On the Theory of Chaotic Rank One Attractors

Qiudong Wang[1]

The theory of chaotic rank one attractors originated from the theory of Benedicks and Carleson on Hénon maps ([2],[3]) and the development that followed ([5], [6]). This theory was developed by Lai-Sang Young and myself in a sequence of articles ([31], [32], [33]). It takes the theory of Hénon-like maps ([20], [4]) as a particular case. Our motivation, in so far as I could interpret it now, was as follows:

(1) We felt that there was a geometric structure (namely the structure of critical regions) that, if added, could make more transparent the overall purpose of the different components of the very complicated *tour de force* analysis of Benedicks and Carleson. It could also induce non-trivial technical simplifications. We also felt that the various technical details skipped in [3], considering the profoundness of the theory in terms of impact, should be filled with care.

(2) We thought it *highly* desirable to have a theory on non-uniformly hyperbolic maps that could be applied to the analysis of concrete differential equations. Based on what was available to us at the time, it was decided that we could achieve such a goal in two steps. First, we could introduce a flexible setting on non-uniformly hyperbolic maps and construct a comprehensive chaos theory by generalizing previous analysis based on [3]. Second, it seemed to us that many natural and engineered systems described by concrete differential equations have a rank one character, and that they offer many possibilities for potential applications for this body of ideas.

We note that before this theory was developed, the scope of application of the theory on the Hénon maps had been extended by Mora and Viana ([20]) to maps with transversal homoclinic tangency through the Newhouse theory ([21], [23]). See also [8].

The rest of this essay is divided into two sections. In the first we discuss the content of the theory and in the second we discuss its application to differential equations.

1. A dynamics theory on chaotic rank one attractors

1.1. **Admissible family of rank one maps.** For $m \geq 2$, let $\mathcal{A} = S \times D^{(m-1)}$ where S is the unit circle and $D^{(m-1)}$ is the unit disk of dimension $m-1$. We consider a 2-parameter family of maps $\{T_{a,b},\ (a,b) \in (a_0, a_1) \times (0, b_0)\}$ where $T_{a,b} : \mathcal{A} \to \mathcal{A}$ is a diffeomorphism from $\mathcal{A}$ to its image. Let $(x_1, y_1) = T_{a,b}(x, y)$ be such that

$$x_1 = F(x, y, a) + bu(x, y, a, b)$$
$$y_1 = bv(x, y, a, b) \tag{1}$$

where $x \in S$, $y \in D^{(m-1)}$. Assume that $F(x, y, a), u(x, y, a, b), v(x, y, a, b)$ are C^3 in (x, y, a, b), and their C^3-norms are uniformly bounded by a constant K_0 that is independent of a and b.

We call the 1D map $f_a : S \to S$ defined by $f_a(x) = F(x, 0, a)$ the *1D singular limit* for $T_{a,b}$. We obtain $\{f_a\}$ by letting $b = 0$ in (1). For small b we could regard $T_{a,b}$

[1]Dept. of Math., University of Arizona, Tucson, AZ 85721, email dwang@math.arizona.edu.

as an m-dimensional map unfolded from the 1D singular limit f_a. Let $C(a) = \{x : f_a'(x) = 0\}$ be the set of critical points for f_a. We assume that $|\partial_y F(x, 0, a)| \neq 0$ for all $x \in C(a)$. This is to say that the unfolding from f_a to $T_{a,b}$ is non-degenerate in the direction of y on $C(a)$. We also assume

$$\frac{\det(DT_{a,b}(z))}{\det(DT_{a,b}(z'))} < K$$

for all $z, z' \in \mathcal{A}$ where K is a constant independent of b. The last is a regularity condition.

