Homoclinic Tangle in Restricted Three-Body Problem

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Time-periodic System and Chaos

– Equation of study: \( x \in \mathbb{R}^2 \)

\[
\frac{dx}{dt} = f(x) + \varepsilon g(x, t)
\]

– Assumptions: (1) \( g(x, t+T) = g(x, t) \); (2) for \( \varepsilon = 0 \), \( x = 0 \) is a saddle point with a homoclinic solution.

– The time-\( T \) map:
Homoclinic Tangle:

– For $\varepsilon = 0$: Homoclinic solution form an invariant loop for time-$T$ map;

– For $\varepsilon \neq 0$: This loop is broken, forming a homoclinic tangle

(Q1) Dynamic Structure of HT.

(Q2) Verification of HT in ODEs.
On (Q1): the dynamics of HT:

- Assume fixed point is dissipative

- Let $\mu = -\ln \varepsilon$, $\mu \to \infty$ as $\varepsilon \to 0$

- Three main dynamical scenarios

  Stable HT; Transient HT; Chaotic HT

- There is a periodic pattern for the alternative occurrence of (A), (B) (C) in $\mu$ as $\mu \to \infty$.

On (Q2): Existence of HT:

- The Poincare/Melnikov Method

- Degenerate Cases (High Order Method)

Today’s Talk: Another degenerate case.
Verification of HT in ODE:

– The Splitting Distance

\[ D(t_0, \varepsilon) = \varepsilon D_0(t_0) + \varepsilon^2 D_1(t_0) + \cdots \]

– Homoclinic Tangle: Exists for small \( \varepsilon \) if there exists \( t_0 \) so that

\[ D_0(t_0) = 0, \quad D'_0(t_0) \neq 0. \]

– Poincaré/Melnikov Integral: Explicit Integral Formula for \( D_0(t_0) \).

– A Historic Fact: HT was discovered in Poincare's study of the 3-body Problem, but his method of verification failed to apply.
Equation of The Planar 3-Body Problem

- Three bodies: $m_1, m_2, m_3$ in $\mathbb{R}^2$.

- Jacobian Coordinates: $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$.

- The Potential Function:
  \[
  U = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}
  \]
  where $r_{ij}$ is the distance from $m_i$ to $m_j$.

- Equations of Motion:
  \[
  \ddot{r}_1 = r_1 \dot{\theta}_1^2 + \mu_1^{-1} \partial_{r_1} U;
  \]
  \[
  \ddot{\theta}_1 = -\frac{2 \dot{r}_1 \dot{\theta}_1}{r_1} + \mu_1^{-1} \frac{1}{r_1^2} \partial_{\theta_1} U;
  \]
  \[
  \ddot{r}_2 = r_2 \dot{\theta}_2^2 + \mu_2^{-1} \partial_{r_2} U;
  \]
  \[
  \ddot{\theta}_2 = -\frac{2 \dot{r}_2 \dot{\theta}_2}{r_2} + \mu_2^{-1} \frac{1}{r_2^2} \partial_{\theta_2} U.
  \]

  where
  \[
  \mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_2 = \frac{m_3 (m_1 + m_2)}{m_1 + m_2 + m_3}.
  \]
Circular Restricted 3-body Problem

- Let $m_3 = 0, \quad m_1 + m_2 = 1, \quad \mu = m_2$.

- Equation for $r_1, \theta_1$:
  \[
  \ddot{r}_1 = r_1 \dot{\theta}_1^2 - \frac{1}{r_1^2}; \quad \ddot{\theta}_1 = -\frac{2\dot{r}_1 \dot{\theta}_1}{r_1}
  \]

- Circular motion $r_1 = 1, \quad \theta_1 = t$.

