ON SEPARATRIX MAP

Abstract.

1. Preliminaries

1.1. Solving the Equations of First Variations. Let \((x, y) = (a(t), b(t))\) be a solution of the second order autonomous equation
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = f(x).
\]
The equations of first variations around \(\ell(t) = (a(t), b(t))\) are
\[
\frac{d\xi}{dt} = \eta; \quad \frac{d\eta}{dt} = f'(a)\xi.
\]
We intend to introduce a specific set of coordinates to solve equation (1.1). Let \(z_1, z_2\) be such that
\[
z_1 = \frac{1}{\sqrt{R}}(b'\xi - b\eta); \quad z_2 = \frac{1}{\sqrt{R}}((R - b'g(a))b^{-1}\xi + g(a)\eta)
\]
where \(R, g\) are arbitrary functions. We have, in reverse,
\[
\xi = \frac{1}{\sqrt{R}}(g(a)z_1 + bz_2); \quad \eta = \frac{1}{\sqrt{R}}(-(R - b'g(a))b^{-1}z_1 + b'z_2).
\]

Lemma 1.1. The equations of first variations (1.1) are transformed in \(z_1, z_2\) to
\[
\frac{dz_1}{dt} = -\frac{R'}{2R}z_1, \quad \frac{dz_2}{dt} = Az_1 + \frac{R'}{2R}z_2
\]
where
\[
A = R'b^{-1}g(a) - g'(a)R - (R - b'g(a))b^{-2}R.
\]
Proof. We note that \(z_1\) is the projection of \((\xi, \eta)\) in the normal direction of the solution \((a, b)\), a variable previously adopted by Melnikov, and \(z_2\) is so designed that the only divisive factor involved in both directions of this change of coordinates is \(\sqrt{R}\). The purpose of this design is to use \(R\) to control the potential singularities introduced by change of coordinates. The function \(g\) is adopted to give us more flexibility in simplifying \(A\).

For \(z_1\) we have
\[
\frac{dz_1}{dt} = \frac{1}{\sqrt{R}}(b''\xi + b'\xi' - b'b\eta - b\eta') - \frac{R'}{2R}z_1 = -\frac{R'}{2R}z_1.
\]
We also have
\[
\frac{dz_2}{dt} = \frac{1}{\sqrt{R}}((R' - b'g(a)b - b''g(a))b^{-1}\xi - (R - b'g(a))b^{-2}b'\xi + (R - b'g(a))b^{-1}\eta)
\]
\[
+ \frac{1}{\sqrt{R}}(g'(a)b\eta + g(a)f'(a)\xi) - \frac{R'}{2R}z_2
\]
\[
= \frac{1}{\sqrt{R}}[(R' - b'g(a)b - b''g(a))b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a)]\xi
\]
\[
+ \frac{1}{\sqrt{R}}[g'(a)b + (R - g(a)b'b^{-1})\eta - \frac{R'}{2R}z_2].
\]
To continue, we have
\[
\frac{dz_2}{dt} = \frac{1}{R} ((R' - b'g(a)b - b''g(a))b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a))(g(a)z_1 + bz_2) \\
+ \frac{1}{R} ([g'(a)b + (R - g(a)b)b^{-1}](-(R - b'g(a))b^{-1}z_1 + b'z_2)) - \frac{R'}{2R}z_2 \\
= \frac{1}{R} ((R' - b'g(a)b - b''g(a))b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a))(g(a)z_1) \\
+ \frac{1}{R} ([g'(a)b + (R - g(a)b)b^{-1}](-(R - b'g(a))b^{-1}z_1)) + \frac{R'}{2R}z_2 \\
= \frac{A}{R}z_1 + \frac{R'}{2R}z_2
\]

where
\[
A = [(R' - b'g'(a)b - b''g(a))]b^{-1} - (R - b'g(a))b^{-2}b' + g(a)f'(a)(g(a)) \\
+ ([g'(a)b + (R - g(a)b)b^{-1}](-(R - b'g(a))b^{-1})) \\
= R'b^{-1}g(a) - g'(a)R - (R - b'g(a))b^{-2}R.
\]

We further remove the dependency of the equation for \(z_2\) on \(z_1\) by letting
\[
w = hz_1 + z_2.
\]
We have
\[
\frac{dw}{dt} = h'z_1 - h \frac{R'}{2R}z_1 + \frac{A}{R}z_1 + \frac{R'}{2R}z_2.
\]
Let \(h = h(t)\) be such that
\[
h' - \frac{R'}{R}h + \frac{A}{R} = 0.
\]
We obtain
\[
\frac{dw}{dt} = \frac{R'}{2R}w.
\]
Re-write \(z_1\) as \(\bar{\xi}\) and \(w\) as \(\bar{\eta}\), we have
\[
\xi = \frac{1}{\sqrt{R}}(b\bar{\eta} - \sqrt{RH}\bar{\xi}); \quad \eta = \frac{1}{\sqrt{R}}(b'\bar{\eta} - \sqrt{R\bar{H}}\bar{\xi})
\]
where
\[
H = \frac{1}{\sqrt{R}}(bh - g(a)); \quad \bar{H} = \frac{1}{\sqrt{R}}(b'h + (R - b'g(a))b^{-1}).
\]
In reverse, we have
\[
\bar{\xi} = \frac{1}{\sqrt{R}}(b'\xi - b\eta); \quad \bar{\eta} = \left(\bar{H}\xi - H\eta\right).
\]