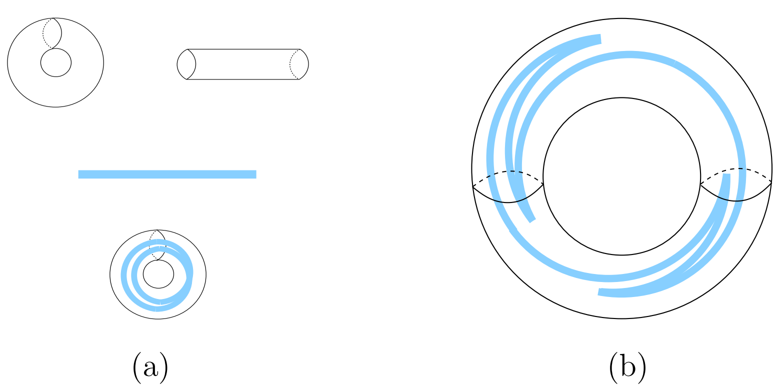
The rest of our conditions are imposed on the 1D singular limit $\{f_a\}$. We assume that there exists an $a_* \in (a_0, a_1)$, so that f_{a_*} is a *Misiurewicz map*. Misiurewicz maps are the simplest 1D maps with non-uniform expansion [19]. For a Misiurewicz map, all critical orbits are kept a fixed distance away from the critical set. We also impose a *parameter transversality condition* on f_a at $a = a_*$ to assure that changing a around a_* changes effectively the dynamics of f_a. The precise formulation of these two conditions are a little tedious and they are fully laid out in [32], to which we would refer the reader who is interested in such detail. What is important for us is that these are *verifiable* conditions. For instance, let

$$f_a(\theta) = \theta + a + L\Phi(\theta) \tag{2}$$

and assume that all critical points of $\Phi(\theta)$ are non-degenerate. It is proved in [34] that, for L sufficiently large, there exists an a_* so that f_{a_*} is a Misiurewicz map and the parameter transversality condition holds at $a = a_*$.

A two parameter family $T_{a,b}$ that assumes the form of (1) and satisfies all assumptions listed above is an *admissible family of rank one maps*.

1.2. **Existence of chaotic rank one attractors.** If the 1D singular limit f_a were *uniformly expanding*, then $T_{a,b}$ would represent an *Axiom A solenoid* [27]. This is to say that $T_{a,b}$ would stretch $\mathcal{A}$ in the x-direction, compress it in the y-direction. The image would then be wrapped a few times and put back in $\mathcal{A}$. In this case, we have a global splitting of stable and unstable directions hence a *uniformly hyperbolic attractor*. See Figure 1(a).



(a) (b)

Figure 1. (a) Axiom A solenoid and (b) rank one attractor

For an admissible family of rank one maps, however, the 1D singular limit is *not* uniformly expanding. See Figure 1(b). For rank one maps, the direction in which the image is wrapped in $\mathcal{A}$ *reverses* at the critical values of f_a, making it impossible for us to attain a global splitting of stable and unstable directions.

The dynamics of the maps of Figure 1(b) is very different from that of the Axiom A solenoid of Figure 1(a) because of the lacking of a global splitting of stable and unstable directions. For the maps of Figure 1(b), periodic sinks, representing stable dynamical behavior, often co-exist with Smale's horseshoe. From a measure theoretic stand point, periodic sinks are *directly observable* but horseshoes are not. This is to say that a sink attracts an open, hence a positive measure, set of orbits but the basin of attraction of a horseshoe is typically a measure zero set. Is there chaos in a *directly observable* form for the maps of Figure 1(b)? How about the existence of such maps that admit a positive Lyapunov exponent on a positive Lebesgue measure set? Smale's horseshoe is now insufficient to offer affirmative answers to these questions.

Historically, these questions were asked for the Hénon maps and were long standing until the ground breaking analysis of Benedicks and Carleson emerged. Our first major task, in order to acquire a more flexible theory on maps for future applications, was to reconstruct the analysis of Benedicks and Carleson for an admissible family of rank one maps. Therefore the following statement is not only the first, but also *the main theorem* for the theory of chaotic rank one attractors.