- Circular restricted 3-body problem
  \[
  \ddot{r}_2 = r_2 \dot{\theta}_2^2 + f; \quad \ddot{\theta}_2 = -\frac{2\dot{r}_2 \dot{\theta}_2}{r_2} + g
  \]
  where
  \[
  f = -\frac{m_1 (r_2 + m_2 \cos(\theta_2 - t))}{r_{13}^3} - \frac{m_2 (r_2 - m_1 \cos(\theta_2 - t))}{r_{23}^3},
  \]
  \[
  g = \frac{m_1 m_2 \sin(\theta_2 - t)}{r_2} \left( \frac{1}{r_{13}^3} - \frac{1}{r_{23}^3} \right);
  \]
More on Restricted 3-body Problem

– The Equations:
\[ \ddot{r}_2 = r_2 \dot{\theta}_2^2 + f; \quad \dot{\theta}_2 = -\frac{2r_2 \dot{\theta}_2}{r_2} + g \]
where \( f, g \) are explicitly given as before.

– The Jacobi Integral:
\[ J = \frac{1}{2} \left( \dot{r}_2^2 + r_2^2 (\dot{\theta}_2 - 1)^2 \right) - \frac{1}{2} r_2^2 - m_1 r_{13}^{-1} - m_2 r_{23}^{-1} \]
where \( J \) is the integral constant.

– With fixed \( J \), rewrite equation in new variables \((X, Y, \theta)\) as
\[
\begin{align*}
\frac{d\theta}{d\tau} &= \sqrt{2} \left( X - |J|^3 U^{-3} X^{-3} \right); \\
\frac{dX}{d\tau} &= Y + \sqrt{2} |J|^{-2} U^2 X^2 G; \\
\frac{dY}{d\tau} &= X - X^3 - X F + \sqrt{2} |J|^{-2} U^2 X Y G
\end{align*}
\]
Applying Poincare-Melnikov Method?

- $F, G, U$ are in terms of $X, Y$, periodic in $\theta$.

- Small parameter: $\varepsilon = J^{-1}$, or $\mu = m_2 << 1$.

- Regarding $(X, Y)$ as phase variable, $\theta$ as time;

- $(X, Y) = (0, 0)$ is a fixed point;

- It admit a homoclinin solution for unperturbed equation.

Hurdles:

(1) How to justify Poincare Melnikov method in degenerate case (fixed point is not a saddle);

(2) Calculating Melnikov function is rather challenging.
Past Results

– First result for HT was from Sitnikov

  • A different Restricted 3-Body Problem.
  
  • Original analysis exceedingly technical.
  
  • Result motivated Smale’s Horseshoe.
  
  • A Clear Presentation in Moser’s book.

– For Circular Restricted 3-Body Problem:


  (B) Zhihong Xia, JDE, 1992

  (C) M. Guardia, etc, Invent. Math., 2016
(A) (Llibre and Simo) There exists $J_0$ sufficiently large, so that for any given $J > J_0$, there exists a $\mu_0(J)$, $0 < \mu < \mu_0(J)$, HT exists.

(B) (Xia) For all but some finite number of values of $\mu$, there is some $J$ such that HT exists.

(C) (Guardia, et al.) For any given mass ratio $\mu$, there exists a $J_0(\mu)$ sufficiently large, so that for all $J > J_0(\mu)$, HT exists.

Remarks: In terms of method,

– (A), (B) Used Poincare-Melnikov method relying on a previous work of McGehee.

– (C) is much more elaborated. Relied on previous theory on exponentially small splitting built over the last forty years.
Main Result: there exists $J_0$ sufficiently large, so that for any given $J > J_0$, HT exists for all but some finite number of values of $\mu$.

This Result is not covered by (A), (B) or (C).

- It covers (A),

- It covers a different region of $J$ from (B).

- Compare to (C):
  - We cover a uniform interval of $J$, but for each $J$ we miss finitely many $\mu$.
  - (C) covers all $\mu$ but the interval of $J$ is NOT uniform.
Remarks on Method of Study:

– **Like (A) and (B)**, I also generalize Poincare-Melnikov method to this degenerate situation.

– **UnLike (A) and (B)**, My method of generalization do not rely on McGehee’s local analysis of degenerate fixed point.

