Power Series Solutions and Integral Manifold of the n-body Problem

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1. Introduction

In this article we discuss our solutions for two of the questions asked in the history of the n-body problem: the construction of the global power series solutions of the n-body problem and the bifurcations of the integral manifold of the spatial three-body problem. We start with a brief discussion on the origins of these two questions.

(a) Global power series solutions: The question on the construction of the global power series solutions was formulated by Weierstrass in 1880's. At the time it was believed that, after being studied intensively for more than one hundred years, the Newtonian n-body problem was about to be resolved analytically. A prize was then established and an announcement was made in volume 7, 1885/86, *Acta Mathematica* for solving the following question formulated by Weierstrass.

Given a system of arbitrarily many mass points that attract each other according to Newton's laws, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

This prize was later awarded to H. Poincaré not for resolving the question but for his invention of the geometric approach in the study of the n-body problem and other differential equations, and its negative implications to the question asked. Poincaré revealed that, in general, mechanical systems are so intrinsically complicated that it is not possible to fully understand the dynamics of Newtonian gravitational systems at his time — a judgment remains unfortunately true till now.

However, to accept Poincare's negative indication on our ability to solve the n-body problem is not the same as to accept that the question posted in *Acta Mathematica* is not solvable. That question is actually solvable and was indeed solved some more than twenty years later by Karl Sundman [14] for the three-body problem. A key idea in Sundman's theory is to regularize the singularities of two-body collision, a task that is not possible for collisions of more than two bodies. Therefore his approach can not be generalized to solve the same question for the problems of more than three bodies. We refer the reader to [11] for a detailed presentation on Sundman's theory

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and to [3] for a complete discussion on history.

In [15] we introduced a set of coordinate change that is a variation of McGehee's transform to deal with the difficulties associated with singularities of collision. The question as stated in *Acta Mathematica* was resolved for all n and the construction of the indicated power series solution turned out to be surprisingly simple.

(b) Birkhoff's conjecture on integral manifold of the three-body problem: The n-body problem admits ten first integrals of linear momentum, angular momentum and energy. Fixing integral constants these first integrals define a (6n-10)-dimensional algebraic variety in a 6n-dimensional phase space. This algebraic variety is the so called integral manifold of the n-body problem. When $n = 3$ integral manifold is an eight dimensional object M_8 in a space of dimension eighteen. The seven dimensional reduced integral manifold is $M_7 = M_8/SO_2$ where SO_2 action is a rotation about the angular momentum vector.

In [2], Birkhoff wrote that

The manifold M_7 has fundamental importance for the problem of three bodies, but so far as I know, it has nowhere been studied, even with respect to the elementary question of connectivity.

As part of a lengthy discussion on the geometry of the integral manifold, Birkhoff showed in [2] that if the integrals are dependent at a point, then it is a point on a central configuration solution. He then stated:

In any case the manifold M_7 can only have a singularity at a point corresponding to an equilateral triangle solution or a straight line solution at constant mutual distances, ...it is only as f (the energy constant) and K (the angular momentum constant) vary through these values that the natural of M_7 from the standpoint of analysis situs can change.

Even though it was given as fact this statement is later known as Birkhoff's conjecture on integral manifolds. In 1941, Wintner [16], who knew Birkhoff's works on integral manifold, observed that "unfortunately, nothing explicit is known as to the topological structure of M_7".

The situation was changed in early 1970's with the papers of Smale [12] [13] and Easton [4], [5]. Smale made an extensive study of the integral manifold in the spirit of Morse theory and proved Birkhoff's conjecture for the plannar n-body problem. Easton used an elementary scheme of projection to study the geometric structure of the integral manifold and obtained a detailed description that independently confirmed Birkhoff's conjecture for the planar three-body problem. However, the original conjecture, which is on the spatial three-body problem, remained open.

In [7], we studied the geometric structure of the integral manifold of the spatial three-body problem along the lines of Easton's scheme. We obtained seven bifurcation values. Four correspond to the solutions of central configurations and three additional ones correspond to restrictions on orientations at infinity. Birkhoff's conjecture on the spatial three-body problem turned out to be false.