Main Theorem *Let $\{T_{a,b},\ (a,b) \in (a_1,a_2)\times(0,b_0)\}$ be an admissible family of rank one maps. Then there exists a positive measure set of parameters $\Delta \subset (a_1,a_2)\times(0,b_0)$, such that for all $(a,b) \in \Delta$, $T_{a,b}$ admits a positive Lyapunov exponent Lebesgue almost everywhere in $\mathcal{A}$.*

The contents of the theory of chaotic rank one attractors are (a) to construct a positive measure set of parameters that fulfills this theorem, and (b) to further understand the dynamics of these chaotic rank one attractors by systematically building a dynamical profile for the maps of the good parameters constructed.

1.3. **Inductive construction of good parameters.** Let $\{T_{a,b}\}$ be an admissible family of rank one maps, and for $T = T_{a,b}$, let

$$\Omega = \cap_{n=0}^{+\infty}\ T^n(\mathcal{A})$$

be the attractor for T. Our main theorem was proved by constructing a positive measure set Δ of good parameters through an elaborate inductive process. The good parameter set Δ is so constructed that, for $(a,b) \in \Delta$, the map $T = T_{a,b}$ possesses a well-defined critical set, which we denote as $\mathcal{C} \subset \Omega$. The critical set $\mathcal{C}$ is such that (i) every point in $\mathcal{C}$ is a point of quadratic tangency of well-defined local stable and unstable manifolds, and (ii) the critical orbits $\cup_{n=-\infty}^{\infty} T^n(\mathcal{C})$ are *all the tangency* in Ω for T.

For the maps of good parameters, the critical set $\mathcal{C}$ is a well structured Cantor set located around the critical points of the 1D singular limit. Let x_0 be a critical point of f_{a_*}, and let $Q(x_0) = [x_0 - \delta, x_0 + \delta] \times D^{(m-1)}$ where $\delta > 0$ is a fixed small number. A conceptual way to comprehend the structure of that Cantor set is to *forever* replace every cylinder by a finite collection of thinner and shorter cylinders inside, starting with $Q(x_0)$ for all $x_0 \in C(a_*)$. See Figure 2.

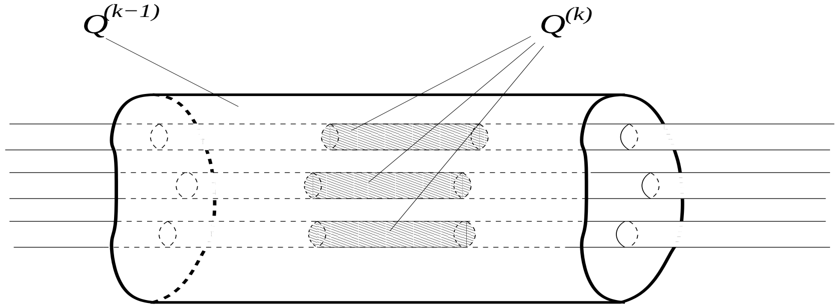


Figure 2. Structure of critical regions

The set of good parameters and the critical sets $\mathcal{C}$ are constructed side by side inductively. Each inductive step starts with a temporary set of good parameters and a collection of small cylinders, each of which we denote as *a critical region* $Q^{(k-1)}$. We construct inductively new critical regions $Q^{(k)}$ inside of $Q^{(k-1)}$, deleting parameters that could potentially ruin the intended quadratic tangency of local stable and unstable manifolds. We prove that, at each step of the induction, the measure of the deleted parameters declines exponentially. At the end we obtain a positive measure set of parameters with a well-defined critical set in $Q(x_0)$ for all $x_0 \in C(a_*)$.

The induction described above and the proof of the main theorem are complicated and long. See [31], [32] and [33].

