Homoclinic Tangle in Restricted Three-Body Problem

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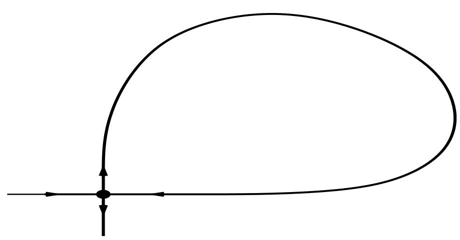
Time-periodic System and Chaos

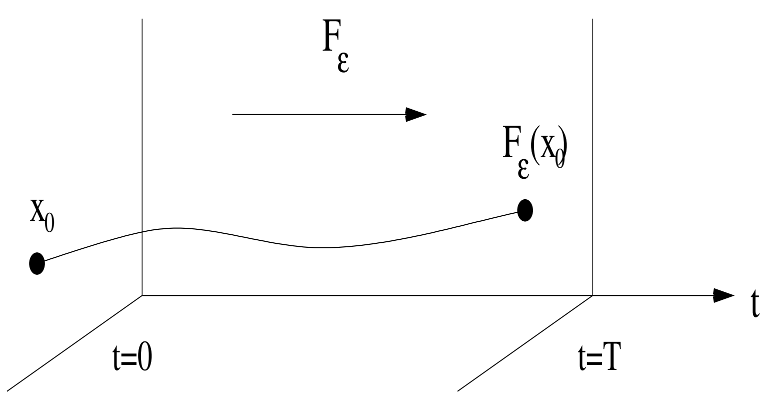
– Equation of study: $x \in \mathbb{R}^2$

$$\frac{dx}{dt} = f(x) + \varepsilon g(x,t)$$

– Assumptions: (1) $g(x, t+T) = g(x,t)$; (2) for $\varepsilon = 0$, $x = 0$ is a saddle point with a homoclinic solution.



– The time-T map:



Homoclinic Tangle:

– For $\varepsilon = 0$: Homoclinic solution form an invariant loop for time-T map;

(a) (b)

– For $\varepsilon \neq 0$: This loop is broken, forming a homoclinic tangle

(a) (b)

(Q1) Dynamic Structure of HT.

(Q2) Verification of HT in ODEs.

On (Q1): the dynamics of HT:

– Assume fixed point is dissipative

– Let $\mu = -\ln \varepsilon$, $\mu \to \infty$ as $\varepsilon \to 0$

– Three main dynamical scenarios

Stable HT; Transient HT; Chaotic HT

– There is a periodic pattern for the alternative occurrence of (A), (B) (C) in μ as $\mu \to \infty$.

On (Q2): Existence of HT:

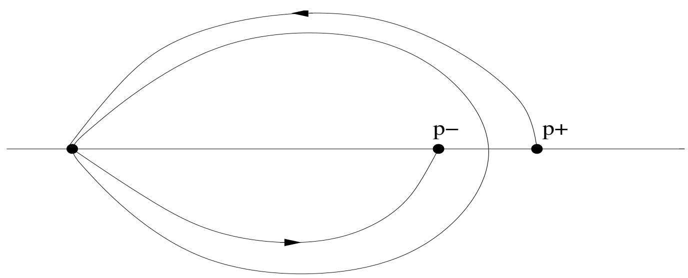
– The Poincare/Melnikov Method

– Degenerate Cases (High Order Method)

Today's Talk: Another degenerate case.

Verification of HT in ODE:

– The Splitting Distance



$$D(t_0, \varepsilon) = \varepsilon D_0(t_0) + \varepsilon^2 D_1(t_0) + \cdots$$

– Homoclinic Tangle: Exists for small ε if there exists t_0 so that

$$D_0(t_0) = 0, \qquad D_0'(t_0) \neq 0.$$

– Poincaré/Melnikov Integral: Explicit Integral Formula for $D_0(t_0)$.

– **A Historic Fact:** HT was discovered in Poincare's study of the 3-body Problem, but his method of verification failed to apply.

Equation of The Planar 3-Body Problem

– Three bodies: m_1, m_2, m_3 in $\mathbb{R}^2$.

– Jacobian Coordinates: $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$.

– The Potential Function:

$$U = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}$$

where r_{ij} is the distance from m_i to m_j.

