies the existence of an invariant set of solutions on which the dynamics are semi-conjugate to the full shift of k symbols. To see this, we prove the following two claims. First, we label all the vertical strips using addresses 1 to k cyclically. Let

$$\boldsymbol{s} = \ldots s_{-1}, s_0; s_1, \ldots$$

be an arbitrary symbolic sequence where $s_i \in \{1, \ldots, k\}$ for all $i \in \mathbb{Z}$. We have

Claim 1. *For any given sequence of* k *symbols* $\boldsymbol{s}$, *and any vertical strip* V *of address* s_0, *there exists a point* $p \in V$, *such that the orbit of* p *is such that* $\mathcal{R}^n(p)$ *is in a vertical strip with address* s_n *for all* $n \in \mathbb{Z}$.

Proof. For a given symbolic sequence $\boldsymbol{s} \ldots s_{-1}, s_0; s_1, \ldots$, let V be a vertical strip of address s_0. Let $\Lambda^+(\boldsymbol{s})$ be all points in V, such that $p \in \Lambda^+(\boldsymbol{s})$ is such that $\mathcal{R}^n(p)$ is in a vertical strip of address s_n for all $n \geqslant 0$. We observe that $\Lambda^+(\boldsymbol{s})$ is a non-empty set. In fact, for any given curve ℓ_h in V that is horizontally crossing, $\ell \cap \Lambda^+(\boldsymbol{s})$ is non-empty. This follows directly from definition 2.1(1).

For the negative direction of time, we define a sequence of vertical strips V_{s_n}, $n \leqslant -1$ by inductively picking $V_{s_{n-1}}$ such that $\mathcal{R}(V_{s_{n-1}})$ horizontally cross V_{s_n}. The existence of $V_{s_{n-1}}$ is ensured by definition 2.1(2) for all $n < 0$. Now for any given $n < 0$, take a curve that is horizontally crossing V_{s_n}. By definition there will be a subsegment ℓ_n, so that $\mathcal{R}^i(\ell_n) \in V_{s_{n+i}}$ for all $0 < i \leqslant -n$. Furthermore, we can make $\mathcal{R}^{-n}(\ell_n)$ cross V_{s_0} horizontally. Denote a point of intersection of $\mathcal{R}^{-n}(\ell_n)$ with $\Lambda^+(\boldsymbol{s})$ by p_n, and let p be an accumulation point of $\{p_n\}$, $n < 0$. p is then a point in V satisfies what is claimed. □

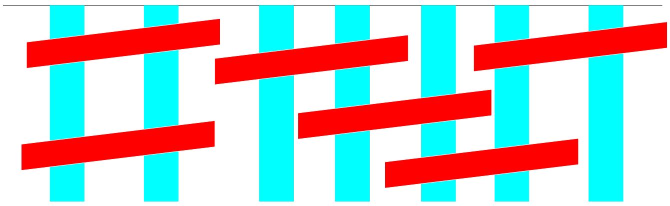

Figure 3. Topological horseshoe in the extended phase space.

Let S be the space of k symbols and $s \in S$. We take a collection of k consecutive vertical strips, which we denote as $V_1, \ldots V_k$, according to their respective addresses. For any given $s \in S$, we obtain a point in $V = \cup_{1 \leqslant i \leqslant k} V_i$, which we denote as $p(s)$. Let

$$\Omega(k) = \cup_{s \in S}\, p(s), \qquad \Omega = \cup_{n \in \mathbb{Z}} \mathcal{R}^n \Omega(k).$$

We now define $h : \Omega \to S$ by letting

$$h(\mathcal{R}^n p(s)) = \sigma^n s.$$

where $\sigma : S \to S$ is the shift operator.

Claim 2. *h is a semi-conjugation of $\mathcal{R}$ on Ω to the shift operation σ on S.*

Proof. h is continuous because $\mathcal{R}$ is continuous. It is surjective from claim 1. For $q \in \Omega$, let $m \in \mathbb{Z}$, $s \in S$, $p \in V$ be such that $q = \mathcal{R}^m(p(s))$. We have

$$\sigma h(q) = \sigma h(\mathcal{R}^m(p(s)) = \sigma^{m+1} s = h(\mathcal{R}^{m+1} p(s)) = h(\mathcal{R} q).$$

This proves that h is a semi-conjugation of $\mathcal{R}$ on Ω and σ on S. □

Remark. We offer the following additional remarks on the dynamics of topological horseshoe as defined in definition 2.1.

(i) (*For time-periodic perturbation*). Assume that $P(x, y, t)$, $Q(x, y, t)$ are time periodic of period T. Then Σ^- is reduced to an annulus by a quotient in the t direction by $\{nT\}$, which we denote as $\mathcal{A}$. Smale's horseshoe on $\mathcal{A}$ for $\mathcal{R}$ is properly defined if there exist two vertical strips in $\mathcal{A}$ such that their respective images under $\mathcal{R}$ horizontally cross both vertical strips. The correspondence of the classical Birkhoff–Melnikov–Smale ensures that, under the assumption that $m < 0 < M$, $\mathcal{R}$ admits a Smale horseshoe on $\mathcal{A}$, from which it follows that there exists a subset of $\mathcal{A}$, the dynamics of which are surjectively projected to a full shift of two symbols.

To view this classical horseshoe in the light of the original physical variables, in particularly, in the direction of time in the extended phase space, let us reverse the process of quotient in the t-direction. Then $\mathcal{A}$ gets back to become Σ^-, and the *two* vertical strips in $\mathcal{A}$ become infinitely many vertical strips in Σ^-. These vertical strips are *periodically spaced* in the t-direction and the geometric conditions for Smale's horseshoe are exactly (1) and (2) in definition 2.1. To demonstrate the symbolic dynamics, one labels all the vertical strips alternatively using the symbols one and two. The solutions are allowed to jump from a given address to any one of the two addresses at the next return to Σ^-.

For $P(x, y, t)$, $Q(x, y, t)$ that are not time periodic, quotient in the t-direction is not allowed so we start with the geometric picture presented in the second paragraph in (i) above. This is to say that we start with an infinitely sequence of vertical strips in Σ^-. The only difference is that this time the vertical strips are *not* necessarily spaced periodically in the t direction.

Observe that the topological horseshoe of definition 2.1 and the *geometrical and dynamical structures* accompanied are identical in the periodic and non-periodic cases. The only difference is that, in periodic case, all vertical strips of the same address are mapped identically to each other through a quotient in the t direction. This is not an essential character as far as the geometry and the dynamics of the corresponding solutions are concerned, and it is the one removed for the non-periodic equations.

(ii) (*Chaotic behaviour of solutions*). The geometric structure of the classical Smale horseshoe and the ones now defined by definition 2.1, as well as the accompanied symbolic dynamics, are technical ways in describing a certain chaotic behaviour of the solutions of given differential equations. We have described the actual dynamical behaviour reflected in these constructions in the paragraph following theorem A in section 1 for $k = \infty$. The chaotic behaviour represented by a horseshoe of k-branches for $k < \infty$ is similar but slightly less dramatic. Instead of the infinitely many independent choices on the length of time in going around the homcolinic loop, we are now given k independent choices. This *intrinsic dynamical description* of chaotic behaviour, as it has turned out, is the same for the solutions of classical horseshoe of Birkhoff–Melnikov–Smale and the ones from definition 2.1.

We are ready to state theorem A in precise terms. Let $\mathcal{W}(t)$ be the Melnikov function in (2.3). Denote

$$m^{\pm} = \liminf_{t\to\pm\infty} \mathcal{W}(t); \qquad M^{\pm} = \limsup_{t\to\pm\infty} \mathcal{W}(t).$$

Theorem 2.2. *Assume that*

$$m^{-}, m^{+} < 0 < M^{-}, M^{+}.$$

Then there exists a $\mu_0 > 0$ *sufficiently small so that for all* $0 < \mu < \mu_0$*, the return map* $\mathcal{R}$ *for equation (2.2) admits a topological horseshoe of infinitely many branches, meaning that for every integer* $k > 1$*, there exists a collection of solutions, the dynamics of which is semi-conjugate to the full shift of* k *symbols.*

Since we are aiming applications to equations forced by sample paths driven by stochastic processes, for which the forcing is usually not differentiable with respect to time, all our theorems (including theorem 2.2) are formulated without the assumption that the Melnikov function is differentiable. The condition of theorem 2.2 implies the existence of infinitely many intersections of the stable and unstable manifold that are topologically transversal, which is consistent with the assumptions in traditional extensions of the classical Birkhoff–Melnikov–Smale for periodically perturbed equations. If one assumes that the forcing function is differentiable in t, then one could replace the assumption of topological transversality of theorem 2.2 by the assumption that there are infinitely many zeros of $\mathcal{W}(t)$, at which the magnitude of the derivatives are uniformly bounded from below by a constant c_0. This assumption is readily to be verified in the analysis of a given set of equations (see the example of section 4). One could then define invariant cones and carry out the corresponding symbolic dynamics in a way that is completely parallel to the classical Birkhoff–Melnikov–Smale theory.

We remark that, for theorem 2.2 to hold, there is no need to impose restrictions on the spacing of the zeros of $\mathcal{W}(t)$ in the t direction. This is because $\mathcal{R}$ is singular at the intersections of stable and unstable manifold, creating infinite expansions in the t-direction that will allow the images of horizontal curves to cross gaps of arbitrarily length (see the proof of theorem 2.2). Restrictions on the spacing of zeros, however, are required in the case where the stable and unstable manifolds do not intersect. See definition 2.4 and theorem 2.5.

Finally, we note that when the theorems of this paper are applied to the periodically perturbed equations, it is necessary to rule out the case that the quotient in the time direction results only *one* vertical strip (the case where the crossing number is one in the terminology of [KY]). This is easily ruled out because the domain of $\mathcal{R}$ must have two vertical boundaries on Σ^- in one period (see [WOk] and the proof of theorem 2.2).

2.5. Other dynamical scenarios

In this paragraph we present in precise terms the rest of scenarios listed in theorem C. In what follows we assume $m, M, m^{\pm}, M^{\pm} \neq 0$. If the forcing functions are almost time periodic, then so are the Melnikov functions $\mathcal{W}(t)$; and this would imply $M = M^{\pm}, m = m^{\pm}$. So for almost periodic equations, non-empty intersections of W^s and W^u would imply the existence of complicated dynamics structures in Λ. This is not the case in general. If $\mathcal{W}(t)$ is such that

$$M^+, M^- < 0 < M.$$

Then $W^u \cap W^s$ is not empty by theorem 2.1(a). In this case, however, the return map $\mathcal{R}$ is only defined on a subset of Σ^- on which $|t|$ is bounded, and Λ is an empty set. As a matter of fact, either $M^- < 0$ or $M^+ < 0$ would cause Λ to be empty. To obtain non-trivial dynamics, we now assume

$$M^-, M^+ > 0. \tag{2.5}$$

Excluding the case $m^-, m^+ < 0$ covered by theorem 2.2, there are three remaining possibilities:

(i) $m^- > 0,\ m^+ < 0$;
(ii) $m^- < 0,\ m^+ > 0$ and
(iii) $m^-, m^+ > 0$.

We now consider these cases separately.

Case (i) $m^- > 0,\ m^+ < 0$. In this case, $\mathcal{R}$ admits a half horseshoe satisfying the following.

Definition 2.2 (Half Horseshoe of type I). *Let $\{V_n\}$ be the same as in definition 2.1. We say that $\mathcal{R}$ admits a half horseshoe of k-branches if*

(1) for every $m > 0$, there exists an $n > m$, such that $\mathcal{R}(V_m)$ crosses $V_n, , \ldots V_{n+k}$ horizontally;
(2) for all m, there exists $\hat{n} < m$, such that $\mathcal{R}(V_{\hat{n}})$ crosses V_m horizontally.

Theorem 2.3. *Assume*

$$m^+ < 0 < m^-, M^{\pm}.$$

Then, there exists a $\mu_0 > 0$ sufficiently small so that for all $0 < \mu < \mu_0$, the return map $\mathcal{R}$ in the extended phase space admits a half horseshoe of infinitely many branches of definition 2.2.

Case (ii) $m^- < 0,\ m^+ > 0$. In this case, we also have a half horseshoe but we need a slightly different definition for it. Instead of the vertical strips $\{V_n\}$, n from $-\infty$ to $+\infty$ in previous definitions, we now assume that n is from $-\infty$ to 0, with $V_0 = \{(t, z) \in \Sigma^- :\in [t_0, +\infty)\}$.

In particular, $\mathcal{R}$ is well defined on V_0. The new definition is as follows.

Definition 2.3 (Half Horseshoe of type II). *Let* $\{V_n\}_{n=-\infty}^{0}$ *be as in the above. We say that* $\mathcal{R}$ *admits a half horseshoe of* k*-branches if*

(1) for every $n < 0$*, there exists an* $m < n - k$*, so that* $\mathcal{R}(V_m), \cdots \mathcal{R}(V_{m+k})$ *crosses* V_n *horizontally;*
(2) there exists an $n < 0$ *so that* $\mathcal{R}(V_n) \subset V_0$*.*

We have the following.

Theorem 2.4. *Assume*

$$m^- < 0 < m^+, M^\pm.$$

Then, there exists a $\mu_0 > 0$ *sufficiently small so that for all* $0 < \mu < \mu_0$*, the return map* $\mathcal{R}$ *admits a half horseshoe of infinitely many branches of definition 2.3.*

Case (iii) $m^+, m^- > 0$. Complicated dynamical structures, in principle, are caused by expansions imposed by the solutions of equation (2.2) in the extended phase space. So far in the above, the expansions responsible for chaos are created by persistent intersections of W^u and W^s. This property is now ruled out by $m^+, m^- > 0$. In this case the structure of Λ could also be complicated. We need a technical phrase before precisely stating our next theorem. We know that there exists a sequence $a_n \to +\infty$ so that $\mathcal{W}(a_n) \to M^+$.

Definition 2.4. *Let* $L^+ > 0$ *be a constant. We say that* M^+ *is* densely approached by an L^+-sequence *if there exists a monotone sequence* $a_n \to +\infty$ *satisfying* $a_{n+1} - a_n < L^+$ *for all* n*, so that* $\mathcal{W}(a_n) \to M^+$*.*

Corresponding definitions for m^+, M^- and m^- are similar.

Theorem 2.5. *Assume that* $M^\pm, m^\pm > 0$*. If both* M^+ *and* m^+ *are densely approached by* L^+*-sequences and* m^+*,* M^+ *and* L^+ *satisfy*

$$M^+ > 2m^+ e^{3\beta k L^+}, \tag{2.6}$$

then there exists $\mu_0 > 0$ *so that for all* $0 < \mu < \mu_0$*,* $\mathcal{R}$ *admits a half horseshoe of* k*-branches of definition 2.2 in* Λ*.*

For inequality (2.6) to hold we must have $M^+ > 2m^+$ so the Melnikov function must oscillate with a non-trivial amplitude as $t \to +\infty$. This inequality is then achieved in two ways. The first is by making m^+ small. This implies that, even though W^u and W^s are eventually separated as $t \to +\infty$, their persistently getting close would generate enough expansion to create complicated structure in Λ. The second way to make inequality (2.6) hold is to make L^+ small. This is to have high frequency oscillations of the Melnikov function between m^+ and M^+ as $t \to +\infty$.