– My study rely on the fact that restricted three-body problem is a **perturbed Duffing equation**.

\[
\begin{align*}
\frac{d\theta}{d\tau} &= \sqrt{2} \left( X - |J|^3 U^{-3} X^{-3} \right); \\
\frac{dX}{d\tau} &= Y + \sqrt{2} |J|^{-2} U^2 X^2 G; \\
\frac{dY}{d\tau} &= X - X^3 - XF + \sqrt{2} |J|^{-2} U^2 XY G
\end{align*}
\]
Global Stable and Unstable Solution

- The homoclinic solution:
  \[ a(\tau) = \frac{2\sqrt{2}}{e^\tau + e^{-\tau}}; \quad b(\tau) = \frac{2\sqrt{2}(e^{-\tau} - e^\tau)}{(e^\tau + e^{-\tau})^2}. \]

- The global stable solution:

\[ \Theta(\tau) = \int_0^\tau \frac{\sqrt{2}S}{\varepsilon^3 U_{\theta_0,\psi}(x + a)^3 a^3} d\tau; \]
\[ M(\tau) = -\frac{1}{a} \int_\tau^{+\infty} \left( b'P + bQ \right) d\tau; \]
\[ W(\tau) = a \int_0^\tau \frac{1}{a} (\tilde{H}P - HQ) d\tau. \]

where \((M, W, \Theta)\) are new variables.
**Main Proposition on Stable Solution:** treating $\varepsilon = J^{-1}$ as parameter of perturbation

There exists an $\varepsilon_0 > 0$ so that for any given $(\theta_0, \rho, \varepsilon) \in D_{\theta_0, \rho, \varepsilon}$ where

$$D_{\theta_0, \rho, \varepsilon} = \mathbb{R} \times (-\varepsilon_0, \varepsilon_0 + 1/2) \times (0, \varepsilon_0),$$

the integral equations in the above admits a unique solution. In addition, $X(0)$ of this solution as a function of $\theta_0, \rho, \varepsilon$ are real analytic on $D_{\theta_0, \rho, \varepsilon}$.

**Remarks on Proof:**

– By using the Contracting Mapping Theorem. Need $\varepsilon_0$ small to make contraction happen.

– **Analytical dependency** on parameters as claimed are critically important for us.

– It works because of rather sophisticated balances of the **singular order** of $X$ in the equation of $\Theta$ and the **order of perturbation** functions. Luckily this is carried through for the restricted 3-body problem.
Existence of HT: Treating $\mu$ as a perturbation parameter.

The Melnikov Integral

$$D_0 = -\frac{\sqrt{2}}{2\epsilon\sqrt{\epsilon}} \int_{-\infty}^{+\infty} (b'a^2 + ab^2) \sin(\theta_0 + \psi)(1 - R_1^{-3}) \, d\tau$$
$$+ \frac{1}{2\epsilon^3\sqrt{\epsilon}} \int_{-\infty}^{+\infty} ab[\epsilon^2 a^2 \cos(\theta_0 + \psi)(2 + R_1^{-3})$$
$$+(1 - R_1^{-3})] \, d\tau$$

where

$$R_1 = \sqrt{1 - 2\epsilon^2 a^2 \cos(\psi + \theta_0) + \epsilon^4 a^4}.$$

$$a(\tau) = \frac{2\sqrt{2}}{e^{\tau} + e^{-\tau}}; \quad b(\tau) = \frac{2\sqrt{2}(e^{-\tau} - e^\tau)}{(e^{\tau} + e^{-\tau})^2}.$$

$$\psi(\tau) = 2 \tan^{-1} \frac{1}{2}(e^{\tau} - e^{-\tau}) - \frac{1}{48\epsilon^3} \left( e^{3\tau} - e^{-3\tau} \right)$$
$$- \frac{3}{16\epsilon^3} \left( e^{\tau} - e^{-\tau} \right).$$
Evaluation of $D_0$

(1) Use Binomial Expansion to write $R_1^{-3}$ as Fourier Expansion in $\psi$.

(2) Use the fact that all integrals containing $e^{im\psi}$ are in the magnitude of $e^{-m\epsilon^{-1}}$ to control all terms of this expansion.

(3) This part is essentially what was in Llibre and Simo.