1.4. **Dynamics of chaotic rank one attractors.** Through the inductive process discussed in the previous subsection we have obtained a new class of chaotic attractors. These chaotic attractors are genuinely non-uniformly hyperbolic in the sense that the stable and unstable directions are allowed to mix. Our focus is now to built a dynamical profile so we could claim a comprehensive understanding on the geometric and dynamical structure of these chaotic rank one attractors. In what follows we let $T = T_{a,b}$ where $(a, b) \in \Delta$ is a good parameter that came out of the long inductive construction. Let $\Omega = \cap_{n=0}^{+\infty} T^n(\mathcal{A})$ be the chaotic rank one attractor, and $\mathcal{C}$ be the critical set for T. The theorems listed in this subsection are proved for 2D chaotic rank one attractors in [31], and the corresponding version in higher dimensions is proved in [32] and [33].

A. Hyperbolic structure and symbolic dynamics We start with

Theorem 1. *For any give $\varepsilon > 0$,*

$$\Lambda_\varepsilon = \{z = (x, y) \in \Omega,\ d(T^n(z), \mathcal{C}) \geq \varepsilon,\ \text{for all } n \in \mathbb{Z}\}$$

is a uniformly hyperbolic invariant subset in Ω for T.

This theorem assures that the critical set $\mathcal{C}$ is in fact the sole source of non-uniform hyperbolicity in Ω.

We next turn to a special coding of orbits on Ω. The following notation will be used: $x_1 < x_2 < \cdots < x_q < x_{q+1} = x_1$ are the critical points of f_{a_*}, and $\mathcal{C}_i$ is the part of $\mathcal{C}$ near $(x_i, 0) \in \mathcal{A}$. Let $\Sigma_q = \Pi_{-\infty}^{\infty}\{1, 2, \cdots, q\}$ be the space of sequences of q-symbols, and let $\sigma : \Sigma_q \to \Sigma_q$ be the shift map.

Theorem 2. *(a) There is a natural partition of $\Omega \backslash \mathcal{C}$ into disjoint sets $A_1, A_2, \cdots, A_q$ so that $z \in A_i$ can be thought of as being "to the right" of $\mathcal{C}_i$ and "to the left" of $\mathcal{C}_{i+1}$.*

(b) Under the additional assumption that $f_{a_*}([x_j, x_{j+1}]) \not\supset S^1$ for any j, there is a closed subset $\Sigma \subset \Sigma_q$ with $\sigma(\Sigma) \subset \Sigma$ and a map $\pi : \Sigma \to \Omega$ with the property that

- for all $\mathbf{s} = (s_i) \in \Sigma$, $\pi(\mathbf{s}) = z$ implies that $T^i z \in \bar{A}_{s_i}$ for all i;

- π is a continuous surjection that is 1-1 except on $\cup_{i=-\infty}^{\infty} T^i \mathcal{C}$, where it is 2-1.

If the assumption in Part (b) does not hold, the statement can be amended by increasing the number of symbols as follows: For f_{a_*}, use the partition given by $C(a_*) \cup f_0^{-1}(C(a_*))$ instead of that by $C(a_*)$, and do likewise with T. See also [33] for more results that follow immediately from the symbolic coding of Theorem 2.

B. Statistical properties Inside a chaotic rank one attractor, orbits jump around in a seemingly random fashion and the future of individual orbits appears entirely *unpredictable*. For these chaotic attractors, however, there are, *laws of statistics* that control the asymptotic distributions of Lebesgue almost all orbits in $\mathcal{A}$.

The precise meaning of the last statement is as follows. Let us first partition $\mathcal{A}$ into a collection of disjoint sub-regions. For a finite orbit of T, we use this partition to define a histogram by counting the proportion of points on this finite orbit in each of the sub-regions. We then take two limits: first we let the orbit go infinitely long and second we refine the partition so the maximal size of all sub-regions goes to zero. In principle, there is no reason *a priori* for the histogram to converge to anything in any one of the two limit processes. For chaotic rank one attractors, however, *not only* the histograms converge, *but also* they converge to only a finite number of probability distributions for Lebesgue almost all orbits in $\mathcal{A}$. These limit distributions are the Sinai-Ruelle-Bowen measures for T.