We also have slightly different versions of theorem 2.5 according to various possibilities for $\mathcal{W}(t)$ as $t \to \pm\infty$. If we assume that m^- and M^- are also densely approached by L^--sequences and

$$M^- > 2m^- e^{3\beta k L^-},$$

then $\mathcal{R}$ would admit half horseshoe of definition 2.3. If similar assumptions are made for $\mathcal{W}(t)$ in both directions of time, then $\mathcal{R}$ would admit a full horseshoe of k-branch. Full horseshoe is also admitted if we assume $m^- < 0 < M^-$ instead of $m^- > 0$ in theorem 2.5; and so on.

3. Chaotic behaviour driven by random forcing

In this section we study chaotic dynamics of differential equations driven by a stationary stochastic process by applying theorem 2.2.

3.1. Chaos in equations perturbed by a stationary stochastic process

Let equation (2.1) be the same as before. Let θ_t be a measurable $\mathbb{P}$-measure preserving flow on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\theta_t \circ \theta_\tau = \theta_{t+\tau} \quad \text{for } t,\ \tau \in \mathbb{R}, \qquad \theta_0 = \mathrm{id}_\Omega.$$

The probability measure $\mathbb{P}$ is assumed to be ergodic and invariant with respect to the flow θ_t. A typical example is the Wiener shift which models the evolution of the white noise [Ar].

To the right-hand side of equation (2.1) we add a random forcing to have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x + f(x, y) + \mu P(x, y, \theta_t\omega), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \beta y + g(x, y) + \mu Q(x, y, \theta_t\omega), \tag{3.1}$$

where μ is a small parameter representing the magnitude of the forcing. We assume the following.

Assumption on the random forcing functions. The functions $P(x, y, \omega)$, $Q(x, y, \omega)$ are C^N in $(x, y) \in U$, measurable in $\omega \in \Omega$ and satisfy

(i) for all $\omega \in \Omega$, $P(0, 0, \omega) = Q(0, 0, \omega) = \partial_x P(0, 0, \omega) = \partial_x Q(0, 0, \omega) = \partial_y P(0, 0, \omega) = \partial_y Q(0, 0, \omega) = 0$;

(ii) for each $(x, y) \in U$ and almost every $\omega \in \Omega$, $P(x, y, \theta_t\omega)$, $Q(x, y, \theta_t\omega)$ as functions of t are continuous, and there exists $M_0 < \infty$, independent of μ and ω, so that

$$|P(x, y, \theta_t\omega)|_{C^N(x,y),\ C^0(x,y,t)}, \qquad |Q(x, y, \theta_t\omega)|_{C^N(x,y),\ C^0(x,y,t)} < M_0.$$

The random forcing we have here is a stationary stochastic process that is called a real noise [Ar] and is degenerated at $(x, y) = (0, 0)$.

We define the *random Melnikov variable* $\mathcal{W}(\omega)$ for $\omega \in \Omega$ by letting

$$\mathcal{W}(\omega) = \int_{-\infty}^{\infty} (v(s)P(a(s), b(s), \theta_s\omega) - u(s)Q(a(s), b(s), \theta_s\omega))\mathrm{e}^{-\int_0^s E(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s. \tag{3.2}$$

Observe that $\mathcal{W}(\omega)$ is well defined because $E(t) \to \beta$ as $t \to +\infty$ and $E(t) \to -\alpha$ as $t \to -\infty$. We have

Theorem 3.1. *If there are two subsets $\Omega_1,\ \Omega_2 \subset \Omega$ of positive measure such that $\mathcal{W}(\omega) > 0$ on Ω_1 and $\mathcal{W}(\omega) < 0$ on Ω_2. Then there exists $\mu_0 > 0$ such that for almost all $\omega \in \Omega$ and all $0 < \mu < \mu_0$, the return map $\mathcal{R}$ induced by equation (3.1) admits a horseshoe of infinitely many branches.*

Proof. Using theorem 2.2, it suffices to verify that there exists a $\mu_0 > 0$ so that, for all $0 < \mu < \mu_0$ and almost all $\omega \in \Omega$,

$$m^{\pm} < 0 < M^{\pm}, \tag{3.3}$$

where

$$M^{\pm} = \limsup_{t\to\pm\infty} \mathcal{W}(\theta_t\omega), \qquad m^{\pm} = \liminf_{t\to\pm\infty} \mathcal{W}(\theta_t\omega).$$

Let Ω_1 and Ω_2 be two subsets of Ω with positive measure such that $\mathcal{W}(\omega) > 0$ on Ω_1 and $\mathcal{W}(\omega) < 0$ on Ω_2. Without loss of generality, let us assume that $\inf_{\omega\in\Omega_1} \mathcal{W}(\omega) = M_0 > 0$

and $\sup_{\omega\in\Omega_2} \mathcal{W}(\omega) = m_0 < 0$. Let $\Xi_{\Omega_1}(\omega)$, $\Xi_{\Omega_2}(\omega)$ be the characteristic function of Ω_1 and Ω_2, respectively. This is to say that $\Xi_{\Omega_1}(\omega) = 1$ if $\omega \in \Omega_1$ and $= 0$ otherwise and $\Xi_{\Omega_2}(\omega)$ is similar. Recall that θ_t is ergodic. This implies that

$$\lim_{T\to\pm\infty} \frac{1}{T} \int_0^T \Xi_{\Omega_i}(\theta_t\omega)\,\mathrm{d}t = \mathbb{P}(\Omega_i) > 0 \tag{3.4}$$

for almost all $\omega \in \Omega$, $i = 1, 2$. Let $\tilde{\Omega}$ be the full measure set in Ω satisfying (3.4), and let $\omega \in \tilde{\Omega}$. From (3.4), it follows that there exists a sequence $t_n \to +\infty$ so that

$$\Xi_{\Omega_1}(\theta_{t_n}\omega) \neq 0,$$

which implies

$$\mathcal{W}(\theta_{t_n}\omega) > M_0.$$

Therefore we obtain

$$\limsup_{t\to+\infty} \mathcal{W}(\theta_t\omega) > M_0 > 0.$$

Similarly we also obtain

$$M^- > M_0, \qquad m^-,\ m^+ < m_0.$$

This verifies (3.3). □

The chaotic behaviour we obtain here is a sample-wise property. The theorem states the randomly forced equation (3.1) has a horseshoe almost surely. Let

$$\mathbb{E}(\mathcal{W}) = \int_{\omega\in\Omega} \mathcal{W}(\omega)\,\mathrm{d}\mathbb{P}, \qquad \mathbb{V}(\mathcal{W}) = \int_{\omega\in\Omega} (\mathcal{W}(\omega) - \mathbb{E}(\mathcal{W}))^2\,\mathrm{d}\mathbb{P}$$

be the expectation and the variance of the random variable $\mathcal{W}(\omega)$. The next proposition provides a way to verify the conditions of theorem 3.1 and its proof is straightforward.

Proposition 3.1. *There are two subsets $\Omega_1,\ \Omega_2 \subset \Omega$ of positive measure such that $\mathcal{W}(\omega) > 0$ on Ω_1 and $\mathcal{W}(\omega) < 0$ on Ω_2 if*

$$\mathbb{E}(\mathcal{W}) = 0, \qquad \mathbb{V}(\mathcal{W}) \neq 0.$$

3.2. Two examples

In this subsection we apply theorem 3.1 first to a Duffing equation driven by a quasiperiodic forcing, then to a Duffing equation driven by a bounded stationary stochastic process given by the increment of a Brownian motion. We start with the autonomous Duffing equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + (\lambda - \gamma q^2)\frac{\mathrm{d}q}{\mathrm{d}t} - q + q^3 = 0, \tag{3.5}$$

where $\lambda > 0$ and γ are parameters. By letting $p = \mathrm{d}q/\mathrm{d}t$, we may write equation (3.5) as a system of first order equations in terms of p and q. We first borrow a result on (3.5) from [HR].

Proposition 3.2 (Dissipative homoclinic saddle). *There exists $\lambda_0 > 0$ sufficiently small, such that for $\lambda \in [0, \lambda_0)$, there exists a γ_λ, $|\gamma_\lambda| < 10\lambda$ such that for $\gamma = \gamma_\lambda$*

(i) equation (3.5) has a homoclinic solution to $(q, p) = (0, 0)$, which we denote as

$$\ell_\lambda = \{\ell_\lambda(t) = (q_\lambda(t), p_\lambda(t)),\ t \in \mathbb{R}\};$$

(ii) for any given $L > 0$*, there exists a* $K(L)$ *independent of* λ*, such that for all* $t \in [-L, L]$*,*

$$|\ell_\lambda(t) - \ell_0(t)| < K(L)\lambda,$$

where $\ell_0(t) = (q_0(t), p_0(t))$*,*

$$q_0(t) = \frac{2\sqrt{2}\mathrm{e}^t}{(1+\mathrm{e}^{2t})}, \qquad p_0(t) = \frac{2\sqrt{2}(\mathrm{e}^t - \mathrm{e}^{3t})}{(1+\mathrm{e}^{2t})^2},$$

is a homoclinic solution of equation (3.5) for $\lambda = \gamma = 0$*.*

Let

$$\alpha = \tfrac{1}{2}(\sqrt{\lambda^2+4}+\lambda), \qquad \beta = \tfrac{1}{2}(\sqrt{\lambda^2+4}-\lambda). \tag{3.6}$$

Then $-\alpha, \beta$ are the two eigenvalues of the linearized part at $(0, 0)$. Since $-\alpha + \beta = -\lambda < 0$, $(0, 0)$ is a dissipative saddle provided that $\lambda > 0$. We fix λ throughout of this section and assume that λ is such that α and β are rationally independent. Let γ_λ be as in proposition 3.2.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ be the metric dynamical system introduced in section 3.1. Let $\mathcal{G} : \Omega \to \mathbb{R}$ be a measurable function. Assume that, for almost every $\omega \in \Omega$, $\mathcal{G}(\theta_t\omega)$ as a function of t is continuous, and there exists $K_0 < \infty$ so that $|\mathcal{G}(\theta_t\omega)| < K_0$ almost surely. We add $\mu q^2\mathcal{G}(\theta_t\omega)$ to the right of equation (3.5) to obtain a randomly forced Duffing equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + (\lambda - \gamma_\lambda q^2)\frac{\mathrm{d}q}{\mathrm{d}t} - q + q^3 = \mu q^2\mathcal{G}(\theta_t\omega). \tag{3.7}$$

We first rewrite equation (3.7) as

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= p, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -(\lambda - \gamma_\lambda q^2)p + q - q^3 + \mu q^2\mathcal{G}(\theta_t). \end{aligned} \tag{3.8}$$

To write the linear part of equation (3.8) in a canonical form, we introduce new variables (x, y) so that

$$q = x + \alpha y, \qquad p = -\alpha x + y, \tag{3.9}$$

where $\alpha = \frac{1}{2}(\lambda + \sqrt{\lambda^2+4})$ is as in (3.6). In reverse we have

$$x = \frac{1}{1+\alpha^2}(q - \alpha p) \qquad y = \frac{1}{1+\alpha^2}(\alpha q + p). \tag{3.10}$$

The new equations in (x, y) are

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\alpha x + f(x, y) + \mu C(x, y)\mathcal{G}(\theta_t\omega), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \beta y + g(x, y) + \mu D(x, y)\mathcal{G}(\theta_t\omega), \end{aligned} \tag{3.11}$$

where $\beta = \alpha^{-1}$ is again as in (3.6), and

$$\begin{aligned} f(x, y) &= \frac{\alpha}{1+\alpha^2}\left(\gamma_\lambda(x+\alpha y)^2(y - \alpha x) + (x+\alpha y)^3\right), \\ g(x, y) &= \frac{-1}{1+\alpha^2}\left(\gamma_\lambda(x+\alpha y)^2(y - \alpha x) + (x+\alpha y)^3\right), \\ C(x, y) &= \frac{\alpha}{1+\alpha^2}(x+\alpha y)^2, \qquad D(x, y) = \frac{-1}{1+\alpha^2}(x+\alpha y)^2. \end{aligned} \tag{3.12}$$

Let ℓ_λ be the homoclinic solution of equation (3.5) from proposition 3.1. In the coordinate (x, y), let us write $\ell_\lambda(t) = (a_\lambda(t), b_\lambda(t))$. Let $(u_\lambda(t), v_\lambda(t))$ be the unit tangent vector of ℓ_λ at $\ell_\lambda(t)$, and

$$\begin{aligned} E_\lambda(t) = {} & v_\lambda^2(t)(-\alpha + \partial_x f(a_\lambda(t), b_\lambda(t))) + u_\lambda^2(t)(\beta + \partial_y g(a_\lambda(t), b_\lambda(t))) \\ & - u_\lambda(t)v_\lambda(t)(\partial_y f(a_\lambda(t), b_\lambda(t)) + \partial_x g(a_\lambda(t), b_\lambda(t))). \end{aligned} \tag{3.13}$$

Then the random Melnikov function for equation (3.8) is

$$\mathcal{W}(\omega) = \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s))\mathcal{G}(\Theta_s\omega)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s, \tag{3.14}$$

in which $C_\lambda(t) = C(a_\lambda(t), b_\lambda(t))$ and $D_\lambda(t)$ is similar.

Denote

$$F(s) = (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s))\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}.$$

We write $\mathcal{W}(\omega)$ as

$$\mathcal{W}(\omega) = \int_{-\infty}^{\infty} F(s)\mathcal{G}(\Theta_s\omega)\,\mathrm{d}s.$$

We consider two examples.

Example 1 (Quasiperiodic forcing). Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus, $\mathcal{F}$ be the Boreal σ-algebra on $\mathbb{T}^n$, and $\mathbb{P}$ be the Haar measure. Then, $(\mathbb{T}^n, \mathcal{F}, \mathbb{P})$ is a probability space. We fix $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and assume that $\lambda_1, \ldots, \lambda_n$ are rationally independent. Consider the measurable flow $\theta_t : \mathbb{T}^n \times \mathbb{R} \to \mathbb{T}^n$ given by

$$\theta_t\omega = \omega + \lambda t = (\omega_1 + \lambda_1 t, \ldots, \omega_n + \lambda_n t). \tag{3.15}$$

The Haar measure $\mathbb{P}$ is an ergodic invariant measure for θ_t.