– Equations of Motion:

$$\begin{aligned}
\ddot{r}_1 &= r_1 \dot{\theta}_1^2 + \mu_1^{-1} \partial_{r_1} U; \\
\ddot{\theta}_1 &= -\frac{2\dot{r}_1 \dot{\theta}_1}{r_1} + \mu_1^{-1} \frac{1}{r_1^2} \partial_{\theta_1} U; \\
\ddot{r}_2 &= r_2 \dot{\theta}_2^2 + \mu_2^{-1} \partial_{r_2} U; \\
\ddot{\theta}_2 &= -\frac{2\dot{r}_2 \dot{\theta}_2}{r_2} + \mu_2^{-1} \frac{1}{r_2^2} \partial_{\theta_2} U.
\end{aligned}$$

where

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \qquad \mu_2 = \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}.$$

Circular Restricted 3-body Problem

– Let $m_3 = 0, \quad m_1 + m_2 = 1, \quad \mu = m_2.$

– Equation for r_1, θ_1:

$$\ddot{r}_1 = r_1\dot{\theta}_1^2 - \frac{1}{r_1^2}; \qquad \ddot{\theta}_1 = -\frac{2\dot{r}_1\dot{\theta}_1}{r_1}$$

– Circular motion $r_1 = 1, \quad \theta_1 = t.$

– Circular restricted 3-body problem

$$\ddot{r}_2 = r_2\dot{\theta}_2^2 + f; \qquad \ddot{\theta}_2 = -\frac{2\dot{r}_2\dot{\theta}_2}{r_2} + g$$

where

$$\begin{aligned} f = & -\frac{m_1\,(r_2 + m_2\cos(\theta_2 - t))}{r_{13}^3} \\ & -\frac{m_2\,(r_2 - m_1\cos(\theta_2 - t))}{r_{23}^3}; \end{aligned}$$

$$g = \frac{m_1 m_2 \sin(\theta_2 - t)}{r_2}\left(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3}\right);$$

More on Restrcited 3-body Problem

– The Equations:

$$\ddot{r}_2 = r_2\dot{\theta}_2^2 + f; \qquad \ddot{\theta}_2 = -\frac{2\dot{r}_2\dot{\theta}_2}{r_2} + g$$

where f, g are explicitly given as before.

– The Jacobi Integral:

$$J = \frac{1}{2}\left(\dot{r}_2^2 + r_2^2(\dot{\theta}_2 - 1)^2\right) - \frac{1}{2}r_2^2 - m_1 r_{13}^{-1} - m_2 r_{23}^{-1}$$

where J is the integral constant.

– With fixed J, rewrite equation in new variables (X, Y, θ) as

$$\begin{aligned}
\frac{d\theta}{d\tau} &= \sqrt{2}\left(X - |J|^3 U^{-3} X^{-3}\right); \\
\frac{dX}{d\tau} &= Y + \sqrt{2}|J|^{-2} U^2 X^2 G; \\
\frac{dY}{d\tau} &= X - X^3 - XF + \sqrt{2}|J|^{-2} U^2 XYG
\end{aligned}$$

Applying Poincare-Melnikov Method?

– F, G, U are in terms of X, Y, periodic in θ.

– Small parameter: $\varepsilon = J^{-1}$, or $\mu = m_2 << 1$.

– Regarding (X, Y) as phase variable, θ as time;

– $(X, Y) = (0, 0)$ is a fixed point;

– It admit a homoclinin solution for unperturbed equation.

Hurdles:

(1) How to justify Poincare Melnikov method in degenerate case (fixed point is not a saddle);

(2) Calculating Melnikov function is rather challenging.

Past Results

– First result for HT was from Sitnikov

- A different Restricted 3-Body Problem.
- Original analysis exceedingly technical.
- Result motivated Smale's Horseshoe.
- A Clear Presentation in Moser's book.

– For Circular Restricted 3-Body Problem:

(A) J. Llibre and C. Simo, Math. Ann., 1980

(B) Zhihong Xia, JDE, 1992

(C) M. Guardia, etc, Invent. Math., 2016

(A) (Llibre and Simo) There exists J_0 sufficiently large, so that for any given $J > J_0$, there exists a $\mu_0(J)$, $0 < \mu < \mu_0(J)$, HT exists.

(B) (Xia) For all but some finite number of values of μ, there is some J such that HT exists.

(C) (Guardia, et al.) For any given mass ratio μ, there exists a $J_0(\mu)$ sufficiently large, so that for all $J > J_0(\mu)$, HT exists.

Remarks: In terms of method,

– **(A), (B)** Used Poincare-Melnikov method relying on a previous work of McGehee.

– **(C)** is much more elaborated. Relied on previous theory on exponentially small splitting built over the last forty years.

Main Result: *there exists J_0 sufficiently large, so that for any given $J > J_0$, HT exists for all but some finite number of values of μ.*

This Result is not covered by **(A), (B)** or **(C)**.

- It covers **(A)**,
- It covers a different region of J from **(B)**.
- Compare to **(C)**:
  - We cover a **uniform interval of J**, but for each J we miss finitely many μ.
  - **(C)** covers all μ but the interval of J is **NOT uniform**.