Let us briefly summarize what we have done so far. In matrix form, the equations of first variations are
\[
(1.5) \quad \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = U \begin{pmatrix} \xi \\ \eta \end{pmatrix}
\]
where
\[
(1.6) \quad U = \begin{pmatrix} 0 & 1 \\ f'(a) & 0 \end{pmatrix}.
\]
Let
\[ A = R' b^{-1} g(a) - g'(a) R - (R - b' g(a)) b^{-2} R. \]
and \( h \) is such that
\[ h' - \frac{R'}{R} h + \frac{A}{R} = 0. \]
(1.7)
Also let
\[ H = \frac{1}{\sqrt{R}} (b h - g(a)); \quad \tilde{H} = \frac{1}{\sqrt{R}} (b' h + (R - b' g(a)) b^{-1}). \]
(1.8)
The new variables \( \tilde{\xi}, \tilde{\eta} \) are defined by letting
\[ \begin{pmatrix} \xi \\ \eta \end{pmatrix} = T \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} \]
where
\[ T = \begin{pmatrix} -H & \frac{b}{\sqrt{R}} \\ -\tilde{H} & \frac{\tilde{b}}{\sqrt{R}} \end{pmatrix}. \]
(1.9)
Lemma 1.2. \( \det(T) = 1 \).
Proof. Recall that
\[ H = \frac{1}{\sqrt{R}} (b h - g(a)); \quad \tilde{H} = \frac{1}{\sqrt{R}} (b' h + (R - b' g(a)) b^{-1}). \]
We have
\[ \det(T) = \frac{1}{\sqrt{R}} \left( -b' H + b \tilde{H} \right) \]
\[ = \frac{1}{\sqrt{R}} \left( -b' \frac{1}{\sqrt{R}} (b h - g(a)) + b \frac{1}{\sqrt{R}} (b' h + (R - b' g(a)) b^{-1}) \right) \]
\[ = \frac{1}{\sqrt{R}} (-b' (b h - g(a)) + b (b' h + (R - b' g(a)) b^{-1})) \]
\[ = 1. \]
\[ \square \]
This lemma implies,
\[ \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = T^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \]
where
\[ T^{-1} = \begin{pmatrix} \frac{b}{\sqrt{R}} & -\frac{b}{\sqrt{R}} \\ H & -\tilde{H} \end{pmatrix}. \]
(1.10)
Proposition 1.1. The equations of first variations (1.5) are transformed in new variables \( (\tilde{\xi}, \tilde{\eta}) \) to
\[ \frac{d\tilde{\xi}}{dt} = -\frac{R'}{2R} \tilde{\xi}; \quad \frac{d\tilde{\eta}}{dt} = \frac{R'}{2R} \tilde{\eta}. \]
(1.11)
Proof. By definition, we have
\[ \begin{pmatrix} \tilde{\xi}' \\ \tilde{\eta}' \end{pmatrix} = B \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} \]
where
\[ B = \left( (T^{-1})' T + T^{-1} UT \right). \]
To calculate $B$, we start with

\[
T^{-1}UT = \begin{pmatrix}
\frac{b'}{\sqrt{R}} & -b & 0 & 1 \\
H & -H & \frac{b}{\sqrt{R}} & -\tilde{H}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
f'(a) & 0
\end{pmatrix}
\begin{pmatrix}
-H & \frac{b}{\sqrt{R}} \\
-\tilde{H} & \frac{b'}{\sqrt{R}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{b'}{\sqrt{R}} & -b & 0 & 1 \\
H & -H & \frac{b}{\sqrt{R}} & -\tilde{H}
\end{pmatrix}
\begin{pmatrix}
-\tilde{H} & \frac{b'}{\sqrt{R}} \\
-Hf'(a) & \frac{b}{\sqrt{R}} f'(a)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\tilde{H} \frac{b'}{\sqrt{R}} + H \frac{b}{\sqrt{R}} f'(a) & \langle b' \rangle^2 R - \frac{b^2}{R} f'(a) \\
-H^2 + H^2 f'(a) & \frac{b'}{\sqrt{R}} H - \frac{b}{\sqrt{R}} H f'(a)
\end{pmatrix}.
\]