For $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $G(X) = G(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and assume that $G(x_1, \ldots, x_n)$ is periodic of period one in x_i, $i = 1, \ldots, n$. Then, G induces a continuous function on $\mathbb{T}^n$, which we denote as $\mathcal{G} : \mathbb{T}^n \to \mathbb{R}$. We assume in addition that the mean of $\mathcal{G}$ is zero, i.e.

$$\int_{[0,1]^n} G(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n = 0. \tag{3.16}$$

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, denote

$$f_m = \int_{-\infty}^{\infty} F(s)\mathrm{e}^{-(m\cdot\lambda s)i}\mathrm{d}s, \qquad g_m = \int_{\omega\in[0,1]^n} G(\omega)\mathrm{e}^{-(m\cdot\omega)i}\,\mathrm{d}\omega.$$

Note that $F(s)$ decays exponentially as $t \to \pm\infty$. The Fourier coefficient f_m is well defined. Clearly, g_m is well defined since G is a continuous function.

Proposition 3.3. *Let $(\mathbb{T}^n, \mathcal{F}, \mathbb{P}, \theta_t)$ and $\mathcal{G} : \mathbb{T}^n \to \mathbb{R}$ be as in the above. If there exists an $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $f_m \cdot g_m \neq 0$, then there exists a constant $\mu_0 > 0$ such that for all $\omega \in \mathbb{T}^n$ and all $0 < \mu < \mu_0$, the return map $\mathcal{R}$ induced by equation (3.7) around the homoclinic loop ℓ_λ admits a topological horseshoe of infinitely many branches.*

Proof. First we verify the assumptions (H) in section 2: (H)(ii) follows from (3.6) and (H)(i) is from the assumption that λ is such that $-\alpha$ and β are rationally independent. We also verify the assumptions (i) and (ii) for the forcing functions in section 3.1: (i) is straightforward from (3.15) and (ii) holds due to the periodicity and the continuity of $G(X)$. To apply theorem 3.1 we observe that

$$\mathcal{W}(\omega) = \int_{-\infty}^{+\infty} F(s)G(\omega + \lambda s)\,\mathrm{d}s.$$

We verify the assumptions of theorem 3.1 using proposition 3.1. For the expectation $\mathbb{E}(\mathcal{W}(\omega))$ we have

$$\mathbb{E}(\mathcal{W}(\omega)) = \int_{[0,1]^n}\int_{-\infty}^{+\infty} F(s)G(\omega + \theta s)\,\mathrm{d}s\,\mathrm{d}\omega = \int_{-\infty}^{+\infty} F(s)\,\mathrm{d}s \cdot \int_{[0,1]^n} G(\omega)\,\mathrm{d}\omega = 0.$$

Here we used (3.16) to obtain the last equality. To compute the variance $\mathbb{V}(\mathcal{W}(\omega))$ we start with

$$\mathbb{V}(\mathcal{W}(\omega)) = \int_{\omega\in[0,1]^n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(s)F(t)G(\omega+\theta s)G(\omega+\theta t)\,\mathrm{d}s\mathrm{d}t\,\mathrm{d}\omega.$$

We expand $G(\omega)$ in terms of Fourier series, then use the orthogonality of triangular functions to obtain

$$\mathbb{V}(\mathcal{W}(\omega)) = \frac{1}{2}\sum_{m\in\mathbb{Z}^n} |f_m|^2|g_m|^2.$$

Under the assumption that there exists an $m \in \mathbb{Z}^n$ such that $f_m \cdot g_m \neq 0$, we have

$$\mathbb{V}(\mathcal{W}(\omega)) > 0.$$

Therefore, by proposition 3.1, the assumptions of theorem 3.1 hold. Note that our claim is for all $\omega \in \mathbb{T}^n$ instead of for almost all $\omega \in \mathbb{T}^n$. This is because all orbits of θ_t are typical as far as the ergodicity of θ_t is concerned. □

Remark. The condition $f_m \cdot g_m \neq 0$ for an $m \in \mathbb{Z}^n$ is rather weak. It requires that the periodic forcing function $G(X)$ does not miss *all* Fourier spectrum of $F(s)$. For a given $G(X)$, this condition is explicitly verifiable. One can first compute the integrals for $\lambda = 0$, then pass the conclusion to $\lambda > 0$ using the continuity of the integrals with respect to λ.

Example 2 (Randomly forced Duffing equation). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classic Wiener probability space, where

$$\Omega = C_0(\mathbb{R},\mathbb{R}) = \left\{\omega(t)\ :\ \omega(\cdot) : \mathbb{R}\to\mathbb{R}\ \text{ is continuous and }\ \omega(0) = 0\right\}$$

has the open compact topology so that Ω is a Polish space, $\mathcal{F}$ is its Borel σ-algebra and $\mathbb{P}$ is the Wiener measure. The Brownian motion takes the form $B_t(\omega) = \omega(t)$. We consider the Wiener shift θ_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is given by

$$\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t). \tag{3.17}$$

It is well known that $\mathbb{P}$ is an ergodic invariant measure for θ_t.

Let $M_0 > 0$ be sufficiently large and $\Delta > 0$ be small. Both M_0 and Δ are fixed below. Let $G : \mathbb{R} \to \mathbb{R}$ be a continuous function defined by

$$G(x) = \begin{cases} \dfrac{M_0}{\Delta} & \text{if } x > M_0,\\ \dfrac{x}{\Delta} & \text{if } |x| \leqslant M_0,\\ -\dfrac{M_0}{\Delta} & \text{if } x < -M_0. \end{cases}$$

Let $\phi(\theta_t\omega)$ denote the stationary stochastic process given by the increment of a Brownian motion

$$\phi(\theta_t\omega) = \omega(t+\Delta) - \omega(t).$$

We consider the stationary stochastic process given by

$$\mathcal{G}(\theta_t\omega) = G(\phi(\theta_t\omega)).$$

We have

$$\mathcal{G}(\theta_t\omega) = \begin{cases} \dfrac{1}{\Delta}M_0 & \text{if } \omega(t+\Delta)-\omega(t) > M_0,\\ \dfrac{1}{\Delta}(\omega(t+\Delta)-\omega(t)) & \text{if } |\omega(t+\Delta)-\omega(t)| \leqslant M_0,\\ -\dfrac{1}{\Delta}M_0 & \text{if } \omega(t+\Delta)-\omega(t) < -M_0. \end{cases} \tag{3.18}$$

We can view $\mathcal{G}(\theta_t\omega)$ as a truncated discrete version of the white noise.

Proposition 3.4. *Let $\Delta > 0$ be small and $M_0 \gg \Delta^{-2}$ be large. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$, $\mathcal{G}(\omega)$ be as in the above. Then there exists a $\mu_0 > 0$ so that for almost all $\omega \in \Omega$ and all $0 < \mu < \mu_0$, the return map $\mathcal{R}$ induced by equation (3.7) around the homoclinic loop ℓ_λ admits a horseshoe of infinitely many branches.*

Proof. We again verify the assumptions of theorem 3.1 through proposition 3.1. For $\mathbb{E}(\mathcal{W})$ we have

$$\begin{aligned}\mathbb{E}(\mathcal{W}) &= \int_\Omega \int_{-\infty}^{+\infty} F(s)\mathcal{G}(\theta_s\omega)\,\mathrm{d}s\,\mathrm{d}\mathbb{P} = \int_{-\infty}^{+\infty} F(s)\left\{\int_\Omega \mathcal{G}(\theta_s\omega)\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s \\ &= \frac{1}{\Delta}\int_{-\infty}^{+\infty} F(s)\,\mathrm{d}s \cdot \frac{1}{\sqrt{2\pi\Delta}}\left\{\int_{-M_0}^{M_0} y\mathrm{e}^{-\frac{y^2}{2\Delta}}\,\mathrm{d}y + \int_{M_0}^{\infty} M_0\mathrm{e}^{-\frac{y^2}{2\Delta}}\,\mathrm{d}y - \int_{-\infty}^{-M_0} M_0\mathrm{e}^{-\frac{y^2}{2\Delta}}\,\mathrm{d}y\right\} \\ &= 0.\end{aligned}$$

We now compute $\mathbb{V}(\mathcal{W})$. Let $(s, t) \in \mathbb{R}^2$ be fixed and denote

$$\begin{aligned}&X_1 = \omega(s+\Delta) - \omega(s), \qquad X_2 = \omega(t+\Delta) - \omega(t), \\ &\Omega(s,t) = \{\omega \in \Omega,\ |X_1| < M_0,\ \text{and } |X_2| < M_0\}.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{V}(\mathcal{W}) &= \int_\Omega \int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\mathcal{G}(\theta_t\omega)\mathcal{G}(\theta_s\omega)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\mathbb{P} \\ &= \int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\left\{\int_\Omega \mathcal{G}(\theta_t\omega)\mathcal{G}(\theta_s\omega)\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\ &= \mathbb{V}_1(\mathcal{W}) + \mathbb{V}_2(\mathcal{W}),\end{aligned}$$

where

$$\begin{aligned}\mathbb{V}_1(\mathcal{W}) &= \frac{1}{\Delta^2}\int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\left\{\int_\Omega X_1X_2\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t, \\ \mathbb{V}_2(\mathcal{W}) &= \int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\left\{\int_{\Omega\setminus\Omega(s,t)} (\mathcal{G}(\theta_t\omega)\mathcal{G}(\theta_s\omega) - \frac{1}{\Delta^2}X_1X_2)\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

We write

$$\omega(s)\omega(t) = -\tfrac{1}{2}(\omega(s)-\omega(t))^2 + \tfrac{1}{2}\omega^2(s) + \tfrac{1}{2}\omega^2(t)$$

and do the same for $\omega(s+\Delta)\omega(t+\Delta)$, $\omega(s)\omega(t+\Delta)$ and $\omega(t)\omega(s+\Delta)$. We have

$$\begin{aligned}\mathbb{V}_1(\mathcal{W}) = &-\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\left\{\int_\Omega (\omega(s+\Delta)-\omega(t+\Delta))^2)\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\ &-\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\left\{\int_\Omega (\omega(s)-\omega(t))^2)\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\ &+\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\left\{\int_\Omega (\omega(t+\Delta)-\omega(s))^2)\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\ &+\frac{1}{2\Delta^2}\int_{(s,t)\in\mathbb{R}^2} F(s)F(t)\left\{\int_\Omega (\omega(t)-\omega(s+\Delta))^2)\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\ = &\ (I) + (II) + (III) + (IV).\end{aligned}$$

By the way the Wiener measure is defined, we have

$$
\begin{aligned}
(I) = & -\frac{1}{2\Delta^2}\int_{t>s} F(s)F(t)\left\{\int_0^{+\infty}(2\pi(t-s))^{-\frac{1}{2}}y^2\mathrm{e}^{-\frac{y^2}{2(t-s)}}\,\mathrm{d}y\right\}\mathrm{d}s\,\mathrm{d}t \\
& -\frac{1}{2\Delta^2}\int_{t<s} F(s)F(t)\left\{\int_0^{+\infty}(2\pi(t-s))^{-\frac{1}{2}}y^2\mathrm{e}^{-\frac{y^2}{2(t-s)}}\,\mathrm{d}y\right\}\mathrm{d}s\,\mathrm{d}t \\
= & -\frac{1}{2\Delta^2}\int_{(t,s)\in\mathbb{R}^2}|t-s|F(s)F(t)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

Similarly, (II) = (I), and

$$
\begin{aligned}
(III) &= \frac{1}{2\Delta^2}\int_{(t,s)\in\mathbb{R}^2}|t-s+\Delta|F(s)F(t)\,\mathrm{d}s\,\mathrm{d}t, \\
(IV) &= \frac{1}{2\Delta^2}\int_{(t,s)\in\mathbb{R}^2}|t-s-\Delta|F(s)F(t)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

Denote

$$
\mathbb{D} = \{(s,t)\in\mathbb{R}^2 : s-\Delta < t < s+\Delta\}.
$$

We have

$$
\begin{aligned}
\mathbb{V}_1(\mathcal{W}) &= \frac{1}{\Delta^2}\int_{\mathbb{D}}(\Delta-|t-s|)F(s)F(t)\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{\Delta^2}\int_{-\infty}^{+\infty}F(s)\left\{\int_{s-\Delta}^{s+\Delta}(\Delta-|t-s|)F(t)\,\mathrm{d}t\right\}\mathrm{d}s \\
&= \int_{-\infty}^{+\infty}F^2(s)\,\mathrm{d}s + \mathcal{O}(\Delta).
\end{aligned}
$$

We now estimate $\mathbb{V}_2(\mathcal{W})$. By definition

$$
\mathbb{V}_2(\mathcal{W}) = (A) + (B),
$$

where

$$
\begin{aligned}
(A) &= \int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_{\Omega\setminus\Omega(s,t)}\mathcal{G}(\theta_t\omega)\mathcal{G}(\theta_s\omega))\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t, \\
(B) &= -\frac{1}{\Delta^2}\int_{(s,t)\in\mathbb{R}^2}F(s)F(t)\left\{\int_{\Omega\setminus\Omega(s,t)}X_1X_2\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

For (A),

$$
\begin{aligned}
|(A)| &\leqslant \int_{(s,t)\in\mathbb{R}^2}|F(s)||F(t)|\left\{\int_{\Omega\setminus\Omega(s,t)}|\mathcal{G}(\theta_t\omega)\mathcal{G}(\theta_s\omega)|\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{\Delta^2}\int_{(s,t)\in\mathbb{R}^2}|F(s)||F(t)|\left\{\int_{|X_1|>M_0,|X_2|>M_0}M_0^2\,\mathrm{d}\mathbb{P}\right. \\
&\quad \left. + \int_{|X_1|>M_0,|X_2|<M_0}M_0|X_2|\,\mathrm{d}\mathbb{P} + \int_{|X_1|<M_0,|X_2|>M_0}M_0|X_1|\,\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\
&\leqslant \frac{M_0^2}{\Delta^2}\int_{(s,t)\in\mathbb{R}^2}|F(s)||F(t)|\left\{2\int_{|X_1|>M_0}\mathrm{d}\mathbb{P} + \int_{|X_2|>M_0}\mathrm{d}\mathbb{P}\right\}\mathrm{d}s\,\mathrm{d}t \\
&= \frac{3M_0^2}{\Delta^2}\int_{M_0}^{\infty}\frac{1}{\sqrt{2\pi\Delta}}\mathrm{e}^{-\frac{y^2}{2\Delta}}\,\mathrm{d}y\cdot\left\{\int_{-\infty}^{\infty}|F(s)|\,\mathrm{d}s\right\}^2.
\end{aligned}
$$

It follows that $|(A)| \to 0$ as $M_0 \to \infty$. We also have for (B),

$$|(B)| \leqslant \frac{1}{\Delta^2} \int_{(s,t)\in\mathbb{R}^2} |F(s)||F(t)| \left\{ \int_{\Omega\setminus\Omega(s,t)} |X_1 X_2| \,\mathrm{d}\mathbb{P} \right\} \mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \frac{1}{\Delta^2} \int_{(s,t)\in\mathbb{R}^2} |F(s)||F(t)| \left\{ \int_{|X_1|>M_0,|X_2|>M_0} |X_1||X_2| \,\mathrm{d}\mathbb{P} \right.$$
$$\left. + \int_{|X_1|>M_0,|X_2|<M_0} M_0|X_1| \,\mathrm{d}\mathbb{P} + \int_{|X_1|<M_0,|X_2|>M_0} M_0|X_2| \,\mathrm{d}\mathbb{P} \right\} \mathrm{d}s\,\mathrm{d}t.$$