Remarks on Method of Study:

– **Like** **(A)** and **(B)**, I also generalize Poincare-Melnikov method to this degenerate situation.

– **UnLike** **(A)** and **(B)**, My method of generalization do not rely on McGehee's local analysis of degenerate fixed point.

– My study rely on the fact that restricted three-body problem is a **perturbed Duffing equation**.

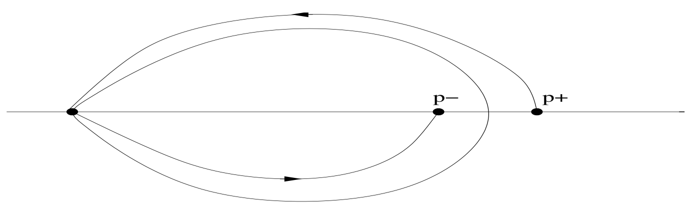
$$
\begin{aligned}
\frac{d\theta}{d\tau} &= \sqrt{2}\left(X - |J|^3 U^{-3} X^{-3}\right); \\
\frac{dX}{d\tau} &= Y + \sqrt{2}|J|^{-2} U^2 X^2 G; \\
\frac{dY}{d\tau} &= X - X^3 - XF + \sqrt{2}|J|^{-2} U^2 XYG
\end{aligned}
$$

Global Stable and Unstable Solution

– The homoclinic solution:

$$a(\tau) = \frac{2\sqrt{2}}{e^{\tau} + e^{-\tau}}; \quad b(\tau) = \frac{2\sqrt{2}\left(e^{-\tau} - e^{\tau}\right)}{(e^{\tau} + e^{-\tau})^2}.$$

–the global stable solution:



– **The integral equations:**

$$\Theta(\tau) = \int_0^{\tau} \frac{\sqrt{2}S}{\varepsilon^3 U^3_{\theta_0,\psi}(x+a)^3 a^3} d\tau;$$
$$M(\tau) = -\frac{1}{a}\int_{\tau}^{+\infty} \left(b'P + bQ\right) d\tau;$$
$$W(\tau) = a\int_0^{\tau} \frac{1}{a}(\tilde{H}P - HQ)d\tau.$$

where (M, W, Θ) are new variables.

Main Proposition on Stable Solution: treating $\varepsilon = J^{-1}$ as parameter of perturbation

There exists an $\varepsilon_0 > 0$ *so that for any given* $(\theta_0, \rho, \varepsilon) \in \mathcal{D}_{\theta_0,\rho,\varepsilon}$ *where*

$$\mathcal{D}_{\theta_0,\rho,\varepsilon} = \mathbb{R} \times (-\varepsilon_0, \varepsilon_0 + 1/2) \times (0, \varepsilon_0),$$

the integral equations in the above admits a unique solution. In addition, $X(0)$ *of this solution as a function of* $\theta_0, \rho, \varepsilon$ *are real analytic on* $\mathcal{D}_{\theta_0,\rho,\varepsilon}$.

Remarks on Proof:

– By using the Contracting Mapping Theorem. Need ε_0 small to make contraction happen.

– Analytical dependency on parameters as claimed are critically important for us.

– It works because of rather sophisticated balances of the singular order of X in the equation of Θ and the order of perturbation functions. Luckily this is carried through for the restricted 3-body problem.

Existence of HT: Treating μ as a perturation parameter.

The Melnikov Integral

$$\begin{aligned}\mathbb{D}_0 \;=\; & -\frac{\sqrt{2}}{2\varepsilon\sqrt{\varepsilon}}\int_{-\infty}^{+\infty}(b'a^2+ab^2)\sin(\theta_0+\psi)\left(1-R_1^{-3}\right)d\tau\\ & +\frac{1}{2\varepsilon^3\sqrt{\varepsilon}}\int_{-\infty}^{+\infty}ab[\varepsilon^2a^2\cos(\theta_0+\psi)(2+R_1^{-3})\\ & +(1-R_1^{-3})]d\tau\end{aligned}$$

where

$$R_1=\sqrt{1-2\varepsilon^2a^2\cos(\psi+\theta_0)+\varepsilon^4a^4}.$$

$$a(\tau)=\frac{2\sqrt{2}}{e^{\tau}+e^{-\tau}};\qquad b(\tau)=\frac{2\sqrt{2}\left(e^{-\tau}-e^{\tau}\right)}{(e^{\tau}+e^{-\tau})^2}.$$

$$\begin{aligned}\psi(\tau)=\;&2\tan^{-1}\frac{1}{2}(e^{\tau}-e^{-\tau})-\frac{1}{48\varepsilon^3}\left(e^{3\tau}-e^{-3\tau}\right)\\ &-\frac{3}{16\varepsilon^3}\left(e^{\tau}-e^{-\tau}\right).\end{aligned}$$

Evaluation of D_0

(1) Use Binomial Expansion to write R_1^{-3} as Fourier Expansion in ψ.