We also have

\[
(T^{-1})' = \begin{pmatrix}
\left( \frac{b'}{\sqrt{R}} \right)' & -\left( \frac{b}{\sqrt{R}} \right)' \\
\tilde{H}' & -H'
\end{pmatrix}
\]

and

\[
(T^{-1})' T = \begin{pmatrix}
\left( \frac{b'}{\sqrt{R}} \right)' & -\left( \frac{b}{\sqrt{R}} \right)' \\
\tilde{H}' & -H'
\end{pmatrix}
\begin{pmatrix}
-H & \frac{b}{\sqrt{R}} \\
-\tilde{H} & \frac{b'}{\sqrt{R}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\left( \frac{b'}{\sqrt{R}} \right)' H + \left( \frac{b}{\sqrt{R}} \right)' \tilde{H} - \tilde{H} \left( \frac{b'}{\sqrt{R}} \right) + H \frac{b}{\sqrt{R}} f'(a) \\
-(\tilde{H})' + H' \tilde{H} - \tilde{H}^2 + H^2 f'(a) & \left( \frac{b'}{\sqrt{R}} \right) H' + \frac{b}{\sqrt{R}} \left( \tilde{H} \right)'
\end{pmatrix}.
\]

This is for us to have

\[
B = (T^{-1})' T + T^{-1} UT = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}
\]

where

\[
B_{11} = -\left( \frac{b'}{\sqrt{R}} \right)' H + \left( \frac{b}{\sqrt{R}} \right)' \tilde{H} - \tilde{H} \left( \frac{b'}{\sqrt{R}} \right) + H \frac{b}{\sqrt{R}} f'(a)
\]

\[
B_{12} = \left( \frac{b'}{\sqrt{R}} \right)' \left( \frac{b}{\sqrt{R}} \right) - \left( \frac{b'}{\sqrt{R}} \right) \left( \frac{b}{\sqrt{R}} \right)' + \langle b' \rangle^2 R - \frac{b^2}{R} f'(a)
\]

\[
B_{21} = -(\tilde{H})' H + H' \tilde{H} - \tilde{H}^2 + H^2 f'(a)
\]

\[
B_{22} = -\left( \frac{b'}{\sqrt{R}} \right) H' + \frac{b}{\sqrt{R}} \left( \tilde{H} \right)' + \frac{b'}{\sqrt{R}} \tilde{H} - \frac{b}{\sqrt{R}} H f'(a).
\]

Recall that

\[
H = \frac{1}{\sqrt{R}} (bh - g(a)); \quad \tilde{H} = \frac{1}{\sqrt{R}} (b'h + (R - b'g(a))b^{-1}).
\]

We have

\[
B_{11} = -\left( \frac{b'}{\sqrt{R}} \right)' H + \left( \frac{b}{\sqrt{R}} \right)' \tilde{H} - \tilde{H} \left( \frac{b'}{\sqrt{R}} \right) + H \frac{b}{\sqrt{R}} f'(a) \left( \frac{1}{2} R^{-3/2} R' \right) (b'H - b\tilde{H})
\]

\[
= \left( \frac{1}{2} R^{-2} R' \right) (b'(bh - g(a)) - b(b'h + (R - b'g(a))b^{-1})) = -\frac{1}{2} R^{-1} R;
\]
\[ B_{12} = \left( \frac{b'}{\sqrt{R}} \right) ' \left( \frac{b}{\sqrt{R}} \right) - \left( \frac{b'}{\sqrt{R}} \right) ' \left( \frac{b}{\sqrt{R}} \right) + \frac{(b')^2}{R} - \frac{b^2}{R} f'(a) \]
\[ = \left( \frac{b''}{\sqrt{R}} - \frac{1}{2} R^{-3/2} b' R' \right) \left( \frac{b}{\sqrt{R}} \right) - \left( \frac{b'}{\sqrt{R}} \right) \left( \frac{b'}{\sqrt{R}} - \frac{1}{2} R^{-3/2} b R' \right) + \frac{(b')^2}{R} - \frac{b^2}{R} f'(a) \]
\[ = -\frac{1}{2} R^{-3/2} b' R' \left( \frac{b}{\sqrt{R}} \right) - \left( \frac{b'}{\sqrt{R}} \right) \left( \frac{1}{2} R^{-3/2} b R' \right) = 0. \]

We also have
\[ B_{21} = -\left( \frac{1}{\sqrt{R}} (b'h + (R - b'g(a))b^{-1}) \right) ' \frac{1}{\sqrt{R}} (bh - g(a)) \]
\[ + \left( \frac{1}{\sqrt{R}} (bh - g(a)) \right) ' \frac{1}{\sqrt{R}} (b'h + (R - b'g(a))b^{-1}) \]
\[ - \left( \frac{1}{\sqrt{R}} (b'h + (R - b'g(a))b^{-1}) \right)^2 + \left( \frac{1}{\sqrt{R}} (bh - g(a)) \right)^2 f'(a) \]
\[ = -\frac{1}{\sqrt{R}} (b'h + (R - b'g(a))b^{-1})' \frac{1}{\sqrt{R}} (bh - g(a)) \]
\[ + \frac{1}{\sqrt{R}} (bh - g(a))' \frac{1}{\sqrt{R}} (b'h + (R - b'g(a))b^{-1}) \]
\[ - \left( \frac{1}{\sqrt{R}} (b'h + (R - b'g(a))b^{-1}) \right)^2 + \left( \frac{1}{\sqrt{R}} (bh - g(a)) \right)^2 f'(a). \]