We have

$$|(B)| \leqslant \frac{1}{\Delta^2} \int_{(s,t)\in\mathbb{R}^2} |F(s)||F(t)| \left\{ \int_{|X_1|>M_0,|X_2|>M_0,|X_1|>|X_2|} |X_1||X_2| \,\mathrm{d}\mathbb{P} \right.$$
$$+ \int_{|X_1|>M_0,|X_2|>M_0,|X_2|>|X_1|} |X_1||X_2| \,\mathrm{d}\mathbb{P}$$
$$\left. + \int_{|X_1|>M_0} M_0|X_1| \,\mathrm{d}\mathbb{P} + \int_{X_2|>M_0} M_0|X_2| \,\mathrm{d}\mathbb{P} \right\} \mathrm{d}s\,\mathrm{d}t.$$
$$\leqslant \frac{1}{\Delta^2} \int_{(s,t)\in\mathbb{R}^2} |F(s)||F(t)| \left\{ \int_{|X_1|>M_0,|X_2|>M_0,|X_1|>|X_2|} |X_1|^2 \,\mathrm{d}\mathbb{P} \right.$$
$$+ \int_{|X_1|>M_0,|X_2|>M_0,|X_2|>|X_1|} |X_2|^2 \,\mathrm{d}\mathbb{P}$$
$$\left. + \int_{|X_1|>M_0} M_0|X_1| \,\mathrm{d}\mathbb{P} + \int_{|X_2|>M_0} M_0|X_2| \,\mathrm{d}\mathbb{P} \right\} \mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \frac{1}{\Delta^2} \int_{(s,t)\in\mathbb{R}^2} |F(s)||F(t)| \left\{ \int_{|X_1|>M_0} |X_1|^2 \,\mathrm{d}\mathbb{P} + \int_{|X_2|>M_0} |X_2|^2 \,\mathrm{d}\mathbb{P} \right.$$
$$\left. + \int_{|X_1|>M_0} M_0|X_1| \,\mathrm{d}\mathbb{P} + \int_{|X_2|>M_0} M_0|X_2| \,\mathrm{d}\mathbb{P} \right\} \mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \frac{1}{\Delta^2} \left\{ 2 \int_{M_0}^{\infty} \frac{1}{\sqrt{2\pi\Delta}} y^2 \mathrm{e}^{-\frac{y^2}{2\Delta}} \,\mathrm{d}y + \int_{M_0}^{\infty} \frac{1}{\sqrt{2\pi\Delta}} M_0 y \mathrm{e}^{-\frac{y^2}{2\Delta}} \,\mathrm{d}y \right\} \cdot \left\{ \int_{-\infty}^{\infty} |F(s)| \,\mathrm{d}s \right\}^2.$$

Again, $|B| \to 0$ as $M_0 \to \infty$. We have proved that $\mathbb{V}(\mathcal{W}) > 0$ for M_0 sufficiently large. □

Proposition 3.4 is theorem B in section 1.

4. Applications to Duffing's equations

In this section, we apply theorems 2.1–2.5 to the Duffing equation driven by a nonautonomous forcing without any time periodicity. We consider a class of C^N function

$$\Phi_{c_1,c_2} : \mathbb{R} \to [\min\{c_1, c_2\}, \max\{c_1, c_2\}]$$

such that

$$\lim_{t\to-\infty} \Phi_{c_1,c_2}(t) = c_1, \qquad \lim_{t\to+\infty} \Phi_{c_1,c_2}(t) = c_2, \tag{4.1}$$

where c_1, c_2 are two real numbers. We assume that the C^N-norm of Φ_{c_1,c_2} is uniformly bounded by a constant, which we denote as $\|\Phi_{c_1,c_2}\|$. We add an external forcing $\mu q^2 \Phi_{\eta^-,\eta^+}(t) \sin\omega t$ to equation (3.5) and perturb its damping by $\mu\Phi_{\tau^-,\tau^+}(t)q^2$ to obtain a non-periodic equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + (\lambda - (\gamma_\lambda + \mu\Phi_{\tau^-,\tau^+}(t))q^2)\frac{\mathrm{d}q}{\mathrm{d}t} - q + q^3 = \mu q^2 \Phi_{\eta^-,\eta^+}(t) \sin\omega t, \tag{4.2}$$

where $\tau^{\pm}$, $\eta^{\pm}$, μ, ω are forcing parameters. The four arbitrary constants $\tau^{\pm}$, $\eta^{\pm}$ are to create arbitrary combinations of $m^{\pm}$, $M^{\pm}$ for the Melnikov functions of the corresponding equations. The use of sine function also ensures that $m^{\pm}$, $M^{\pm}$ are densely approached by L-sequences where $L = 4\pi\omega^{-1}$.

We rewrite equation (4.2) as

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= p, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -(\lambda - \gamma_\lambda q^2)p + q - q^3 + \mu(\Phi_{\tau^-,\tau^+}(t)q^2 p + q^2\Phi_{\eta^-,\eta^+}(t)\sin\omega t). \end{aligned} \tag{4.3}$$

To put the linear part of equation (4.3) in a canonical form, we again introduce new variables (x, y) using (3.9) and (3.10). The new equations for (x, y) are

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\alpha x + f(x, y) + \mu(A(x, y)\Phi_{\tau^-,\tau^+}(t) + C(x, y)\Phi_{\eta^-,\eta^+}(t)\sin\omega t), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \beta y + g(x, y) + \mu(B(x, y)\Phi_{\tau^-,\tau^+}(t) + D(x, y)\Phi_{\eta^-,\eta^+}(t)\sin\omega t), \end{aligned} \tag{4.4}$$

where $\beta = \alpha^{-1}$ is again as in (3.6), and $f(x, y)$, $g(x, y)$, $C(x, y)$, $D(x, y)$ are the same as in (3.12) and

$$A(x, y) = \frac{\alpha}{1+\alpha^2}(x+\alpha y)^2(y-\alpha x), \qquad B(x, y) = \frac{-1}{1+\alpha^2}(x+\alpha y)^2(y-\alpha x).$$

Observe that the functions f, g, A, B, C, D are of order at least two at $(x, y) = (0, 0)$. Again, let $(a_\lambda(t), b_\lambda(t))$ be the homoclinic solution and $E_\lambda(t)$ be as in (3.13). Then the Melnikov function for equation (4.4) is

$$\mathcal{W}(t) = \mathcal{W}_1(t) + \mathcal{W}_2(t), \tag{4.5}$$

where

$$\begin{aligned} \mathcal{W}_1(t) &= \int_{-\infty}^{+\infty} (v_\lambda(s)A_\lambda(s) - u_\lambda(s)B_\lambda(s))\Phi_{\tau^-,\tau^+}(s+t)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s, \\ \mathcal{W}_2(t) &= \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s))\Phi_{\eta^-,\eta^+}(s+t)\sin\omega(s+t)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s, \end{aligned} \tag{4.6}$$

in which $A_\lambda(t) = A(a_\lambda(t), b_\lambda(t))$. The quantities $B_\lambda(t)$, $C_\lambda(t)$, $D_\lambda(t)$ are similar.

Denote

$$\begin{aligned} J &= \int_{-\infty}^{+\infty} (v_\lambda(s)A_\lambda(s) - u_\lambda(s)B_\lambda(s))\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s, \\ J_s &= \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s))\sin(\omega s)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s, \\ J_c &= \int_{-\infty}^{+\infty} (v_\lambda(s)C_\lambda(s) - u_\lambda(s)D_\lambda(s))\cos(\omega s)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s. \end{aligned} \tag{4.7}$$

Recall that $m^{\pm} = \liminf_{t\to\pm\infty} \mathcal{W}(t)$, $M^{\pm} = \limsup_{t\to\pm\infty} \mathcal{W}(t)$. We have the following.

Proposition 4.1. *Let $R > 0$ be fixed and assume $\omega \in (0, R)$. Then there exists λ_0 sufficiently small, depending on R, such that for all $\lambda \in (0, \lambda_0)$,*

(a) $J > 0$, $J_s^2 + J_c^2 \neq 0$;
(b) $m^{\pm} = J\tau^{\pm} - \sqrt{J_s^2 + J_c^2}\,\eta^{\pm}$, $M^{\pm} = J\tau^{\pm} + \sqrt{J_s^2 + J_c^2}\,\eta^{\pm}$ and
(c) $m^{\pm}$, $M^{\pm}$ are all densely approached by L-sequences where $L = 4\pi\omega^{-1}$.

Proof. Item (a) follows from a straightforward computation in the case of $\lambda = 0$ and the fact that J, J_c, J_s are continuous in λ. For items (b) and (c) we start from $\lim_{t\to+\infty} \Phi_{\tau^-,\tau^+}(t) = \tau^+$. For a given $\varepsilon > 0$, there exists t_0 sufficiently large so that

$$|\Phi_{\tau^-,\tau^+}(t) - \tau^+| < \varepsilon$$

for all $t \geqslant t_0$. Write

$$\begin{aligned}\mathcal{W}_1(t) = &\int_{-\infty}^{-t+t_0} (v(s)A_\lambda(s) - u(s)B_\lambda(s))\Phi_{\tau^-,\tau^+}(s+t)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \\ &+ \int_{-t+t_0}^{+\infty} (v(s)A_\lambda(s) - u(s)B_\lambda(s))\Phi_{\tau^-,\tau^+}(s+t)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s.\end{aligned}$$

As $t \to +\infty$, $-t + t_0 \to -\infty$, and the first integral goes to zero. For the second integral we observe that, for all $s \geqslant -t + t_0$, $|\Phi_{\tau^-,\tau^+}(t+s) - \tau^+| < \varepsilon$. Hence the value of the second integral is $K\varepsilon$-close to the value of the integral in which we replace $\Phi_{\tau^-,\tau^+}(s+t)$ in the same integral by τ^+. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{t\to+\infty} \mathcal{W}_1(t) = \tau^+ J.$$

Similarly, we have

$$\lim_{t\to-\infty} \mathcal{W}_1(t) = \tau^- J. \tag{4.8}$$

We write $\mathcal{W}_2(t)$ as

$$\mathcal{W}_2(t) = \cos(\omega t) J_s(t) + \sin(\omega t) J_c(t), \tag{4.9}$$

where

$$J_s(t) = \int_{-\infty}^{+\infty} (v(s)C_\lambda(s) - u(s)D_\lambda(s))\Phi_{\eta^-,\eta^+}(s+t)\sin(\omega s)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s,$$

$$J_c(t) = \int_{-\infty}^{+\infty} (v(s)C_\lambda(s) - u(s)D_\lambda(s))\Phi_{\eta^-,\eta^+}(s+t)\cos(\omega s)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s.$$

Similarly,

$$\lim_{t\to\pm\infty} J_s(t) = \eta^\pm J_s, \qquad \lim_{t\to\pm\infty} J_c(t) = \eta^\pm J_c. \tag{4.10}$$

Combining (4.9) and (4.10) we obtain

$$\liminf_{t\to\pm\infty} \mathcal{W}_2(t) = -\eta^\pm\sqrt{J_s^2 + J_c^2}; \qquad \limsup_{t\to\pm\infty} \mathcal{W}_2(t) = \eta^\pm\sqrt{J_s^2 + J_c^2}. \tag{4.11}$$

Items (b) and (c) follow directly from (4.8), (4.9) and (4.11). □

We also have the following estimates on $m = \inf_{t\in\mathbb{R}} \mathcal{W}(t)$ and $M = \sup_{t\in\mathbb{R}} \mathcal{W}(t)$.

Proposition 4.2. *Let $\mathcal{W}(t)$ be as in (4.5). We have*

$$m \geqslant \tfrac{1}{2}\left((\tau^+ + \tau^-)J - (\eta^+ + \eta^-)\sqrt{J_s^2 + J_c^2}\right) - K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|),$$
$$M \leqslant \tfrac{1}{2}\left((\tau^+ + \tau^-)J + (\eta^+ + \eta^-)\sqrt{J_s^2 + J_c^2}\right) + K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|),$$

where

$$K = \int_{-\infty}^{+\infty} (|v(s)A_\lambda(s) - u(s)B_\lambda(s)| + |v(s)C_\lambda(s) - u(s)D_\lambda(s)|)\mathrm{e}^{-\int_0^s E_\lambda(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s. \tag{4.12}$$

Proof. By definition

$$|\mathcal{W}(t) - \tau^{\pm} J - \eta^{\pm}(J_s \cos\omega t + J_c \sin\omega t)| \leqslant K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|),$$

from which the estimates on m and M directly follow. □

Applications of theorem 2.1. To apply theorem 2.1(a) it suffices to have

$$\min(m^+, m^-) < 0 < \max(M^+, M^-).$$

By proposition 4.1(b), this is

$$\begin{aligned} &\min\left(J\tau^- - \sqrt{J_s^2 + J_c^2}\,\eta^-, \qquad J\tau^+ - \sqrt{J_s^2 + J_c^2}\,\eta^+\right) < 0, \\ &\max\left(J\tau^- + \sqrt{J_s^2 + J_c^2}\,\eta^-, \qquad J\tau^+ + \sqrt{J_s^2 + J_c^2}\,\eta^+\right) > 0. \end{aligned} \tag{4.13}$$

We deduce from (4.13) sufficient conditions for scenario (a) in three particular cases. Note that if $c = c_1 = c_2$, then $\Phi_{c_1,c_2}(t) = c$ is a constant.

Case (i). $\eta^+ = \eta^- = 0$. In this case equation (4.2) becomes

$$\frac{d^2 q}{dt^2} + (\lambda - (\gamma_\lambda + \mu\Phi_{\tau^-,\tau^+}(t))q^2)\frac{dq}{dt} - q + q^3 = 0. \tag{4.14}$$

For this equation to have homoclinic solutions, it suffice to have

$$\tau^+ \cdot \tau^- < 0$$

from (4.13). For equation (4.14), however, $\mathcal{W}_2(t) = 0$. Hence $m^- = M^- = \tau^- J$, $m^+ = M^+ = \tau^+ J$. The Melnikov function does not oscillate as $t \to \pm\infty$. No complicated dynamics are expected.

Case (ii). $\tau^+ = \tau^- = 0$. In this case the sufficient condition for theorem 2.1(a) derived from (4.13) is $\eta^{\pm} \neq 0$. Equation (4.2) becomes

$$\frac{d^2 q}{dt^2} + (\lambda - \gamma_\lambda q^2)\frac{dq}{dt} - q + q^3 = \mu q^2 \Phi_{\eta^-,\eta^+}(t) \sin\omega t. \tag{4.15}$$

Equation (4.15) is not time periodic if $\eta^- \neq \eta^+$.