Our Conclusion:

(1) For all given $\epsilon$ sufficiently small, we have $D_0(0) = 0, D'_0(0) > 0$;

(2) It then follows that there exists $\mu > 0$ sufficiently small so that the splitting distance $D(0) = 0, D'(0) > 0$;

(3) But $D$ as a function in $\mu$ is analytic: $D'(0) \neq 0$ for all but finitely many $\mu$. 
On Global Stable Solution

– The integral equation for stable solution

\[ \Theta(\tau) = \int_0^\tau \frac{\sqrt{2S}}{\varepsilon^3 U^3_{\theta_0, \psi}(x + a)^3 a^3} d\tau; \]

\[ M(\tau) = -\frac{1}{a} \int_{\tau}^{+\infty} (b' P + b Q) d\tau; \]

\[ W(\tau) = a \int_0^\tau \frac{1}{a} (\tilde{H} P - H Q) d\tau. \]

– Let \( \mathcal{V} = (M(\tau), W(\tau), \Theta(\tau)), \quad \tau \in [0, +\infty). \)

– Let

\[ \mathcal{F}_\Theta(\mathcal{V}) := \int_0^\tau \frac{\sqrt{2S}}{\varepsilon^3 U^3_{\theta_0, \psi}(x + a)^3 a^3} d\tau; \]

\[ \mathcal{F}_M(\mathcal{V}) := -\frac{1}{a} \int_{\tau}^{+\infty} (b' P + b Q) d\tau; \]

\[ \mathcal{F}_W(\mathcal{V}) := a \int_0^\tau \frac{1}{a} (\tilde{H} P - H Q) d\tau. \]
– Let $\mathcal{F}$ be such that

$$\mathcal{F}(\mathcal{V}) := (\mathcal{F}_\Theta(\mathcal{V}), \mathcal{F}_M(\mathcal{V}), \mathcal{F}_W(\mathcal{V})).$$

– Initialization: $\Theta_0(\tau) = 0$.

$$\Theta_0(\tau) = 0$$

$$M_0(\tau) = \frac{\sqrt{\varepsilon}}{a^2} \int_\tau^{+\infty} a^4 (b' + b^2 a^{-1}) \sin 2(\theta_0 + \psi) d\tau;$$

$$W_0(\tau) = -\int_0^\tau a^3 (\bar{H} - ba^{-1}H) \sin 2(\theta_0 + \psi) d\tau.$$

– Inductively define

$$\mathcal{V}_{n+1}(\tau) = \mathcal{F}(\mathcal{V}_n(\tau))$$

– Estimate $\|\mathcal{V}_{n+1} - \mathcal{V}_n\|$ to conclude $\{\mathcal{V}_n\}$ is a normal family.
Key technical point of control

\[ F_\Theta(V) := \int_0^\tau \frac{\sqrt{2S}}{\varepsilon^3 U_{\theta_0,\psi}^3(x + a)^3} a^3 d\tau; \]

\[ F_M(V) := -\frac{1}{a} \int_\tau^{+\infty} (b'P + bQ) d\tau; \]

\[ F_W(V) := a \int_0^\tau \frac{1}{a} (\tilde{H}P - HQ) d\tau. \]

– Two competing factors:

(1) Singularity in the equation for \( \Theta \) at \( a = 0 \).

(2) \( P, Q \) are perturbations of high enough order in \( a \).

– The terms in \( S \) that is \( \Theta \)-dependent is not singular in \( a \);

– the order of \( P, Q \) in \( a \) is high enough to cancel the singularity brought in by the equation of \( \Theta \).

– This is a rather detailed check. It barely worked out for us here (By some luck, I suppose)
Work in Progress

Develop a more general Melnikov Method that is not Duffing dependent for degenerate saddle case.

– I have developed a way to derive integral equations for splitting distance, and to formulate a setting that is more general to cover other degenerate cases. The issue here is how to find good application other than restricted 3-body problem.

– To derive the separatrix map to study the dynamics of homoclinic tangles. In particular, in the case of degenerate saddles with dissipation.

Thanks You!