We have
\[ RB_{21} = -(b'h + (R - b'g(a))b^{-1})' (bh - g(a)) + (bh - g(a))' (b'h + (R - b'g(a))b^{-1}) \]
\[ - (b'h + (R - b'g(a))b^{-1})^2 + (bh - g(a))^2 f'(a) \]
\[ = -(b''h + b'h') + R'b^{-1} - b''g(a)b^{-1} - b'g'(a)bb^{-1} - Rb^{-2}b' + b'g(a)b^{-2}b'(bh - g(a)) \]
\[ + (b'h + bh' - g'(a)b)(b'h + (R - b'g(a))b^{-1}) \]
\[ - (b'h + (R - b'g(a))b^{-1})^2 + (bf'(a)h - g(a)f'(a))(bh - g(a)) \]
\[ = -(b'h' + R'b^{-1} - b''g(a)bb^{-1} - Rb^{-2}b' + b'g(a)b^{-2}b')(bh - g(a)) \]
\[ + (bh' - g'(a)b - (R - b'g(a))b^{-1})(b'h + (R - b'g(a))b^{-1}) \]
\[ = -(b'h' + R'b^{-1} - b''g(a)bb^{-1} - Rb^{-2}b' + b'g(a)b^{-2}b')(bh) \]
\[ - (b'h' + R'b^{-1} - b''g(a)bb^{-1} - Rb^{-2}b' + b'g(a)b^{-2}b'(g(a)) \]
\[ + (bh' - g'(a)b - (R - b'g(a))b^{-1})(b'h) \]
\[ + (bh' - g'(a)b - (R - b'g(a))b^{-1})(Rb^{-1}) + (g'(a)b - (R - b'g(a))b^{-1})(-b'g(a)b^{-1}) \]
\[ = Rh' - R'h + R'b^{-1}g(a) - g'(a)R - R^2b^{-2} + b'g(a)b^{-2}R. \]

Recall that, by definition,
\[ A = R'b^{-1}g(a) - g'(a)R - (R - b'g(a))b^{-2}R. \]
and \( h \) is such that
\[ \frac{h'}{R} h + \frac{A}{R} = 0. \]
We conclude that
\[ B_{21} = h' - \frac{R'}{R} h + \frac{A}{R} = 0. \]
Finally,

\[
B_{22} = -\left( \frac{b'}{\sqrt{R}} \right) H' + \frac{b}{\sqrt{R}} \left( \frac{\dot{H}}{R} \right)' + \frac{b'}{\sqrt{R}} \dot{H} - \frac{b}{\sqrt{R}} H f'(a)
\]

\[
= -\left( \frac{b'}{\sqrt{R}} \right) \left( \frac{1}{\sqrt{R}} (b h - g(a)) \right)' + \frac{b}{\sqrt{R}} \left( \frac{1}{\sqrt{R}} (b' h + (R - b' g(a)) b^{-1}) \right)'
+ \frac{b'}{\sqrt{R}} \left( b' h + (R - b' g(a)) b^{-1} \right) - \frac{b}{\sqrt{R}} \left( b h - g(a) \right) f'(a)
\]

\[
= -\left( \frac{b'}{\sqrt{R}} \right) \left( \frac{1}{\sqrt{R}} (b h - g(a))' - \frac{R'}{2 R} (b h - g(a)) \right)
+ b \left( b' h + (R - b' g(a)) b^{-1} \right)' - \frac{R'}{2 R} (b' h + (R - b' g(a)) b^{-1})
+ b' (b' h + (R - b' g(a)) b^{-1}) - b (b h - g(a)) f'(a)
\]

Note that

\[ b'' = f'(a) b. \]

We have

\[
RB_{22} = -b' \left( b' h + bh' - g'(a) b \right)
+ b \left( f(a) b h + b' h' - (R - b' g(a)) b^{-2} b' + (R' - b' g(a) b^{-1} b) b^{-1} \right) - \frac{R'}{2}
+ b' (b' h + (R - b' g(a)) b^{-1}) - b (b h - g(a)) f'(a)
\]

\[
= \frac{R'}{2}.
\]

In conclusion, we have

\[
B = \left( (T^{-1})' T + T^{-1} UT \right) = \begin{pmatrix} -\frac{R'}{2 R} & 0 \\ 0 & \frac{R'}{2 R} \end{pmatrix}.
\]

We also include an important technical equality for future use.

**Lemma 1.3.** We have

\[
\left( \frac{H}{\sqrt{R}} \right)' = \frac{\dot{H}}{\sqrt{R}}.
\]
Proof.

\[
\left( \frac{H}{\sqrt{R}} \right)' = -\frac{1}{R^2} (bh - g(a))R' + \frac{1}{R} (b'h + bh' - g'(a)b) \\
= -\frac{1}{R^2} (bh - g(a))R' + \frac{1}{R} (b'h + b \left( \frac{R'}{R} h - \frac{A}{R} \right) - g'(a)b) \\
= -\frac{1}{R^2} (bh - g(a))R' \\
+ \frac{1}{R} \left( b'h + b \left( \frac{R'}{R} h - \frac{R' b^{-1} g(a) - g'(a) R - (R - b' g(a)) b^{-2} R}{R} \right) - g'(a)b \right) \\
= \frac{1}{R} (b'h + b \left( (R - b' g(a)) b^{-1} R \right)) = \tilde{H}/\sqrt{R}.
\]

\[\square\]