Theorem 3. *There are at least one, and at most q, ergodic SRB measures for T where q is the number of critical points of the 1D singular limit.*

See [37] and the reference therein for more on the theory of SRB measures. In addition to Theorem 3, we have also proved many refined dynamical properties of chaotic rank one attractors, including a central limit theorem, an exponential mixing rate and a principle of large deviations by applying the theories developed previously in [36] and [25]. See [33].

2. Application to differential equations

The theory of chaotic rank one attractors has been applied to the analysis of many naturally and engineered systems described by concrete differential equations. In this section we present three applications. The first application is on periodically perturbed homoclinic solutions [29]. The second and the third applications are both under the framework of periodically kicked stable limit cycles [34]. The second is on the dynamics of periodically kicked equations with Hopf bifurcation [35] and the third is an application to PDE arena [17]. For more examples of application see also [15], [14], [11], [30].

2.1. **Periodically perturbed homoclinic solutions.** Periodically-forced second order equations, including the forced nonlinear pendulum, Duffing equation, and van der Pol oscillator, have been studied extensively in history ([9], [7], [13] [10], [28]). When a homoclinic solution in a given second order equation is periodically perturbed, transversal intersections of stable and unstable manifolds occur within a certain range of the forcing parameter, generating homoclinic tangles and chaotic dynamics ([24], [26]). There also exist values of parameter for which the stable and unstable manifolds of the perturbed saddle are pulled apart. For these two cases, Figure 3 schematically illustrates the time-T maps for the perturbed equations, where T is the period of the perturbation.

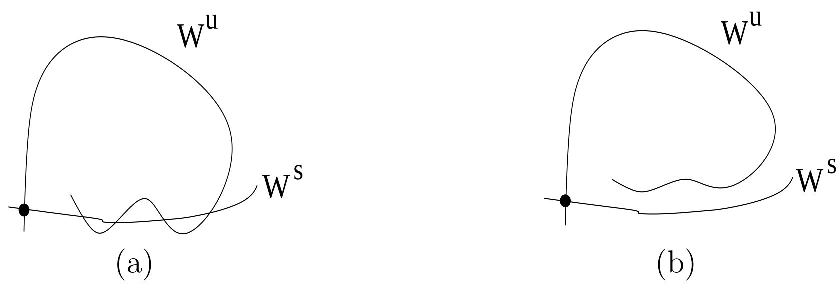


Figure 3. (a) Homoclinic intersection and (b) separated invariant manifold.

Historically, the homoclinic tangles of Figure 3(a) have been regarded as a major venue for chaos to be manifested in differential equations. Proving chaos through the intersection of the stable and unstable manifolds ([18], [26]) has become a standard practice in the study of differential equations. This is to say that we often prove chaos in a given equation via the elimination of the scenario of Figure 3(b).

Afraimovich and Shilnikov, on the other hand, argued for chaos *in the scenario of Figure 3(b)* in [1]. They proposed to study the dynamics of periodically perturbed homoclinic solutions through the *separatrix map*, a return map around the unperturbed homoclinic solution in the extended phase space. Afraimovich and Shilnikov illustrated that, when a homoclinic solution is periodically perturbed, then horseshoe exists as well in the scenario of Figure 3(b) provided that the forcing frequency is sufficiently large.