(2) Use the fact that all integrals containing $e^{im\psi}$ are in the magnitude of $e^{-m\varepsilon^{-1}}$ to control all terms of this expansion.

(3) This part is essentially what was in Llibre and Simo.

Our Conclusion:

(1) For all given ε sufficiently small, we have $D_0(0) = 0, D_0'(0) > 0$;

(2) It then follows that there exists $\mu > 0$ sufficiently small so that the splitting distance $D(0) = 0, D'(0) > 0$;

(3) But D as a function in μ is analytic: $D'(0) \neq 0$ for all but finitely many μ.

On Global Stable Solution

– The integral equation for stable solution

$$\Theta(\tau) = \int_0^\tau \frac{\sqrt{2}S}{\varepsilon^3 U^3_{\theta_0,\psi}(x+a)^3 a^3} d\tau;$$

$$M(\tau) = -\frac{1}{a}\int_\tau^{+\infty} \left(b'P + bQ\right) d\tau;$$

$$W(\tau) = a\int_0^\tau \frac{1}{a}(\tilde{H}P - HQ) d\tau.$$

– Let $\mathcal{V} = (\mathbb{M}(\tau), \mathbb{W}(\tau), \Theta(\tau)),\ \tau \in [0, +\infty).$

– Let

$$\mathcal{F}_\Theta(\mathcal{V}) := \int_0^\tau \frac{\sqrt{2}S}{\varepsilon^3 U^3_{\theta_0,\psi}(x+a)^3 a^3} d\tau;$$

$$\mathcal{F}_\mathbb{M}(\mathcal{V}) := -\frac{1}{a}\int_\tau^{+\infty} \left(b'P + bQ\right) d\tau;$$

$$\mathcal{F}_\mathbb{W}(\mathcal{V}) := a\int_0^\tau \frac{1}{a}(\tilde{H}P - HQ) d\tau.$$

– Let $\mathcal{F}$ be such that

$$\mathcal{F}(\mathcal{V}) := (\mathcal{F}_{\Theta}(\mathcal{V}),\ \mathcal{F}_{\mathbb{M}}(\mathcal{V}),\ \ \mathcal{F}_{\mathbb{W}}(\mathcal{V}))\,.$$

– Initialization: $\Theta_0(\tau) = 0$.

$$\begin{aligned}
\Theta_0(\tau) =& 0\\
\mathbb{M}_0(\tau) =& \frac{\sqrt{\varepsilon}}{a^2}\int_{\tau}^{+\infty} a^4(b' + b^2a^{-1})\sin 2(\theta_0 + \psi))d\tau;\\
\mathbb{W}_0(\tau) =& -\int_0^{\tau} a^3\left(\tilde{H} - ba^{-1}H\right)\sin 2(\theta_0 + \psi)d\tau.
\end{aligned}$$

– Inductively define

$$\mathcal{V}_{n+1}(\tau) = \mathcal{F}(\mathcal{V}_n(\tau))$$

– Estimate $\|\mathcal{V}_{n+1} - \mathcal{V}_n\|$ to conclude $\{\mathcal{V}_n\}$ is a normal family.

Key technical point of control

$$\mathcal{F}_{\Theta}(\mathcal{V}) := \int_0^{\tau} \frac{\sqrt{2}S}{\varepsilon^3 U^3_{\theta_0,\psi}(x+a)^3 a^3} d\tau;$$

$$\mathcal{F}_{\mathbb{M}}(\mathcal{V}) := -\frac{1}{a}\int_{\tau}^{+\infty} \left(b' P + bQ\right) d\tau;$$

$$\mathcal{F}_{\mathbb{W}}(\mathcal{V}) := a\int_0^{\tau} \frac{1}{a}(\tilde{H}P - HQ)d\tau.$$

– Two competing factors:

(1) Singularity in the equation for Θ at $a = 0$.

(2) P, Q are perturbations of high enough order in a.

– The terms in S that is Θ-dependent is not singular in a;

– the order of P, Q in a is high enough to cancel the singularity brought in by the equation of Θ.

– This is a rather detailed check. It barely worked out for us here (By some luck, I suppose)

Work in Progress

Develop a more general Melnikov Method that is not Duffing dependent for degenerate saddle case.

– I have developed a way to derive integral equations for splitting distance, and to formulate a setting that is more general to cover other degenerate cases. The issue here is how to find good application other than restricted 3-body problem.

– To derive the separatrix map to study the dynamics of homoclinic tangles. In particular, in the case of degenerate saddles with dissipation.

Thanks You!