1.2. Equations and Solutions Around Unperturbed Homoclinic Loop. We study the dynamics of solutions of non-autonomously perturbed equations around a homoclinic solution of the unperturbed equation. We write the perturbed equations as

\[
\begin{align*}
\frac{dx}{dt} &= y + \varepsilon P(x, y, t), \\
\frac{dy}{dt} &= f(x) + \varepsilon Q(x, y, t).
\end{align*}
\]

The unperturbed equation is that of \( \varepsilon = 0 \). We assume that \((x, y) = (0, 0)\) is a saddle fixed point of the unperturbed equation and \((a(t), b(t))\) is a homoclinic solution. This is to say that we have

\[f(0) = 0,\]

and the homoclinic solution \((a(t), b(t))\) is such that

\[\lim_{t \to \pm \infty} (a(t), b(t)) = (0, 0).\]

We also assume, without loss of generality, that

\[b(0) = 0, \quad a(0) > 0.\]

This is for us to have, by symmetry,

\[a(t) = a(-t), \quad b(t) = -b(-t).\]

Let

\[\ell = \{(a(t), b(t)) : \quad t \in (-\infty, +\infty)\}\]

and \(D_{\varepsilon}\) be the small neighborhood of \(\ell\) of size \(K\varepsilon\).

The surfaces \(\Sigma_0\) and \(\Sigma_1\): In what follows, let \(I_0\) be a small segment of the \(x\)-axis around \(x = a(0)\) and \(I_1\) be a small segment whose left end is \(x = 0\). This is to say we let

\[I_0 \in (-\varepsilon + a(0), K\varepsilon + a(0)), \quad I_1 = (0, K\varepsilon)\]

We let

\[\Sigma_0 := \{(x, 0, t) : \quad x \in I_0, \quad y = 0, \quad t \in (-\infty, +\infty)\}\]

\[\Sigma_1 := \{(x, 0, t) : \quad x \in I_1, \quad y = 0, \quad t \in (-\infty, +\infty)\}\]

in the extended phase space. Our purpose is to compute the return time

\[\mathcal{F} : \Sigma_0 \to \Sigma_1\]

defined by the perturbed equation. For this purpose, we need to calculate the solution of equation satisfies the initial condition \(x(t_0) = x_0, y(t_0) = 0\) for an arbitrarily given \(t_0\) and \(x_0 \in I_0\). From this point on, we focus on this specific solution.
Let \((\hat{x}(t), \hat{y}(t))\) be the solution of the perturbed equation satisfying \((\hat{x}(t_0), \hat{y}(t_0)) = (x_0, y_0) \in I\). First we let \((x(t), y(t)) = (\hat{x}(t + t_0), \hat{y}(t + t_0))\). Then, \((x(t), y(t))\), as far as it stays inside of \(D_\varepsilon\), satisfies

\[
\begin{align*}
\frac{dx}{dt} &= y + \varepsilon P(x, y, t + t_0); \\
\frac{dy}{dt} &= f(x) + \varepsilon Q(x, y, t + t_0).
\end{align*}
\]

It is such that

\[
x(0) = x_0, \quad y(0) = 0.
\]

Let \(X, Y\) be such that

\[
\varepsilon X(t) = x(t) - a(t); \quad \varepsilon Y(t) = y(t) - b(t).
\]

We also denote

\[
X(t) = X, \quad Y(t) = Y, \quad x(t) = x, \quad y(t) = y, \quad a = a(t), \quad b = b(t).
\]

We have

\[
\begin{align*}
\frac{dX}{dt} &= Y + \varepsilon P(\varepsilon X + a, \varepsilon Y + b, t + t_0); \\
\frac{dY}{dt} &= f'(a)X + \varepsilon S(t, \varepsilon, X) + Q(\varepsilon X + a, \varepsilon Y + b, t + t_0).
\end{align*}
\]

where

\[
S(t, \varepsilon, X) = \varepsilon^{-2} \left( f(\varepsilon X + a) - f(a) - \varepsilon f'(a)X \right).
\]

This is to have

\[
\frac{d}{dt} \left( \begin{array}{c} X \\ Y \end{array} \right) = U \left( \begin{array}{c} X \\ Y \end{array} \right) + \left( \begin{array}{c} 0 \\ \varepsilon S(t, \varepsilon, X) \end{array} \right) + \left( \begin{array}{c} P(\varepsilon X + a, \varepsilon Y + b, t + t_0) \\ Q(\varepsilon X + a, \varepsilon Y + b, t + t_0) \end{array} \right)
\]

where

\[
U = \left( \begin{array}{cc} 0 & 1 \\ f'(a) & 0 \end{array} \right).
\]