Following the method proposed in [1], Ott and I rigorously derived the separatrix map for periodically perturbed equation with a homoclinic saddle in [29]. We verified that, under the assumption that the saddle fixed point is dissipative and non-resonant, *the separatrix map is a family of rank one maps* with a 1D singular limit in the form of

$$f_a(\theta) = \theta + a + \frac{\omega}{\beta} \ln M(\theta)$$

where ω is the forcing frequency; β is the unstable eigenvalue of the saddle fixed point; $M(\theta)$ is the classical Melnikov function, which is *explicitly computable* for a given equation; and a is such that

$$a = \frac{\omega}{\beta} \ln \mu^{-1} + K,$$

in which μ is a parameter representing the magnitude of the perturbation and K is a constant from the given equation. Therefore if the Melnikov function $M(\theta)$ is such that $\min_\theta M(\theta) > 0$ (which puts us directly into the scenario of Figure 3(b)), and all critical points of $M(\theta)$ are *non-degenerate*, then the separatrix map is an admissible family of rank one maps provided that the forcing frequency is sufficiently large. In this case we obtain chaotic rank one attractors for a positive measure set of μ.

As a concrete example, we applied the theory of chaotic rank one attractors to the Duffing equation in the form of

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^3 = \mu \sin \omega t. \tag{3}$$

It is well-known that, for every $\lambda_0 > 0$ sufficiently small, there exists a γ_0 such that equation (3) has a homoclinic solution when $\mu = 0$. This homoclinic solution we denote as ℓ_0. Our next theorem asserts the existence of a *directly observable global chaotic attractor* in the neighborhood of ℓ_0.

Theorem 4. *Let $\lambda_0 > 0$ be sufficiently small and γ_0, ℓ_0 be as in the above. Then there exists a positive measure set Δ for (γ, ω, μ) close to $(\gamma_0, \infty, 0)$ such that for $(\gamma, \omega, \mu) \in \Delta$, the Duffing equation (3) admits a unique chaotic rank one attractor in an open neighborhood of ℓ_0.*

The results reported in this subsection are due to Ott and myself [29].

2.2. **Periodically kicked super-critical Hopf bifurcations.** Our next application is about periodically kicked supper critical Hopf bifurcations. *Hopf bifurcation* is a dynamical scenario of historical and practical importance in both theory and application. External actions in the form of an impulse function are often modelled by *periodic kicks* in various applications, including in switch controlled circuits and in neuroscience and biology.

Let us start with the theory on Hopf bifurcation. For $x \in \mathbb{R}^m,\ m \geq 2$, consider the following μ-dependent system of differential equations

$$\frac{dx}{dt} = A_\mu x + f_\mu(x) \tag{4}$$

where A_μ is an $m \times m$ matrix and $f_\mu(x)$ is a vector valued function of order ≥ 2 at $x = 0$. We assume that, around $\mu = 0$, all eigenvalues of A_μ except a conjugating pair, which we denote as $\lambda_{1,2}$, are with negative real part. We assume that $\lambda_{1,2} = a(\mu) \pm \omega(\mu)\sqrt{-1}$ are such that $a(0) = 0,\ \omega(0) \neq 0$. Corresponding to $\lambda_{1,2}$, equation (4) has a 2-dimensional local center manifold W^c at $x = 0$, and the equation induced on W^c can be written in a complex variable z in a normal form as

$$\dot{z} = (a(\mu) + i\omega(\mu))z + k_1(\mu)z^2\bar{z} + k_2(\mu)z^3\bar{z}^2 + \cdots \tag{5}$$

where $k_1(\mu), k_2(\mu), \cdots$ are complex numbers. We have a *generic Hopf bifurcation* for equation (4) if $Re(k_1(0)) \neq 0$. A Hopf bifurcation is *sup-critical* if $Re(k_1(0)) < 0$ and it is *sub-critical* if $Re(k_1(0)) > 0$. In the rest of this subsection we assume $Re(k_1(0)) < 0$ so we are interested in a sup-critical Hopf bifurcation, in which a *stable* periodic solution comes out of $x = 0$.

We now add a time-dependent forcing term to equation (4) to form a new equation

$$\frac{dx}{dt} = A_\mu x + f_\mu(x) + \varepsilon\Phi(x) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \tag{6}$$

where ε is a small parameter representing the magnitude of the forcing, $\Phi(x)$ is such that $\Phi(0) = 0$, and $\delta(t)$ is the standard δ-function.