Case (iii). $\tau^- = \tau^+ := \rho$, and we assume $\eta^- = \eta^+ = 1$. In this case equation (4.2) becomes

$$\frac{d^2 q}{dt^2} + (\lambda - (\gamma_\lambda + \mu\rho))q^2)\frac{dq}{dt} - q + q^3 = \mu q^2 \sin\omega t, \tag{4.16}$$

and the sufficient condition for theorem 2.1(a) deduced from (4.13) is

$$|\rho| < \frac{\sqrt{J_s^2 + J_c^2}}{J}.$$

For the assumptions of theorem 2.1(b) and (c) we use estimates in proposition 4.2. We obtain a sufficient condition for theorem 2.1(b) as

$$(\tau^+ + \tau^-)J > (\eta^+ + \eta^-)\sqrt{J_s^2 + J_c^2} + 2K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|)$$

and a sufficient condition for theorem 2.1(c) as

$$(\tau^+ + \tau^-)J < -(\eta^+ + \eta^-)\sqrt{J_s^2 + J_c^2} - 2K(|\tau^+ - \tau^-| + |\eta^+ - \eta^-|).$$

From these general inequalities, various conditions for equations (4.14), (4.15) and (4.16) are deduced accordingly. We skip the details.

Applications of theorem 2.2. The condition for theorem 2.2 is $\max(m^+, m^-) < 0 < \min(M^+, M^-)$. Using proposition 4.1(b), this is transformed to

$$\begin{aligned} &\max(J\tau^+ - \sqrt{J_s^2 + J_c^2}\eta^+, J\tau^- - \sqrt{J_s^2 + J_c^2}\eta^-) < 0, \\ &\min(J\tau^+ + \sqrt{J_s^2 + J_c^2}\eta^+, J\tau^- + \sqrt{J_s^2 + J_c^2}\eta^-) > 0. \end{aligned} \tag{4.17}$$

It follows from (4.17) that (i) theorem 2.2 cannot be applied to equation (4.14), for the inequalities of (4.17) become self-conflicting if $\eta^+ = \eta^- = 0$, and (ii) theorem 2.2 always applies to equation (4.15), for it is trivial for (4.17) to hold if $\tau^+ = \tau^- = 0$. A necessary condition for theorem 2.2 to apply to equation (4.16) is, denoting $\tau^+ = \tau^- := \rho$,

$$|\rho| < \frac{\sqrt{J_s^2 + J_c^2}}{J}.$$

Applications of theorems 2.3 and 2.4. The first condition for theorem 2.3 is $m^+ < 0 < m^-$. Using proposition 4.1(b), this requires

$$\tau^+ < \frac{\sqrt{J_s^2 + J_c^2}}{J}\eta^+; \qquad \tau^- > \frac{\sqrt{J_s^2 + J_c^2}}{J}\eta^-.$$

We also need $M^+ > 0$, which requires

$$\tau^+ > -\frac{\sqrt{J_s^2 + J_c^2}}{J}\eta^+.$$

Putting these two together we obtain

$$|\tau^+| < \frac{\sqrt{J_s^2 + J_c^2}}{J}\eta^+; \qquad \tau^- > \frac{\sqrt{J_s^2 + J_c^2}}{J}\eta^-.$$

These are the conditions for theorem 2.3 to apply to equation (4.2). We note that these inequalities become self-conflicting if $\tau^- = \tau^+$, $\eta^+ = \eta^-$. This agrees with our previous observation that theorem 2.3 only applies to equations that have no periodicity. Application of theorem 2.4 is similar.

Applications of theorem 2.5. For theorem 2.5 to apply we first need $m^\pm > 0$, which implies

$$\tau^+ > \frac{\sqrt{J_s^2 + J_c^2}}{J}\eta^+; \qquad \tau^- > \frac{\sqrt{J_s^2 + J_c^2}}{J}\eta^-. \tag{4.18}$$

We also need

$$M^+ > 2m^+ \mathrm{e}^{3k\beta L}.$$

Using the values of m^+ and M^+ from proposition 4.1(b) with $L = 4\pi\omega^{-1}$ from proposition 4.1(c), we obtain

$$\frac{\tau^+ + \frac{\sqrt{J_s^2+J_c^2}}{J}\eta^+}{\tau^+ - \frac{\sqrt{J_s^2+J_c^2}}{J}\eta^+} > 2\mathrm{e}^{12\pi\beta\omega^{-1}k}.$$

From this we obtain

$$\omega^{-1} < \frac{1}{12\pi\beta}\left(\ln\frac{\tau^+ + \frac{\sqrt{J_s^2+J_c^2}}{J}\eta^+}{\tau^+ - \frac{\sqrt{J_s^2+J_c^2}}{J}\eta^+} - \ln 2\right). \tag{4.19}$$

(4.18) and (4.19) together are sufficient conditions for theorem 2.5 to be applied to equation (4.2).

5. Poincaré return maps in the extended phase space

In this section, we estimate the Poincaré return map in the extended phase space and give its leading terms. In section 5.1 we introduce a coordinate change that linearizes equation (2.2) in B_ε. In section 5.2 we derive a normal form for equation (2.2) around the entire length of the homoclinic loop ℓ out of $B_{\frac{1}{4}\varepsilon^2}$. Poincaré sections $\Sigma^\pm$ are introduced in precise terms in section 5.3. We then compute the return maps $\mathcal{R} : \Sigma^- \to \Sigma^-$ based on the equations derived in sections 5.1 and 5.2.

5.1. Local linearization

We consider a time-dependent transformation

$$\begin{aligned} x &= X + \mathbb{P}(X, Y) + \mu\tilde{\mathbb{P}}(X, Y, t),\\ y &= Y + \mathbb{Q}(X, Y) + \mu\tilde{\mathbb{Q}}(X, Y, t), \end{aligned} \tag{5.1}$$

where $\mathbb{P}, \mathbb{Q}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$ as functions of X and Y are C^r for $r \geqslant 2$ on $|(X, Y)| < 2\varepsilon$ for all $t \in \mathbb{R}$ and the values of these functions and their first derivatives with respect to X and Y at $(X, Y) = (0, 0)$ are all zero. The functions $\mathbb{P}$ and $\mathbb{Q}$ are independent of t and μ. The functions $\tilde{\mathbb{P}}(X, Y, t)$ and $\tilde{\mathbb{Q}}(X, Y, t)$ are continuous in t.

Proposition 5.1. *For each integer $r > 0$, there exists an integer $N_0 = N_0 > r$ such that if f, g, P, Q are C^N for $N \geqslant N_0$ with uniformly bounded derivatives and $-\alpha, \beta$ satisfies the nonresonant conditions up to order N_0, then there exists a time-dependent transformation in the form of (5.1) defined on $B_\varepsilon \times \mathbb{R} \times [-\mu_0, \mu_0]$ that transforms equation (2.2) into*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\alpha X, \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = \beta Y$$

where B_ε is a small neighbourhood of $(X, Y) = (0, 0)$ and μ_0 is a positive constant. Moreover, the C^r-norms of $\mathbb{P}, \mathbb{Q}, \tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$ as functions of X, Y are all uniformly bounded by a constant K that is independent of both ε and μ on $(X, Y) \in B_\varepsilon$ for all $t \in \mathbb{R}$.

Proposition 5.1 is proved in appendix A.

5.2. A standard form around homoclinic loop

In this subsection we derive a standard form for equation (2.2) around the homoclinic loop of equation (2.1) outside of $B_{\frac{1}{4}\varepsilon^2}$.

Two small scales. Two small quantities $\mu \ll \varepsilon \ll 1$ represent two small scales of different magnitude. Let ε be the size of a small neighbourhood of $(x, y) = (0, 0)$ that makes the linearization of section 5.1 valid. Associated with ε is the small neighbourhood

$$B_\varepsilon = \{(x, y) : x^2 + y^2 < 4\varepsilon^2\}, \qquad \mathcal{B}_\varepsilon = B_\varepsilon \times \mathbb{R},$$

and L^+ and $-L^-$, the respective times the homoclinic solution $\ell(t)$ enters $B_{\frac{1}{2}\varepsilon}$ in the positive and the negative directions of time. The quantities L^+ and L^- are related, both are completely determined by ε and $\ell(t)$. The parameter $\mu(\ll\varepsilon)$ controls the magnitude of the nonautonomous perturbation.

Notation. The letter K is used throughout to generically represent constants that are *independent* of μ. The precise value of K is allowed to change from line to line. In occasions, a specific constant is used in different places. There are also times we need to distinguish two K in the same line. We will then use subscripts to denote them as $K_0, K_1, \ldots$. We will also make distinctions between constants that are dependent of ε and those do not by making such dependence explicit. A constant that depends on ε is written as $K(\varepsilon)$. A constant written as K is independent of ε.

For the homoclinic solution $\ell(t) = (a(t), b(t))$ we regard t not as time, but as a parameter that parametrizes the curve ℓ in the (x, y)-space. We replace t by s to write this homoclinic loop as $\ell(s) = (a(s), b(s))$. We have

$$\frac{\mathrm{d}a(s)}{\mathrm{d}s} = -\alpha a(s) + f(a(s), b(s)), \qquad \frac{\mathrm{d}b(s)}{\mathrm{d}s} = \beta b(s) + g(a(s), b(s)). \tag{5.2}$$

By definition,

$$\begin{aligned} u(s) &= \frac{-\alpha a(s) + f(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}, \\ v(s) &= \frac{\beta b(s) + g(a(s), b(s))}{\sqrt{(-\alpha a(s) + f(a(s), b(s)))^2 + (\beta b(s) + g(a(s), b(s)))^2}}. \end{aligned} \tag{5.3}$$

Let

$$e(s) = (v(s), -u(s)).$$

We now introduce new variables (s, z) such that

$$(x, y) = \ell(s) + ze(s).$$

This is to say that

$$x = x(s, z) := a(s) + v(s)z, \qquad y = y(s, z) := b(s) - u(s)z. \tag{5.4}$$

We derive the equations for (2.2) in new variables (s, z) defined through (5.4). Differentiating (5.4) we obtain

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= (-\alpha a(s) + f(a(s), b(s)) + v'(s)z)\frac{\mathrm{d}s}{\mathrm{d}t} + v(s)\frac{\mathrm{d}z}{\mathrm{d}t}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (\beta b(s) + g(a(s), b(s)) - u'(s)z)\frac{\mathrm{d}s}{\mathrm{d}t} - u(s)\frac{\mathrm{d}z}{\mathrm{d}t}, \end{aligned} \tag{5.5}$$

where $u'(s) = \mathrm{d}u(s)/\mathrm{d}s$, $v'(s) = \mathrm{d}v(s)/\mathrm{d}s$. Let us denote

$$\begin{aligned} F(s, z) &= -\alpha(a(s) + zv(s)) + f(a(s) + zv(s), b(s) - zu(s)), \\ G(s, z) &= \beta(b(s) - zu(s)) + g(a(s) + zv(s), b(s) - zu(s)), \\ P(s, z, t) &= P(a(s) + zv(s), b(s) - zu(s), t), \\ Q(s, z, t) &= Q(a(s) + zv(s), b(s) - zu(s), t). \end{aligned}$$

Using equation (2.2), we obtain from equation (5.5) the new equations for s, z as

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{u(s)F(s, z) + v(s)G(s, z) + \mu(u(s)P(s, z, t) + v(s)Q(s, z, t))}{\sqrt{F(s, 0)^2 + G(s, 0)^2} + z(u(s)v'(s) - v(s)u'(s))}, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= v(s)F(s, z) - u(s)G(s, z) + \mu(v(s)P(s, z, t) - u(s)Q(s, z, t)). \end{aligned}$$

We rewrite these equations as

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= 1 + zw_1(s, z, t) + \frac{\mu(u(s)P(s, z, t) + v(s)Q(s, z, t))}{\sqrt{F(s, 0)^2 + G(s, 0)^2}}, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= E(s)z + z^2 w_2(s, z) + \mu(v(s)P(s, z, t) - u(s)Q(s, z, t)), \end{aligned} \tag{5.6}$$

where

$$E(s) = v^2(s)(-\alpha + \partial_x f(a(s), b(s))) + u^2(s)(\beta + \partial_y g(a(s), b(s))) \\ - u(s)v(s)(\partial_y f(a(s), b(s)) + \partial_x g(a(s), b(s))).$$

Also in the rest of this section we let $K_0(\varepsilon)$ be a given constant independent of μ and regard equation (5.6) as been defined on

$$\{s \in [-2L^-, 2L^+],\ t \in \mathbb{R}\ |z| < K_0(\varepsilon)\mu\}.$$

The C^N-norms of $w_1(s, z, t)$ and $w_2(s, z)$ with respect to s, z and the C^0-norm of $w_1(s, z, t)$ with with respect to t are bounded above by a constant $K(\varepsilon)$.

Finally we re-scale the variable z by letting

$$Z = \mu^{-1} z. \tag{5.7}$$

We arrive at the following equations

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= 1 + \mu\tilde{w}_1(s, Z, t), \\ \frac{\mathrm{d}Z}{\mathrm{d}t} &= E(s)Z + \mu\tilde{w}_2(s, Z, t) + (v(s)P(s, 0, t) - u(s)Q(s, 0, t)), \end{aligned} \tag{5.8}$$

where (s, Z, t) is defined on

$$\boldsymbol{D} = \{(s, Z, t) :\ s \in [-2L^-, 2L^+],\ |Z| \leqslant K_0(\varepsilon),\ t \in \mathbb{R}\}.$$

Here we assume that μ is sufficiently small so that

$$\mu \ll \min_{s\in[-2L^-,2L^+]} (F(s, 0)^2 + G(s, 0)^2).$$

Again, the C^N-norms of $\tilde{w}_1$, $\tilde{w}_2$ in s, Z and the C^0-norm in t are uniformly bounded by a constant $K(\varepsilon)$ on $\boldsymbol{D}$. Equation (5.8) is the one we need. Note that

$$P(s, 0, t) = P(a(s), b(s), t), \qquad Q(s, 0, t) = Q(a(s), b(s), t).$$

5.3. Poincaré sections $\Sigma^\pm$

We define $\Sigma^\pm$ inside of $\mathcal{B}_\varepsilon \cap \boldsymbol{D}$ by letting

$$\Sigma^- = \{(X, Y, t) :\ Y = \varepsilon,\ |X| < \mu,\ t \in \mathbb{R}\}$$

and

$$\Sigma^+ = \{(X, Y, t) :\ X = \varepsilon,\ |Y| < K_1(\varepsilon)\mu,\ t \in \mathbb{R}\}.$$

$K_1(\varepsilon)$ will be precisely defined momentarily.