We introduce new variables \((M, W)\) by letting

\[
\left( \begin{array}{c} X \\ Y \end{array} \right) = T \left( \begin{array}{c} M \\ W \end{array} \right); \quad \left( \begin{array}{c} M \\ W \end{array} \right) = T^{-1} \left( \begin{array}{c} X \\ Y \end{array} \right)
\]

where

\[
T = \left( \begin{array}{cc} -H & \frac{b}{\sqrt{R}} \\ -\bar{H} & \frac{b}{\sqrt{R}} \end{array} \right); \quad T^{-1} = \left( \begin{array}{cc} \frac{\bar{b}}{\sqrt{R}} & -\bar{b} \\ \bar{H} & -H \end{array} \right).
\]

We have

\[
\begin{align*}
X &= -HM + \frac{b}{\sqrt{R}} W, \quad Y = -\bar{H}M + \frac{b'}{\sqrt{R}} W; \\
M &= \frac{b'}{\sqrt{R}} X - \frac{b}{\sqrt{R}} Y, \quad W = \bar{H}X - HY.
\end{align*}
\]

We also recall that

\[
H = \frac{1}{\sqrt{R}}(bh - g(a)); \quad \bar{H} = \frac{1}{\sqrt{R}}(b'h + (R - b'g(a))b^{-1}).
\]

The new equations for \(M, W\) are

\[
\frac{d}{dt} \left( \begin{array}{c} M \\ W \end{array} \right) = \left( \begin{array}{cc} -\frac{R'}{2R} & 0 \\ 0 & \frac{R'}{2R} \end{array} \right) \left( \begin{array}{c} M \\ W \end{array} \right) + T^{-1} \left( \begin{array}{c} 0 \\ \varepsilon S(t, X, \varepsilon) \end{array} \right) + T^{-1} \left( \begin{array}{c} P(\varepsilon X + a, \varepsilon Y + b, t + t_0) \\ Q(\varepsilon X + a, \varepsilon Y + b, t + t_0) \end{array} \right).
\]
This is to have

\[
\frac{dM}{dt} = -\frac{R'}{2R} M - \varepsilon \frac{b}{\sqrt{R}} S + \frac{b'}{\sqrt{R}} P - \frac{b}{\sqrt{R}} Q
\]

\[
\frac{dW}{dt} = \frac{R'}{2R} W - \varepsilon H S(t, \varepsilon, X) + \tilde{H} P - HQ
\]

where

\[ S = S(t, \varepsilon, X), \quad P = P(\varepsilon X + a, \varepsilon Y + b, t + t_0), \quad Q = Q(\varepsilon X + a, \varepsilon Y + b, t + t_0) \]

and

\[ X = -HM + \frac{b'}{\sqrt{R}} W, \quad Y = -\tilde{H} M + \frac{b'}{\sqrt{R}} W \]

Finally, we let \((M, W)\) be such that

\[ M = \sqrt{RM}, \quad W = \left(\sqrt{R}\right)^{-1} W. \]

In new variables \(M, W\), we have

\[
\frac{dM}{dt} = -\varepsilon b S + b' P - bQ
\]

\[
\frac{dW}{dt} = -\varepsilon H S + \frac{1}{\sqrt{R}} \tilde{H} P - \frac{1}{\sqrt{R}} HQ
\]

We have, as far as this solution stays inside of \(D_\varepsilon\), that

\[
M(t) = M_0 - \int_0^t (-\varepsilon b S + b' P - bQ) \, dt
\]

\[
W(t) = W_0 - \int_0^t \left(-\varepsilon H S + \frac{1}{\sqrt{R}} \tilde{H} P - \frac{1}{\sqrt{R}} HQ \right) \, dt
\]

We let

\[ g(a) = b', \quad R = b^2 + (b')^2. \]

We have

\[ A = ((b')^2 - b^2)(f'(a) + 1), \]

and, from

\[ h' - \frac{R'}{R} h + \frac{A}{R} = 0, \]

\[ h = (b^2 + (b')^2) \int_0^t \frac{((b')^2 - b^2)(f'(a) + 1)}{(b^2 + (b')^2)^2} \, dt. \]

and

\[ H = \frac{bh - b'}{\sqrt{b^2 + (b')^2}}; \quad \tilde{H} = \frac{b'h + b}{\sqrt{b^2 + (b')^2}}. \]

In particular, we have

\[ R(0) = (f(a(0)))^2, \quad h(0) = 0, \]

\[ H(0) = -1, \quad \tilde{H}(0) = 0. \]

Now for an initial point \((x_0, 0, t_0)\sigma\), we have

\[
M_0 = \frac{1}{\varepsilon} (x_0 - a(0)),
\]

\[ W_0 = 0. \]
This solution is then defined by

\[ M(t) = M(0) + \int_0^t (-\varepsilon b S + b'P - bQ) \, dt \]

(1.28)

\[ W(t) = \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} HS + \frac{1}{\sqrt{R}} HP - \frac{1}{\sqrt{R}} HQ \right) \, dt \]

1.3. **Primary Stable and Unstable Solution.** From this section on, we consider the case in which

\[ f(x) = -x + x^3, \quad P = 0, \quad Q = x^2 \sin t \]

In this case we have

\[ S(t, \varepsilon, X) = \varepsilon^{-2} \left( f(\varepsilon X + a) - f(a) - \varepsilon f'(a)X \right) \]