We computed the time-T map *around* the stable periodic solution coming out of $x = 0$ for small μ. It turned out that the time-T map is a family of rank one maps with a singular 1D limit, as $T \to \infty$, in the form of

$$f_a(\theta) = \theta + a + L\phi(\theta)$$

where $a \approx \omega(0)T \ mod(2\pi)$; $\phi(\theta) = \phi_0(\theta) + \mathcal{O}(\sqrt{\mu}) + \mathcal{O}(\varepsilon)$ in which $\phi_0(\theta)$ is a function of θ determined by A_0 and $D\Phi(0)$; and $L \approx \varepsilon \cdot \tau$ and

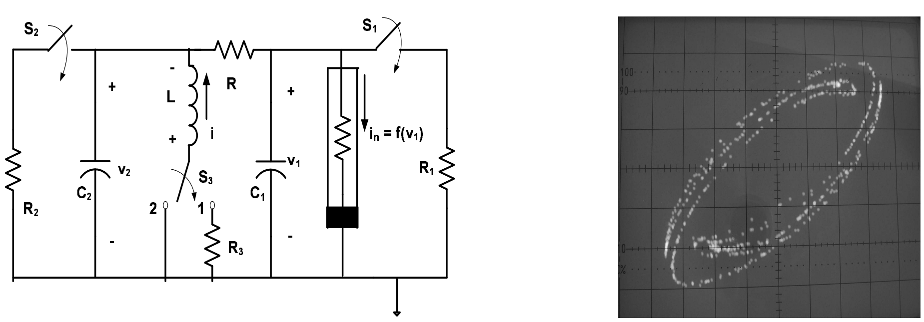
$$\tau = \left| \frac{Im(k_1(0))}{Re(k_1(0))} \right|.$$

So to find chaotic rank one attractors for equation (6),

(i) first we compute $\phi_0(\theta)$, and verify that all its critical points are non-degenerate;
(ii) second we compute the normal form (5) to find $k_1(0)$, and we look for equations for which τ is large;
(iii) let μ and ε be sufficiently small so that $\phi(\theta)$ is close to $\phi_0(\theta)$; we also let $L = \tau \cdot \varepsilon$ be sufficiently large;
(iv) we then conclude that chaotic rank one attractors exist for a positive measure set of T.

We skip the precise statement and refer the reader to [35], [17] and [30] for more details.

To provide a concrete application we applied this theory to a periodically kicked Chua's circuit. The circuit is shown in Figure 4(a) and the periodic kicks are implemented by switches being turned on and off periodically. We analyzed the differential equation for the circuit following the steps (i)-(iv) above, and found parameters for which the chaotic rank one attractors are likely to occur. We then physically built the circuit. Figure 4(b) is a picture of chaotic rank one attractor that came out of lab simulations. See [30] and [22].



(a) (b)

Figure 4. (a) Switch controlled Chua circuit and (b) a chaotic rank one attractor

The theory on periodically kicked Hopf bifurcation is due to Young and myself [35]; the application to Chua's circuit is due to Oksasoglu and myself [30]; and the circuit is built by Demirkol, Ozoguz, and Akgul [22].

2.3. **Application to a parabolic partial differential equation.** The *Brusselator*, as described by the equations below, is a simplified model of an autocatalytic chemical reaction with diffusion [16]:

$$\begin{aligned} u_t &= d_1 \Delta u + a - (b+1)u + u^2 v, \\ v_t &= d_2 \Delta v + bu - u^2 v. \end{aligned} \tag{7}$$

We consider this model in one physical dimension, i.e. $u = u(x,t)$ and $v = v(x,t)$ for $x \in [0,1]$, and let $\Delta = \partial_{xx}$. Here a and b are constants representing concentrations of certain initial substances, u and v are variables representing concentrations of two intermediates, and $d_1, d_2 > 0$ are their respective diffusion coefficients. The term $u^2 v$ represents the autocatalytic step in the reaction.