Let $q \in \Sigma^+$ or Σ^-. We can also use (s, Z, t)-coordinate to represent q, for which the defining equations for $\Sigma^\pm$ are not as direct. To compute the return maps, we need to first address two issues that are technical in nature. First, we need to derive the defining equations on $\Sigma^\pm$ for (s, Z, t). Second, we need to be able to change coordinates from (X, Y, t) to (s, Z, t) and vice versa on $\Sigma^\pm$. We start with some preparations in notation.

Notation. The intended formula for the return maps would inevitably contain terms that are explicit and terms that are implicit. Implicit terms are usually 'error' terms, and the usefulness of a derived formula would depend completely on how well the error terms are controlled. We aim at C^r-control on all error terms with respect to phase variables and C^0-control with respect to time t. The derivations of the return maps involve a composition of maps and multiple coordinate changes. To facilitate our presentation, from this point on we adopt specific conventions for indicating controls on magnitude. For a given constant, we write $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\mu)$ to indicate that the magnitude of the constant is bounded by K, $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. For a function of a set V of variables on a specific domain, we write $\mathcal{O}_V(1)$, $\mathcal{O}_V(\varepsilon)$ or $\mathcal{O}_V(\mu)$ to indicate that the C^r-norm of the function on the specified domain with respect to the phase variables in V and the C^0-norm with respect to t (if t is in V) are bounded by K, $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. We chose to specify the domain in the surrounding text rather than explicitly involving it in the notation. For example, $\mathcal{O}_{Z,t}(\mu)$ represents a function of Z, t, the C^r-norm of which with respect to Z and the C^0-norm of which with respect to t are bounded above by $K(\varepsilon)\mu$.

In what follows we let

$$\mathbb{X} = \mu^{-1}X, \qquad \mathbb{Y} = \mu^{-1}Y.$$

Proposition 5.2. *Coordinate conversions on $\Sigma^\pm$ are as follows:*

(a) On Σ^+, (i) $s = L^+ + \mathcal{O}_{Z,t}(\mu)$, (ii) $\mathbb{Y} = (1 + \mathcal{O}(\varepsilon))Z + \mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu)$.
(b) On Σ^-, (i) $s = -L^- + \mathcal{O}_{Z,t}(\mu)$, (ii) $Z = (1 + \mathcal{O}(\varepsilon))\mathbb{X} + \mathcal{O}_t(1) + \mathcal{O}_{\mathbb{X},t}(\mu)$.

Proposition 5.2 is proved in appendix B.

5.4. *The map $\mathcal{M} : \Sigma^- \to \Sigma^+$*

Let

$$\mathcal{W}_L(t) = \int_{-L^-}^{L^+} (v(s)P(a(s), b(s), s + t) - u(s)Q(a(s), b(s), s + t))\mathrm{e}^{-\int_0^s E(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s. \qquad (5.9)$$

We also write

$$P_L = \mathrm{e}^{\int_{-L^-}^{L^+} E(s)\,\mathrm{d}s}, \qquad P_L^+ = \mathrm{e}^{\int_0^{L^+} E(s)\,\mathrm{d}s}. \qquad (5.10)$$

Note that for P_L we integrate from $s = -L^-$ to $s = L^+$, while for P_L^+ the integration starts from $s = 0$. First we have the following.

Lemma 5.1. $$P_L \sim \varepsilon^{\frac{\alpha}{\beta}-\frac{\beta}{\alpha}} \ll 1, \qquad P_L^+ \sim \varepsilon^{-\frac{\beta}{\alpha}} \gg 1.$$

Proof. By the definition of $L^\pm$ we have

$$\varepsilon \sim \mathrm{e}^{-\alpha L^+} \sim \mathrm{e}^{-\beta L^-}.$$

We also have

$$P_L \sim \mathrm{e}^{\beta L^+ - \alpha L^-}, \qquad P_L^+ \sim \mathrm{e}^{\beta L^+}.$$

Lemma 5.1 follows directly from these estimates. □

For $q = (s^-, Z, t_0) \in \Sigma^-$, the value of s^- is uniquely determined by that of (Z, t_0) through proposition 5.2(b)(i). So we can use (Z, t_0) for q. Let $(s(t), Z(t))$ be the solution of equation (5.8) initiated from (s^-, Z) at $t = t_0$, and $\hat{t}$ be the time $(s(\hat{t}), Z(\hat{t}))$ hits Σ^+. In what follows we write

$$s^+ = s(\hat{t}), \qquad \hat{Z} = Z(\hat{t}).$$

Proposition 5.3. *Denote $(\hat{Z}, \hat{t}) = \mathcal{M}(Z, t_0)$. We have*

$$\begin{aligned} \hat{Z} &= P_L^+ \mathcal{W}_L(t_0 + L^-) + P_L Z + \mathcal{O}_{Z,t_0}(\mu), \\ \hat{t} &= t_0 + L^+ + L^- + \mathcal{O}_{Z,t_0}(\mu). \end{aligned} \qquad (5.11)$$

Proof. We rewrite equation (5.8) as

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}s} &= E(s)Z + (v(s)P(a(s), b(s), \theta_t\omega) - u(s)Q(a(s), b(s), \theta_t\omega)) + \mathcal{O}_{s,Z,t}(\mu),\\ \frac{\mathrm{d}t}{\mathrm{d}s} &= 1 + \mathcal{O}_{s,Z,t}(\mu)\end{aligned} \tag{5.12}$$

on $\boldsymbol{D}$ where

$$\boldsymbol{D} = \{(s, Z, t) : \ s \in [-2L^-, 2L^+], \ |Z| < K_1(\varepsilon), \ t \in \mathbb{R}\}.$$

From the second item of (5.12) we obtain

$$t = t_0 + s - s^- + \int_{s^-}^{s} \mathcal{O}_{s,Z,t}(\mu)\,\mathrm{d}s,$$

from which the claim on $\hat{t}$ follows. Substituting it into the first item of (5.12) we obtain

$$\begin{aligned}\frac{\mathrm{d}Z}{\mathrm{d}s} = E(s)Z + \Bigg\{ & v(s)P\left(a(s), b(s), t_0 + s - s^- + \int_{s^-}^{s} \mathcal{O}_{s,Z,t}(\mu)\,\mathrm{d}s\right)\\ & -u(s)Q\left(a(s), b(s), t_0 + s - s^- + \int_{s^-}^{s} \mathcal{O}_{s,Z,t}(\mu)\,\mathrm{d}s\right)\Bigg\} + \mathcal{O}_{s,Z,t_0}(\mu),\end{aligned}$$

from which it follows that

$$\hat{Z} = P_L(Z + \Phi_L(t_0)) + \mathcal{O}_{Z,t_0}(\mu)$$

where P_L is as in (5.10) and

$$\begin{aligned}\Phi_L(t) = \int_{s^-}^{s(\hat{t})} \Bigg\{ & v(s)P\left(a(s), b(s), t + s - s^- + \int_{s^-}^{s} \mathcal{O}_{s,Z,t}(\mu)\,\mathrm{d}s\right)\\ & -u(s)Q\left(a(s), b(s), t + s - s^- + \int_{s^-}^{s} \mathcal{O}_{s,Z,t}(\mu)\,\mathrm{d}s\right)\Bigg\} \cdot \mathrm{e}^{-\int_{s^-}^{s} E(\tau)\mathrm{d}\tau}\,\mathrm{d}s.\end{aligned} \tag{5.13}$$

We caution that, since $P(x, y, t)$, $Q(x, y, t)$ are only continuous in t, we need an argument here to allow us to drop the $\mathcal{O}(\mu)$ terms inside of the t-argument, and regard the resulted error term as $\mathcal{O}(\mu)$. To make this argument we let

$$T = t + s - s^- + \int_{s^-}^{s} \mathcal{O}_{s,Z,t}(\mu)\,\mathrm{d}s,$$

and write the integral as

$$\begin{aligned}\Phi_L(t) = \int_{t}^{t+L^++L^-} & \left\{v(T - t - L^-)P\left(a(T - t - L^-), b(T - t - L^-), T\right)\right.\\ & \left.-u(T - t - L^-)Q\left(a(T - t - L^-), b(T - t - L^-), T\right)\right\}\\ & \cdot\,\mathrm{e}^{-\int_{-L^-}^{T-t-L^-} E(\tau)\,\mathrm{d}\tau}\,\mathrm{d}T + \mathcal{O}_{Z,t}(\mu).\end{aligned}$$

Here we used the fact that P, Q are C^N in (x, y) to push the $\mathcal{O}(\mu)$ terms in the (x, y)-argument out of the integral. Now let

$$s = T - t - L^-$$

to rewrite this integral as

$$\begin{aligned}\Phi_L(t) = \int_{-L^-}^{L^+} & \left\{v(s)P\left(a(s), b(s), s + t + L^-\right)\right.\\ & \left.-u(s)Q\left(a(s), b(s), s + t + L^-\right)\right\} \cdot \mathrm{e}^{-\int_{-L^-}^{s} E(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s + \mathcal{O}_{Z,t}(\mu).\end{aligned}$$

We also used proposition 5.2 to replace s^- by $-L^-$ and $s(\hat{t})$ by L^+. Also observe that

$$P_L\Phi_L(t) = P_L^+ \cdot \mathcal{W}_L(t+L^-) + \mathcal{O}_{Z,t}(\mu).$$

This proved the line for $\hat{Z}$.

Let

$$K_1(\varepsilon) = \max_{t\in\mathbb{R},\ s\in[-2L^-,2L^+]} P_s(2+|\Phi_s(t)|), \tag{5.14}$$

where P_s and Φ_s are obtained by replacing L^+ with s in P_L and Φ_L. $K_1(\varepsilon)$ is the one we use for $\boldsymbol{D}$ and Σ^+. The solutions of (5.12) initiated on Σ^- will stay inside of $\boldsymbol{D}$ before hitting Σ^+. □

5.5. *The return map $\mathcal{R}$*

First we compute $\mathcal{N} : \Sigma^+ \to \Sigma^-$. For $(\mathbb{X}, \mathbb{Y}, t) \in \Sigma^+$ we have $\mathbb{X} = \varepsilon\mu^{-1}$. Similarly, for $(\mathbb{X}, \mathbb{Y}, t) \in \Sigma^-$ we have $\mathbb{Y} = \varepsilon\mu^{-1}$. Denote a point on Σ^+ using $(\mathbb{Y}, t)$ and a point on Σ^- using $(\mathbb{X}, t)$. For $(\mathbb{Y}, t) \in \Sigma^+$, let

$$(\tilde{\mathbb{X}}, \tilde{t}) = \mathcal{N}(\mathbb{Y}, t).$$

Proposition 5.4. *We have for* $(\mathbb{Y}, t) \in \Sigma^+$,

$$\begin{aligned} \tilde{\mathbb{X}} &= (\mu\varepsilon^{-1})^{\frac{\alpha}{\beta}-1}\mathbb{Y}^{\frac{\alpha}{\beta}}, \\ \tilde{t} &= t + \frac{1}{\beta}\ln(\varepsilon\mu^{-1}) - \frac{1}{\beta}\ln\mathbb{Y}. \end{aligned} \tag{5.15}$$

Proof. Let T be the time it takes for the solution of the linear equation of proposition 5.1 from $(\varepsilon, Y, t) \in \Sigma^+$ to get to $(\tilde{X}, \varepsilon, \tilde{t}) \in \Sigma^-$. We have

$$\tilde{X} = \varepsilon e^{-\alpha T}, \qquad \varepsilon = Ye^{\beta T}, \qquad \tilde{t} = t + T$$

from which (5.15) follows. □

We are now ready to compute the return map $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$. We use $(\mathbb{X}, t)$ to represent a point on Σ^- and denote $(\tilde{\mathbb{X}}, \tilde{t}) = \mathcal{R}(\mathbb{X}, t)$.

Proposition 5.5. *The map* $\mathcal{R} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \to \Sigma^-$ *is given by*

$$\begin{aligned} \tilde{\mathbb{X}} &= (\mu\varepsilon^{-1})^{\frac{\alpha}{\beta}-1}[(1+\mathcal{O}(\varepsilon))P_L^+\mathbb{F}(\mathbb{X}, t)]^{\frac{\alpha}{\beta}}, \\ \tilde{t} &= t + (L^+ + L^-) + \frac{1}{\beta}\ln\mu^{-1}\varepsilon(1+\mathcal{O}(\varepsilon))P_L^+ + \mathcal{O}_{\mathbb{X},t}(\mu) - \frac{1}{\beta}\ln\mathbb{F}(\mathbb{X}, t), \end{aligned} \tag{5.16}$$

where

$$\mathbb{F}(\mathbb{X}, t) = \mathcal{W}_L(t+L^-) + P_L(P_L^+)^{-1}(1+\mathcal{O}(\varepsilon))\mathbb{X} + (P_L^+)^{-1}(1+P_L)\mathcal{O}_t(1) + \mathcal{O}_{\mathbb{X},t}(\mu), \tag{5.17}$$

and $\mathcal{W}_L(t)$ *and* P_L, P_L^+ *are as in (5.9) and (5.10).*

Proof. Using proposition 5.4 and proposition 5.2(b)(ii), we have

$$\begin{aligned} \hat{Z} &= P_L(1+\mathcal{O}(\varepsilon))\mathbb{X} + P_L^+\mathcal{W}_L(t+L^-) + P_L\mathcal{O}_t(1) + \mathcal{O}_{\mathbb{X},t}(\mu), \\ \hat{t} &= t + (L^+ + L^-) + \mathcal{O}_{\mathbb{X},t}(\mu). \end{aligned}$$

Let $\hat{\mathbb{Y}}$ be the $\mathbb{Y}$-coordinate for $(\hat{Z}, \hat{t})$, we have from proposition 5.2(a)(ii),

$$\hat{\mathbb{Y}} = (1+\mathcal{O}(\varepsilon))P_L^+\mathbb{F}(\mathbb{X}, t), \tag{5.18}$$

where $\mathbb{F}(\mathbb{X}, t)$ is as in (5.17). We then obtain (5.16) using (5.15). □

6. Dynamics of the Poincaré return maps

In section 5 we estimated the Poincaré return map $\mathcal{R} : \Sigma^- \to \Sigma^-$ and gave explicitly its leading terms in proposition 5.5. The proofs of theorems 2.1–2.5 will be based exclusively on the forms given in this proposition. Before moving on to these proofs, let us take a pause to make the return map appear more transparent. Let $(t, \mathbb{X}) \in \Sigma^-$, and $(t_1, \mathbb{X}_1) = \mathcal{R}(t, \mathbb{X})$. We have from proposition 5.5

$$\begin{aligned} t_1 &= t + \boldsymbol{a} - \frac{1}{\beta} \ln \mathbb{F}(t, \mathbb{X}, \mu) + \mathcal{O}_{\mathbb{X},t}(\mu), \\ \mathbb{X}_1 &= \boldsymbol{b}[\mathbb{F}(t, \mathbb{X}, \mu)]^{\frac{\alpha}{\beta}}, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \boldsymbol{a} &= \frac{1}{\beta} \ln \mu^{-1} + (L^+ + L^-) + \frac{1}{\beta} \ln(\varepsilon(1 + \mathcal{O}(\varepsilon)) P_L^+), \\ \boldsymbol{b} &= (\mu\varepsilon^{-1})^{\frac{\alpha}{\beta}-1}[(1 + \mathcal{O}(\varepsilon)) P_L^+]^{\frac{\alpha}{\beta}}, \end{aligned} \tag{6.2}$$

and

$$\mathbb{F}(t, \mathbb{X}, \mu) = \mathcal{W}(t) + \boldsymbol{k}\mathbb{X} + \mathbb{E}(t, \mu) + \mathcal{O}_{t,\mathbb{X}}(\mu), \tag{6.3}$$

in which $\mathcal{W}(t)$ is as in (2.4),

$$\boldsymbol{k} = P_L(P_L^+)^{-1}(1 + \mathcal{O}(\varepsilon)), \tag{6.4}$$

and

$$\mathbb{E}(t, \mu) = (P_L^+)^{-1}(1 + P_L)\mathcal{O}_t(1) + \mathcal{W}_L(t) - \mathcal{W}(t). \tag{6.5}$$

We note that, instead of $\mathcal{W}(t + L^-)$, we write $\mathcal{W}(t)$ in (6.3) and (6.5). This is achieved by a simple change of variable from $t \to t - L^-$. Nothing else is affected. We have

(i) $\boldsymbol{a} \approx \frac{1}{\beta} \ln \mu^{-1}$. $\boldsymbol{a} \to +\infty$ as $\mu \to 0$.
(ii) $\boldsymbol{b} \sim \mu^{\frac{\alpha}{\beta}-1}$. $\boldsymbol{b} \to 0$ as $\mu \to 0$ by (H)(ii).
(iii) $\boldsymbol{k} \sim \varepsilon^{\frac{\alpha}{\beta}}$.
(iv) $\mathbb{E}(t, \mu) \sim \varepsilon^{\frac{\beta}{\alpha}} \mathcal{O}_t(1)$.