(1.29)

\[ = \varepsilon^{-2} \left( -\varepsilon X + a \right) = \varepsilon^{-2} \left( -\varepsilon X + a \right) \]

We also have

\[ Q = (a + \varepsilon X)^2 \sin(t + t_0) \]

We have the solution formula for \( M, W \) as

\[ M(t) = M(0) + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + t_0) \right) \, dt \]

(1.30)

\[ W(t) = \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin(t + t_0) \right) \, dt \]

We begin with

**Proposition 1.2.** For every \( t_0 \in (-\infty, \infty) \), there exists a unique \( M^s(t_0) \), such that \((M^s(t_0), 0, t_0) \in \Sigma \) is such that \((x(t), y(t)) \to (0, 0) \) as \( t \to +\infty \). There also exists unique \( M^u(t_0) \), such that \((M^u(t_0), 0, t_0) \in \Sigma \) is such that \((x(t), y(t)) \to (0, 0) \) as \( t \to -\infty \).

We denote the corresponding primary stable solution as \( M^s(t, t_0), W^s(t, t_0), t \in [0, +\infty) \), and the primary unstable solution as \( M^u(t, t_0), W^u(t, t_0), t \in (-\infty, 0) \). We have

\[ M^s(t_0) = -\int_0^{+\infty} \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + t_0) \right) \, dt \]

(1.31)

\[ W^s(t_0) = 0 \]

The corresponding quantities for primary unstable solutions are similar.

We now assume the stable curve \((t_0, M^s(t_0))\) and the unstable curve \((t_0, M^u(t_0))\) intersect at \( t_0 = 0 \). This is to say that we have

\[ M^s(0) = M^u(0), \]

**New Variables** \( \xi, \eta \): We now introduce new variable \((\xi, \eta)\) by letting

\[ M_0 = M^s(\xi) + M^u(\eta) - M^s(0); \quad t_0 = \xi + \eta \]

This is a change of variable in between \((\xi, \eta)\) and \((t_0, M_0)\) on \( \Sigma \). The corresponding solution initiated at \( M_0, t_0 \) in \( \xi, \eta \) we denote as

\[ M(t, \xi, \eta), W(t, \xi, \eta) \]

We have

\[ M(t, \xi, \eta) = M^s(\xi) + M^u(\eta) - M^s(0) + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + \xi + \eta) \right) \, dt \]

(1.32)

\[ W(t, \xi, \eta) = \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin(t + \xi + \eta) \right) \, dt \]
where
\[ X = -HM(t, \xi, \eta) + \frac{b}{\sqrt{R}} W(t, \xi, \eta) \]
We also recall
\[ Y = -\tilde{H}M(t, \xi, \eta) + \frac{b'}{\sqrt{R}} W(t, \xi, \eta) \]

**Primary stable solutions: \( \eta = 0 \)**
\[
M^s(t, \xi, 0) = M^s(\xi) + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + \xi) \right) dt
\]
\[
W^s(t, \xi, 0) = \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin(t + \xi) \right) dt
\]
where
\[ X = -HM^s(t, \xi, 0) + \frac{b}{\sqrt{R}} W^s(t, \xi, 0) \]

The homoclinic solution is
\[
M^s(t, 0, 0) = M^s(0) + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin t \right) dt
\]
\[
W^s(t, 0, 0) = \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin t \right) dt
\]
\[ X = -HM^s(t, 0, 0) + \frac{b}{\sqrt{R}} W^s(t, 0, 0) \]

**Primary unstable solutions: \( \xi = 0 \)**
\[
M^u(t, 0, \eta) = M^u(\eta) + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + \eta) \right) dt
\]
\[
W^u(t, 0, \eta) = \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin(t + \eta) \right) dt
\]
where
\[ X = -HM^u(t, 0, \eta) + \frac{b}{\sqrt{R}} W^u(t, 0, \eta) \]

1.4. **From \( \Sigma_0 \) to \( \Sigma_1 \).** Recall that
\[
M(t, \xi, \eta) = M^s(\xi) + M^u(\eta) - M^s(0) + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + \xi + \eta) \right) dt
\]
\[ = M^s(\xi) + M^u(\eta) - M^s(0) + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a^2 + 2aX + \varepsilon^2 X^2) \sin(t + \xi + \eta) \right) dt
\]
\[ = M^s(\xi) + M^u(\eta) - M^s(0) + \int_0^t b a^2 \sin(t + \xi + \eta) dt
\]
\[ + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(2aX + \varepsilon^2 X^2) \sin(t + \xi + \eta) \right) dt
\]
\[ W(t, \xi, \eta) = \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin(t + \xi + \eta) \right) dt
\]
Recall that
\[ X = -HM(t, \xi, \eta) + \frac{b}{\sqrt{R}} W(t, \xi, \eta) \]
We have

\[ X = -H \left( M^u(\xi) + M^v(\eta) - M^u(0) - \int_0^t ba^2 \sin(t + \xi + \eta) dt \right) \]
\[ + \int_0^t \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(2\varepsilon aX + \varepsilon^2 X^2) \sin(t + \xi + \eta) \right) dt \]
\[ + \frac{b}{\sqrt{R}} \left( \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - H(a + \varepsilon X) \sin(t + \xi + \eta) \right) dt \right) \]