One sees immediately that $(u(t), v(t)) \equiv (a, ba^{-1})$ is a stationary solution. Letting $U = u - a$, $V = v - ba^{-1}$, and $\mathbf{u} = (U, V)$, we write (7) as an evolutionary equation

$$\dot{\mathbf{u}} = \mathcal{A}_{a,b}\mathbf{u} + f_{a,b}(\mathbf{u}), \tag{8}$$

where

$$\mathcal{A}_{a,b} = \begin{pmatrix} d_1 \frac{\partial^2}{\partial x^2} + (b-1) & a^2 \\ -b & \theta d_1 \frac{\partial^2}{\partial x^2} - a^2 \end{pmatrix}, \qquad \theta = \frac{d_2}{d_1}, \quad d_1 \neq 0;$$

and

$$f_{a,b}(\mathbf{u}) = \begin{pmatrix} UV^2 + ba^{-1}U^2 + 2aUV \\ -UV^2 - ba^{-1}U^2 - 2aUV. \end{pmatrix}.$$

It is well known that Hopf bifurcation occurs as the parameters a and b are varied. See [12].

We study periodically kicked equations satisfying the Neumann boundary condition and the Dirichelet boundary condition. The forcing functions are chosen to respect the corresponding boundary condition.

Neumann boundary condition.

$$\begin{aligned} u_t &= d_1 \Delta u + a - (b+1)u + u^2 v + \rho(1 + \cos \pi x)\ p_{T,\iota}(t), \\ v_t &= d_2 \Delta v + bu - u^2 v\ ; \\ \partial_x u(0,t) &= \partial_x u(1,t) = 0, \qquad \partial_x v(0,t) = \partial_x v(1,t) = 0 \end{aligned} \tag{9}$$

Dirichelet boundary condition.

$$\begin{aligned} u_t &= d_1 \Delta u + a - (b+1)u + u^2 v + \rho \sin \pi x\ p_{T,\iota}(t), \\ v_t &= d_2 \Delta v + bu - u^2 v\ ; \\ u(0,t) &= u(1,t) = a, \qquad v(0,t) = v(1,t) = ba^{-1}. \end{aligned} \tag{10}$$

In both cases, $\rho \geq 0$ and $0 < \iota << 1 << T$ are constants and

$$p_{T,\iota}(t) = \sum_{n=-\infty}^{\infty} p_\iota(t - nT) \quad \text{with} \quad p_\iota(t) = \begin{cases} \iota^{-1} & 0 \leq t < \iota, \\ 0 & \text{elsewhere.} \end{cases}$$

We use $p_{T,\iota}(t)$ in the place of a periodic delta function to avoid certain regularity problems. Our plan in [17] is to first develop a theory on chaotic rank one attractors for periodically kicked sup-critical Hopf bifurcation in the context of infinite dimensional systems. We then apply this theory to equations (9) and (10).

Theorem 5. (Neumann Boundary condition) *Let $d_1 > 0$, $\theta < 1$ be fixed. Then for a sufficiently large, there is an open set of ρ, ι and b (depending on d_1, θ and a), $b \approx a^2 + 1$, for which the time-T map of equation (9) has a chaotic rank one attractor for a positive measure set of large T.*

The computation needed for the Dirichelet boundary condition is considerably more involved than those for Neumann boundary condition. We limited ourselves in [17] to the special case $d_1 = \pi^{-2}$.

Theorem 6. (Dirichelet boundary condition) *Let $d_1 = \pi^{-2}$, $0 < \theta << 1$. Then there are open sets of a, b, ρ and ι, for which the time-T map of equation (10) has a chaotic rank one attractor for a positive measure set of large T.*

We refer the interested reader to [17] for details on the infinite dimensional theory and the application. The results reported in this subsection are due to Lu, Young and myself [17].

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