We can think $\mathcal{R}$ as a 2D family of maps unfolded from the 1D maps

$$f(t) = t + \boldsymbol{a} - \frac{1}{\beta} \ln(\mathcal{W}(t) + \mathbb{E}(t, 0)). \tag{6.6}$$

Since $\boldsymbol{k} \gg \mu$, the first derivative of $\mathbb{F}(t, \mathbb{X}, \mu)$ with respect to $\mathbb{X}$ is approximately $\boldsymbol{k}$ and the unfolding from $f(t)$ to $\mathcal{R}$ in the $\mathbb{X}$-direction is determined mainly by the linear term $\boldsymbol{k}\mathbb{X}$. Note that when ε is sufficiently small, $\mathbb{E}(t, \mu)$ is a C^0-small perturbation to $\mathcal{W}(t)$.

In what follows we also write $\mathbb{F}(t, \mathbb{X}, \mu)$ as $\mathbb{F}(t, \mathbb{X})$ and $\mathbb{E}(t, \mu)$ as $\mathbb{E}(t)$.

6.1. Proof of theorem 2.1

(a) Assume $m < 0 < M$. We make ε sufficiently small so that

$$\sup_{t\in\mathbb{R}} |\mathbb{E}(t, \mu)| \ll \min\{|m|, \ M\}.$$

It then follows that there exist values of t so that

$$\mathbb{F}(t, 0, \mu) = \mathcal{W}(t) + \mathbb{E}(t, \mu) + \mathcal{O}_t(\mu) = 0, \tag{6.7}$$

where $\mathbb{F}(t, \mathbb{X}, \mu)$ is as in (6.3). Observe that local unstable manifold of $(X, Y) = (0, 0)$ is defined by $\mathbb{X} = 0$ and the local stable manifold is defined by $\mathbb{Y} = 0$. Let $(\hat{t}(t, \mathbb{X}), \hat{\mathbb{Y}}(t, \mathbb{X})) = \mathcal{M}(t, \mathbb{X})$. For t satisfying (6.7), we have

$$\hat{\mathbb{Y}}(t, 0) = 0.$$

This proves $W^u \cap W^s \neq \emptyset$.

(b) Assume $m > 0$. We again make ε sufficiently small so that

$$\sup_{t\in\mathbb{R}} |\mathbb{E}(t)| \ll m.$$

Let

$$\Sigma^- = \{(t, \mathbb{X}) : t \in \mathbb{R},\ \mathbb{X} \in [0, 1]\}.$$

Then for any given $(t, 0) \in \Sigma^-$, the $\mathbb{Y}$-coordinates of $\mathcal{R}^n(t, 0)$ will be positive for all $n \geqslant 1$ according to (6.1) for $\mathcal{R}$. This rules out the possibility for such orbit to be part of W^s.

(c) If $M < 0$, then all solutions starting from the line $\mathbb{X} = 0$ on Σ^- will hit Σ^+ with a negative $\mathbb{Y}$-coordinate. Such solutions will hit $\mathbb{Y} = -\varepsilon$ then get out of $\mathcal{U}_\varepsilon$. These solutions, together with the surface defined by $\mathbb{Y} = -\varepsilon$ and $\mathbb{X} = 0$, form a 2D surface in the extended phase space that prevents the possibility of any of these solutions to intersect W^s in the future. □

6.2. *Proof of theorem 2.2*

Assume that ε is sufficiently small so that

$$\sup_{t\in\mathbb{R}} |\mathbb{E}(t, \mu)| \ll \min\{|m^\pm|,\ M^\pm\}.$$

Let $\{a_k\}_{k=-\infty}^{+\infty}$ be a monotone bi-infinite sequence. $a_k \to \pm\infty$ as $k \to \pm\infty$ is such that $\lim_{k\to\pm\infty} \mathcal{W}(a_k) = M^\pm$. Similarly, let $\{b_k\}_{k=-\infty}^{\infty}$, $b_k \to \pm\infty$ as $k \to \pm\infty$ be such that $\lim_{k\to\pm\infty} \mathcal{W}(b_k) = m^\pm$. Without loss of generality we assume

$$\mathcal{W}(a_k) > \frac{99}{100} M^+, \qquad \mathcal{W}(b_k) < \frac{99}{100} m^+$$

for all $k \geqslant 0$; and

$$\mathcal{W}(a_k) > \frac{99}{100} M^-, \qquad \mathcal{W}(b_k) < \frac{99}{100} m^-$$

for all $k < 0$. We also assume that

$$b_{k-1} < a_k < b_k$$

for all $k \in \mathbb{Z}$. We let $\Sigma^- = \{(t, \mathbb{X}) : 0 \leqslant \mathbb{X} \leqslant 1\}$ and denote

$$D_k = \{(t, \mathbb{X}) \in \Sigma^-,\ a_k \leqslant t \leqslant b_k\}.$$

First we prove that there exists a non-self intersecting, continuous curve ξ_k inside of D_k, connecting $\mathbb{X} = 0$ and $\mathbb{X} = 1$ and satisfying

$$\mathbb{F}(t, \mathbb{X}) = 0.$$

This claim holds because

(a) the set of points inside of D_k satisfying $\mathbb{F} = 0$ is the intersection of D_k with the pre-image of $\mathcal{M}$ of the stable manifold $\mathbb{Y} = 0$ in Σ^+, therefore can consists of at most finitely many non-self intersecting continuous curves;

(b) these curve segments can only end at either $\mathbb{X} = 0$ or $\mathbb{X} = 1$ because $\mathbb{F}(a_k, \mathbb{X}) > 0$, $\mathbb{F}(b_k, \mathbb{X}) < 0$ for $0 \leqslant \mathbb{X} \leqslant 1$ and

(c) if none of these continuous segments connecting $\mathbb{X} = 1$ and $\mathbb{X} = 0$, then we could find a continuous path in D_k connecting $t = a_k$ and $t = b_k$, on which $\mathbb{F} \neq 0$, but this is not possible because the values of $\mathbb{F}$ at the end of this path has opposite sign.

We further argue that there must be a ξ_k as in the above so that $\mathbb{F}$ assumes opposite sign on different sides of ξ_k. This is because if $\mathbb{F}$ for ξ_k constructed in last paragraph assumes the same sign on both side, then it can be used as a vertical boundary together with either $t_k = a_k$ or $t_k = b_k$ to define a new D_k. Then a new ξ_k is constructed. This process must end.

We define V_k as the vertical strip bounded by ξ_k and a slight shift of ξ_k to the positive side of $\mathbb{F}$. The vertical strips $\{V_k\}_{k=-\infty}^{+\infty}$ now serve as the bi-infinite sequence of vertical strips of definition 2.1. For definition 2.1 to fulfil, it suffices to observe that

$$t_1(\xi_k) = +\infty$$

for all k. □

6.3. *Proof of theorem 2.3 and 2.4*

First we prove theorem 2.3. With the assumption that $m^+ < 0 < M^+$, we assume the same sequences $a_k, b_k \to +\infty$ in the proof of theorem 2.2 but only for $k > 0$. The vertical strips V_k for $k > 0$ are defined the same as in the proof of theorem 2.2. To construct V_k for $k < 0$ in definition 2.2, we first define two monotonically decreasing sequence $b_k, a_k \to -\infty$ as $k \to -\infty$ inductively so that

$$b_{k-1} < a_k < b_k$$

and we assume that

$$t_1(b_{k-1}, 0) > b_k, \qquad t_1(a_{k-1}, 0) < a_k.$$

Note that we can start with a b_0 sufficiently negative so that

$$\mathbb{F}(t, \mathbb{X}) > 0$$

for all $t \in (-\infty, b_0)$, $0 \leqslant \mathbb{X} \leqslant 1$, and we make the image of $t = b_0$ under $\mathcal{R}$ to be on the right side of V_1 by letting μ small. We then take a_0 sufficiently negative than b_0 so that the image of $t = a_0$ is on the left side of V_1. Finally, for $k < 0$, we let

$$V_k = \{(t, \mathbb{X}) \in \Sigma^-, \ a_k < t < b_k\}.$$

The vertical strips $\{V_k\}_{k=-\infty}^{+\infty}$ now serve as the bi-infinite sequence of vertical strips of definition 2.2.

To prove theorem 2.4 we use the same V_k constructed in the proof of theorem 2.2 for $k < 0$. We take $V_0 = \{(t, \mathbb{X} \in \Sigma^-, t > a_0\}$ where a_0 is such that $\mathbb{F}(t, \mathbb{X}) > 0$ on V_0. □

6.4. *Proof of theorem 2.5*

Again, let ε be sufficiently small so that

$$\sup_{t \in \mathbb{R}} |\mathbb{E}(t)| \ll \min\{m^{\pm}, \ M\}.$$

Let $\{a_n\}_{n \geqslant 0}$, $\{b_n\}_{n \geqslant 0}$, $a_n, b_n \to +\infty$ be monotone sequences such that $\lim_{n \to +\infty} \mathcal{W}(a_n) = M^+$, $\lim_{n \to +\infty} \mathcal{W}(b_n) = m^+$. We can assume without loss of generality that

$$b_n < a_n < b_{n+1}$$

for all $n > 0$, and further that $a_{n+1} - a_n,\ b_{n+1} - b_n < L^+$ for some fixed $L^+ > 0$ because of the assumption that both m^+ and M^+ are densely approached by L^+-sequences. Let

$$\tilde{M} = \max\left\{M^+ - \tfrac{1}{10}(M^+ - m^+),\ \tfrac{99}{100}M^+\right\}$$

and

$$\tilde{m} = \min\left\{m^+ + \tfrac{1}{10}(M^+ - m^+),\ \tfrac{101}{100}m^+\right\}.$$

We can also assume that

$$\mathcal{W}(a_n) > \tilde{M}, \qquad \mathcal{W}(b_n) < \tilde{m}$$

for all $n \geqslant 0$.

To construct V_n for $n > 0$, first we denote

$$D_n = \{(t, \mathbb{X}) \in \Sigma^-,\ b_n < t < a_n\},$$

and claim that there exists a non-self intersecting continuous curve ξ_n in D_n that connects $\mathbb{X} = 0$ and $\mathbb{X} = 1$, such that

$$\mathbb{F}(t, \mathbb{X}) = \tfrac{3}{2}\tilde{m} \tag{6.8}$$

on ξ_k. To prove this claim we follow the same argument as in the proof of theorem 2.2, using again the fact that the solutions of (6.8) is the pre-image of $\mathcal{M}$ of the horizontal line $\mathbb{Y} = (1 + \mathcal{O}(\varepsilon))P_L^+\tilde{m}$ in Σ^+. Let the vertical strip defined by ξ_n and $t = a_n$ be V_n.

For definition 2.2(i), it suffices for us to have, for $n < 0$,

$$t_1(\xi_n, \delta) - t_1(a_n, \delta) > 2kL^+.$$

where (ξ_n, δ) is a point on ξ_n. From (6.1)

$$t_1(\xi_n, \delta) - t_1(a_n, \delta) = \xi_n - a_n + \beta^{-1}(\ln\tilde{M} - \ln\tfrac{3}{2}\tilde{m}) + \mathcal{O}(\mu).$$

So it suffices to have

$$\frac{\tilde{M}}{\tilde{m}} > \tfrac{3}{2}\mathrm{e}^{3k\beta L^+}.$$

By $\tilde{M} > \frac{99}{100}M^+, \tilde{m} < \frac{101}{100}m^+$, it then suffices that

$$M^+ > 2m^-\mathrm{e}^{3\beta kL^+}.$$

The vertical strips V_n for $n \leqslant 0$ are constructed the same as in the proof of theorem 2.3. Again by making μ sufficiently small, we can make $\boldsymbol{a}$ in (6.1) large enough so that $\mathcal{R}(V_0)$ horizontally cross some $V_n, n > 0$. □

Appendix A. Proof of proposition 5.1

In this appendix, we give results on smooth linearization for a nonautonomous differential equation around an equilibrium point. Consider a nonautonomous differential equation in $\mathbb{R}^d$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(x, t, \mu), \tag{A.1}$$

where A is a $d \times d$ real matrix, f is a nonlinear function of high order in x, and μ is a parameter.

For the matrix A, we assume the following.

Hypothesis A. A is hyperbolic, that is, A has no eigenvalues on the imaginary axis.

This condition implies that there exists an invariant splitting of the phase space $\mathbb{R}^d = E_u \oplus E_s$ with the associated projections Π_u and Π_s and positive constants α, β and K such that

$$\begin{aligned} &||e^{At}\Pi_s|| \leqslant K e^{-\beta t} \qquad \text{for } t \geqslant 0,\\ &||e^{At}\Pi_u|| \leqslant K e^{\beta t} \qquad \text{for } t \leqslant 0,\\ &||e^{At}|| \leqslant K e^{\alpha|t|} \qquad \text{for } t \in \mathbb{R}. \end{aligned} \tag{A.2}$$

We assume the following for the nonlinear term $f(t, x, \mu)$.