\[ \tilde{X} = -H \left( M^u(\xi) - \int_0^t ba^2 \sin(t + \xi) dt \right) \]
\[ + \int_0^t \left( -\varepsilon b(3a\tilde{X}^2 - \varepsilon \tilde{X}^3) - b(2\varepsilon a\tilde{X} + \varepsilon^2 \tilde{X}^2) \sin(t + \xi) \right) dt \]
\[ + \frac{b}{\sqrt{R}} \left( \int_0^t \left( -\frac{\varepsilon}{\sqrt{R}} H(3a\tilde{X}^2 - \varepsilon \tilde{X}^3) - H(a + \tilde{X}) \sin(t + \xi) \right) dt \right) \]

where \( X = X(t, \xi, \eta), \tilde{X} = X(t, \xi, 0) \). Let

\[ \Delta X = X - \tilde{X} = X(t, \xi, \eta) - X(t, \xi, 0) \]

and let \( T = T(\xi, \eta) \) be the time the solution hit \( \Sigma_1 \). We study \( \Delta X \) for \( t \in (0, T) \). We start with

\[ \Delta X = X_0(t) + \varepsilon H \int_0^t F_1 \Delta X dt + \varepsilon \frac{b}{\sqrt{R}} \int_0^t F_2 \Delta X dt \]

where

\[ X_0(t) = H(\xi) - H(0) - \int_0^t ba^2 \sin(t + \xi + \eta) dt \]
\[ - \int_0^t \frac{1}{\sqrt{R}} Ha^2 \sin(t + \xi + \eta) dt + \varepsilon \eta H \int_0^t G_1 dt + \varepsilon \frac{b}{\sqrt{R}} \int_0^t G_2 dt \]

and \( F_1, F_2, G_1, G_2 \) are functions of \( X, \tilde{X} \). The details of these functions are not important.

**Iterations:** We write

\[ \Delta X = X_0(t) + \mathcal{F}(\Delta X) \]

where

\[ \mathcal{F}(\Delta X) = \varepsilon H \int_0^t F_1 \Delta X dt + \varepsilon \frac{b}{\sqrt{R}} \int_0^t F_2 \Delta X dt \]

and define \( X_n(t) \) by letting

\[ X_{n+1}(t) = X_0(t) + \mathcal{F}(X_n(t)) \]

**Proposition 1.3.** We prove

1. Convergence of this iteration process;
2. \( X_0(t) = F_0(t) \eta \) where \( F_0(t) \) is such that \( 0 < K^{-1} < F_0(t) < K \) for a constant \( K \);
3. \( X_n(t) = (F_0(t) + O(\varepsilon)) \eta \) for all \( n \).

From these items we conclude

\[ \Delta X = (F_0(t) + O(\varepsilon)) \eta. \]

**Proof.** (1) is by contraction provided by small \( \varepsilon \); (2) need some checking but should be straightforward; (3) Should also follows by contracting mapping principle. \( \square \)
1.5. **Time it takes from** $\Sigma_0$ **to** $\Sigma_1$: Let $T = T(\xi, \eta)$ be the time solution hit $\Sigma_1$. We have $Y(T) = 0$. Recall that

$$Y(t, \xi, \eta) = -\ddot{H}M(t, \xi, \eta) + \frac{b'}{\sqrt{R}} W(t, \xi, \eta).$$

We have at $t = T$,

$$\ddot{H}M(T, \xi, \eta) = \frac{b'}{\sqrt{R}} W(T, \xi, \eta)$$

Recall

(1.39)

$$M(T, \xi, \eta) = M^s(\xi) + M^u(\eta) - M^s(0) + \int_0^T \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + \xi + \eta) \right) dt$$

$$\mathbb{W}(T, \xi, \eta) = \int_0^T \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin(t + \xi + \eta) \right) dt$$

We still need the reference point at $\eta = 0$ by using

$$\ddot{H}M(T, \xi, 0) = \frac{b'}{\sqrt{R}} W(T, \xi, 0) - Y(T, \xi, 0)$$

where

(1.40)

$$M(T, \xi, 0) = M^s(\xi) + \int_0^T \left( -\varepsilon b(3aX^2 - \varepsilon X^3) - b(a + \varepsilon X)^2 \sin(t + \xi) \right) dt$$

$$\mathbb{W}(T, \xi, 0) = \int_0^T \left( -\frac{\varepsilon}{\sqrt{R}} H(3aX^2 - \varepsilon X^3) - \frac{1}{\sqrt{R}} H(a + \varepsilon X)^2 \sin(t + \xi) \right) dt$$

We have

$$\ddot{H}[M(T, \xi, \eta) - M(T, \xi, 0)] = \frac{b'}{\sqrt{R}} [W(t, \xi, \eta) - W(t, \xi, 0)] - Y(T, \xi, 0)$$

We now put the integral formulas into this equality to solve for $T$. The key here is $M^u(\eta) - M^u(0) \approx \eta$. This is also the key in proving item (2) in previous proposition.

**To be continued....**