Hypothesis B. There are an open neighbourhood U of 0 in $\mathbb{R}^d$ and $\mu_0 > 0$ such that

(i) $f : U \times \mathbb{R} \times [-\mu_0, \mu_0] \to \mathbb{R}^d$ is C^N in x for some integer $N \geqslant 2$ and continuous in (t, μ) with uniformly bounded derivatives

$$\sup_{(x,t,\mu)\in U\times\mathbb{R}\times[-\mu_0,\mu_0]} ||D_x^k f(x, t, \mu)|| \leqslant K_1,$$

where K_1 is a positive constant;

(ii) $f(0, t, \mu) = 0$ and $D_x f(0, t, \mu) = 0$.

In order to construct the transformation to linearizing equation (A.1), we use the standard cut-off function to modify the nonlinearity $f(t, x, \mu)$.

Let $\sigma(s)$ be a C^∞ function from $(-\infty, \infty)$ to $[0, 1]$ with

$$\sigma(s) = 1 \text{ for } |s| \leqslant 1, \qquad \sigma(s) = 0 \text{ for } |s| \geqslant 2,$$
$$\sup_{s\in\mathbb{R}} |\sigma'(s)| \leqslant 2.$$

Let ρ be a positive constant such that the ball $B(0, \rho) \subset U$. We consider a modification of $f(t, x, \mu)$. Let

$$\tilde{f}(x, t, \mu) = \sigma_\rho(|x|) f(x, t, \mu), \qquad \text{where} \quad \sigma_\rho(|x|) = \sigma\left(\frac{|x|}{\rho}\right).$$

An elementary calculation gives

(i) $\tilde{f}(x, t, \mu) = f(x, t, \mu)$, for $|x| \leqslant \rho$;

(ii) there exists a positive constant K_2 such that

$$||D_x \tilde{f}(x, t, \mu)|| \leqslant 10K_1\rho \qquad \text{for all } (x, t, \mu) \in \mathbb{R}^d \times \mathbb{R} \times [-\mu_0, \mu_0];$$
$$\sup_{(x,t,\mu)\in\mathbb{R}^d\times\mathbb{R}\times[-\mu_0,\mu_0]} ||D_x^k \tilde{f}(x, t, \mu)|| \leqslant \tilde{K}_2 \text{ for } 2 \leqslant k \leqslant N.$$

Let $x(t, x_0, \omega_1, \mu)$ denote the solution of

$$\frac{dx}{dt} = Ax + \tilde{f}(x, t + \omega_1, \mu), \qquad x(0) = x_0.$$

Clearly, $x(t, x_0, \omega_1, \mu)$ exists for all $t \in \mathbb{R}$, $\omega_1 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ and $\mu \in [-\mu_0, \mu_0]$ and satisfies

$$x(t, x_0, \omega_1, \mu) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(x(s, x_0, \omega_1, \mu), s + \omega_1, \mu)\, ds.$$

Note that $x(t, x_0, \omega_1, \mu)$ forms a nonautonomous dynamical system. This together with the metric dynamical system $\theta^t\omega = (t + \omega_1, \omega_2)$ forms a cocycle, where we denote μ by ω_2. We will first consider the corresponding time discrete nonautonomous dynamical system $\phi(n, \omega, x) = x(n, x, \omega_1, \mu)$. We write its time-one map as

$$\varphi(\omega, x) := \Phi x + F(\omega, x),$$

where $\Phi = \mathrm{e}^A$ and $F(\omega, x) = \int_0^1 \mathrm{e}^{A(1-s)} f(x(s, \omega_1, x_0, \mu), s + \omega_1, \mu)\,\mathrm{d}s$. Note that

$$F(\omega, 0) = 0 \qquad \text{and} \qquad D_x F(\omega, 0) = 0,$$
$$\sup_{(\omega x)\in(\mathbb{R}\times[-\mu_0,\mu_0])\times\mathbb{R}^d} ||D_x^k F(\omega, x)|| \leqslant \tilde{K}_3 \text{ for } 2 \leqslant k \leqslant N \tag{A.3}$$

for some positive constant K_3. We choose $\rho > 0$ such that

$$\rho \leqslant \min\left\{\frac{\beta}{60K^2K_3}, \qquad \frac{1}{20K\mathrm{e}^\alpha K_3}\right\}. \tag{A.4}$$

This implies that $\psi(\omega)x := \Phi x + F(\omega, x)$ is a C^N diffeomorphism on $\mathbb{R}^d$. $\psi(\omega)$ generates a C^N nonautonomous dynamical system $\phi(n, \omega, x)$. We note that each sequence x_n satisfies $x_n = \phi(n, \omega, x_0)$ if and only if x_n satisfies

$$x_{n+1} = \Phi x_n + F(n + \omega, x_n). \tag{A.5}$$

For the time discrete nonautonomous dynamical system generated by $\varphi(\omega, x)$, using theorem 4.12 from [LL] with a mild modification, we have the following theorem of linearization.

Theorem A.1. *Assume hypotheses (A) and (B) hold. For each integer $k > 0$, there exists an integer $N_0 = N_0(k, \alpha, \beta)$ such that if φ is C^N for $N \geqslant N_0$ and the real parts of eigenvalues of A, $\lambda_1, \ldots, \lambda_p$, satisfy the nonresonant conditions up to order N_0,*

$$\lambda_i \neq (\tau, \lambda), \quad \text{for all} \quad 1 \leqslant i \leqslant p, \qquad 2 \leqslant |m| \leqslant N_0,$$

then there is a C^k local diffeomorphism $x = h(\omega, y) = y + \tilde{h}(\omega, y)$ such that

$$h(\theta\omega, \cdot) \circ \varphi(\omega, x) = \Phi h(\omega, x),$$

where $\tilde{h} : \mathbb{R} \times [-\mu_0, \mu_0] \times V \to \mathbb{R}^d$ is a C^k function in x with all bounded derivatives and continuous in ω and $\tilde{h}(\omega, 0) = 0$ and $D_x\tilde{h}(\omega, 0) = 0$, V is an open neighbourhood of 0.

The statement holding for all ω instead almost surely is due to the fact that A is a constant matrix. The continuity instead of measurability in ω follows from the fact that the stable and unstable manifolds and invariant foliations are continuous in ω.

Letting

$$H(x, \omega) = \int_0^1 \mathrm{e}^{-As} h\big(\theta^s\omega, \phi(s, \omega, x)\big)\,\mathrm{d}s$$

we have the following corollary.

Corollary A.1. *Assume hypotheses (A) and (B) hold. For each integer $k > 0$, there exists an integer $N_0 = N_0(k, \alpha, \beta)$ such that if f is C^N for $N \geqslant N_0$ and the real parts of eigenvalues of A, $\lambda_1, \ldots, \lambda_p$, satisfy the nonresonant conditions up to order N_0,*

$$\lambda_i \neq (\tau, \lambda), \quad \text{for all } 1 \leqslant i \leqslant p, \qquad 2 \leqslant |m| \leqslant N_0,$$

where $(\tau, \lambda) = \sum_{j=1}^p \tau_j\lambda_i$, $(\tau_1, \ldots, \tau_p) \in \mathbb{N}^p$, $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $|m| = \sum_{j=1}^p m_j$, then there is an invertible transformation $y = H(x, t, \mu) = x + \tilde{H}(x, t, \mu)$ which transforms equation (A.1) to the linear equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ay$$

where $\tilde{H} : V \times \mathbb{R} \times [-\mu_0, \mu_0] \to \mathbb{R}^d$ is C^k in y and continuous in (t, μ) with all bounded derivatives and $\tilde{H}(0, t, \mu) = 0$ and $D_x\tilde{H}(0, t, \mu) = 0$, V is an open neighbourhood of 0.

Proposition 5.1 follows from this corollary.

Appendix B. Proof of proposition 5.2

In this appendix we prove proposition 5.2. We start with the defining equations for Σ^+ in (s, Z, t).

Lemma B.1. *We have for* $(s, Z, t) \in \Sigma^+$

$$s = L^+ + \mathcal{O}_{Z,t}(\mu).$$

Proof. We have on Σ^+,

$$\begin{aligned} a(s) + v(s)z &= \varepsilon + \mathbb{P}(\varepsilon, Y) + \mu\tilde{\mathbb{P}}(\varepsilon, Y, t), \\ b(s) - u(s)z &= Y + \mathbb{Q}(\varepsilon, Y) + \mu\tilde{\mathbb{Q}}(\varepsilon, Y, t). \end{aligned} \tag{B.1}$$

By the definition

$$\begin{aligned} a(L^+) &= \varepsilon + \mathbb{P}(\varepsilon, 0), \\ b(L^+) &= \mathbb{Q}(\varepsilon, 0). \end{aligned} \tag{B.2}$$

Let

$$\begin{aligned} W_1 &= a(s) - a(L^+) + v(s)z - \mu\tilde{\mathbb{P}}(\varepsilon, 0, t), \\ W_2 &= b(s) - b(L^+) - u(s)z - \mu\tilde{\mathbb{Q}}(\varepsilon, 0, t). \end{aligned} \tag{B.3}$$

We have from (B.1) and (B.2),

$$\begin{aligned} W_1 &= \mathbb{P}(\varepsilon, Y) - \mathbb{P}(\varepsilon, 0) + \mu(\tilde{\mathbb{P}}(\varepsilon, Y, t) - \tilde{\mathbb{P}}(\varepsilon, 0, t)), \\ W_2 &= Y + \mathbb{Q}(\varepsilon, Y) - \mathbb{Q}(\varepsilon, 0) + \mu((\tilde{\mathbb{Q}}(\varepsilon, Y, t) - \tilde{\mathbb{Q}}(\varepsilon, 0, t)) \end{aligned}$$

which we rewrite as

$$\begin{aligned} W_1 &= (\mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))Y + \mathcal{O}_{Y,t}(1)Y^2, \\ W_2 &= (1 + \mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))Y + \mathcal{O}_{Y,t}(1)Y^2. \end{aligned} \tag{B.4}$$

We first obtain

$$Y = (1 + \mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2 \tag{B.5}$$

by inverting the second line in (B.4). We then substitute it into the first line in (B.4) to obtain

$$\begin{aligned} W_1 &= (\mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))((1 + \mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2) \\ &\quad + \mathcal{O}_{Y,t}(1)((1 + \mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2)^2 \\ &= (\mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2. \end{aligned}$$

Consequently,

$$F(s, Z, t) := W_1 - (\mathcal{O}(\varepsilon) + \mu\mathcal{O}_t(1))W_2 + \mathcal{O}_{W_2,t}(1)W_2^2 = 0, \tag{B.6}$$

where W_1, W_2 as function of s, Z, t are defined by (B.3). To rewrite W_1 and W_2, we let

$$\xi = s - L^+ \tag{B.7}$$

and expand $a(s)$ in terms of ξ as

$$a(s) = a(L^+) + a'(L^+)\xi + \sum_{i=2}^{\infty} a_i(L^+)\xi^i.$$

Expansions for $b(s)$, $u(s)$ and $v(s)$ are similar. We have

$$\begin{aligned} W_1 &= a'(L^+)\xi + \sum_{i=2}^{\infty} a_i(L^+)\xi^i + v(L^+)z + (v'(L^+)\xi + \sum_{i=2}^{\infty} v_i(L^+)\xi^i)z - \mu\tilde{\mathbb{P}}(\varepsilon, 0, t), \\ W_2 &= b'(L^+)\xi + \sum_{i=2}^{\infty} b_i(L^+)\xi^i - u(L^+)z - (u'(L^+)\xi + \sum_{i=2}^{\infty} u_i(L^+)\xi^i)z - \mu\tilde{\mathbb{Q}}(\varepsilon, 0, t). \end{aligned} \tag{B.8}$$

We now put (B.8) for W_1 and W_2 back into equation (B.6) and replace z by μZ. We obtain

$$(a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+) + h(t, \xi)\xi)\xi = \mathcal{O}_{Z,t}(\mu)$$

where the C^r norm of $h(t, \xi)$ is bounded from above by $K(\varepsilon)$. Also note that $a'(L^+) \approx -\alpha\varepsilon$, $b'(L^+) = \mathcal{O}(\varepsilon^2)$. We finally obtain

$$s = L^+ + \mathcal{O}_{Z,t}(\mu)$$

by solving ξ. This completes the proof of lemma B.1. □

Lemma B.1 is not precise enough. We need the following refinement.

Lemma B.2. *We have on* Σ^+,

$$s - L^+ = -\frac{v(L^+) + \mathcal{O}(\varepsilon)u(L^+)}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)}z + \frac{\mu}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)}\mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu^2).$$

Proof. It suffices for us to drop all terms that are $\mathcal{O}_{Z,t}(\mu^2)$ in equation (B.6) to solve for ξ. From lemma B.1 we conclude that all terms in ξ, z of degree higher than one are $\mathcal{O}_{Z,t}(\mu^2)$. With these terms all dropped, (B.6) becomes

$$(a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+))\xi + (v(L^+) + \mathcal{O}(\varepsilon)u(L^+))z = \mu\mathcal{O}_t(1), \tag{B.9}$$

from which the estimates of lemma B.2 on Σ^+ follows. □

Recall that $\mathbb{X} = \mu^{-1}X$, $\mathbb{Y} = \mu^{-1}Y$.

Lemma B.3. *On* Σ^+ *we have*

$$\mathbb{Y} = (1 + \mathcal{O}(\varepsilon))Z + \mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu).$$

Proof. We have

$$\begin{aligned}
Y &= (1 + \mathcal{O}(\varepsilon))(b'(L^+)\xi - u(L^+)z - \mu\tilde{\mathbb{Q}}(\varepsilon, 0, t)) + \mathcal{O}_{Z,t}(\mu^2) \\
&= (1 + \mathcal{O}(\varepsilon))\left(-\left(u(L^+) + b'(L^+)\frac{v(L^+) + \mathcal{O}(\varepsilon)u(L^+)}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)}\right)z\right. \\
&\quad \left. + \frac{\mu b'(L^+)}{a'(L^+) - \mathcal{O}(\varepsilon)b'(L^+)}\mathcal{O}_t(1) - \mu\tilde{\mathbb{Q}}(\varepsilon, 0, t)\right) + \mathcal{O}_{Z,t}(\mu^2) \\
&= (1 + \mathcal{O}(\varepsilon))z + \mu\mathcal{O}_t(1) + \mathcal{O}_{Z,t}(\mu^2),
\end{aligned} \tag{B.10}$$

where the first equality follows from using (B.5), (B.8) and lemma B.1; the second equality from using lemma B.2. To obtain the third equality we use $u(L^+) = -1 + \mathcal{O}(\varepsilon)$, $a'(L^+) \approx -\alpha\varepsilon$, $b'(L^+) = \mathcal{O}(\varepsilon^2)$. □

Lemma B.1 is proposition 5.2(a)(i) and lemma B.3 is proposition 5.2(a)(ii). Proposition 5.2(b) follows from parallel computations.

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