2. Global Power Series Solutions

(1) Consider the system of n gravitational particles $(m_1, \cdots, m_n)$ moving in $\mathbb{R}^3$. Let $r_i = (x_i, y_i, z_i)$ be the position vector for m_i. The equations of motion for the n-body problem are written as

$$m_i \frac{d^2 r_i}{dt^2} = \sum_{j \neq i} \frac{m_i m_j (r_j - r_i)}{|r_j - r_i|^3}. \tag{1}$$

Let $q = (r_1, \cdots, r_n)$, $p = (\frac{1}{m_1}\frac{dr_1}{dt}, \cdots \frac{1}{m_n}\frac{dr_n}{dt})$. We have from (1) that

$$\frac{dq}{dt} = M^{-1} p, \qquad \frac{dp}{dt} = DU(q) \tag{2}$$

where $M = diag(m_1, \cdots, m_n)$, $DU(q)$ is the gradient of the potential function $U(q)$ and

$$U(q) = \sum_{i<j} \frac{m_i m_j}{|r_i - r_j|}.$$

Let Δ_{ij} be the set of $q \in \mathbb{R}^{3n}$ such that $r_i = r_j$, and $\Delta = \cup \Delta_{ij}$. (2) is analytically defined on $(q, p) \in \hat{R}$ where $\hat{R} = \{\mathbb{R}^{3n} \setminus \Delta\} \times \mathbb{R}^{3n}$.

Definition 1 *For any given $t_0 \in \mathbb{R}$ and $(q_0, p_0) \in \hat{R}$, we call the solution of (2) satisfying $q(t_0) = q_0, p(t_0) = p_0$ that is fully extended on both directions of the real t-axis as a* **global solution**.

If a global solution is defined on (t_-, t_+) and $t_+ < +\infty$, then $q(t_+) \to \Delta$ as $t \to t_+$, and the solution will end up with a singularity characterized by $U(q) \to \infty$. So under the assumption that a solution avoids collisions of all kind, $t_\pm = \pm\infty$. Let the time interval of the global solution satisfying $q(t_0) = q_0, p(t_0) = p_0$ be denoted as $(t_-(t_0, q_0, p_0), t_+(t_0, q_0, p_0))$. For t_0 fixed we write

$$\mathcal{C} = \{(q_0, p_0) \in \hat{R} : \ t_-(t_0, q_0, p_0) > -\infty \ or \ t_+(t_0, q_0, p_0) < +\infty\}.$$

Weierstrass's question can now be reformulated as to construct a power series solution that converges on the entire t-axis for any given initial condition that is in $\hat{R} \setminus \mathcal{C}$.

It is rather easy to see that the power series solution indicated in Weierstrass's formulation exists: Regarding time t as a complex variable, any solution in $\hat{R} \setminus \mathcal{C}$ is analytic on a simply connected domain of the real t-axis. This domain can be mapped onto the unit disk of a new complex variable, resulting a power series solution that converges for all real time. What was asked by Weierstrass is not the existence but a step by step **construction** of a power series solution, a question that is entirely different in natural. For a given initial condition (t_0, q_0, p_0), the first obstacle in constructing the indicated power series solutions is to decide if (q_0, p_0) is in $\mathcal{C}$. Unfortunately, it is not possible to tell if the solution of a given initial value would run into a singularity of collision in the future. Our inability in separating set $\mathcal{C}$ from the rest of the initial values is a major obstacle.

Sundman observed that, when $n = 3$, total collision of the three bodies is not possible for solutions of non-zero angular momentum. Therefore the only possible singularities for these solutions are collisions of two bodies. He further observed that the singularities of two-body collision are algebraic branch point in t. By writing equations of motion in new phase and time variables (the variables of regularization), the singularities of two-body collision are removed, allowing solutions to extend beyond the time of collision. Consequently, Sundman proved the following theorem that clearly resolves the question posted in *Acta Mathematica* for the three-body problem with non-zero angular momentum.

Theorem 1 (Sundman) *For any given initial condition t_0, q_0, p_0 of the three bodies, under the assumption that the total angular momentum is non-zero, there is a new time variable s* **explicitly defined** *and a constant $w > 0$* **explicitly given** *in q_0, p_0 and the masses, such that the time t and the positions q of the three bodies, as functions of s, are analytic on $|Im(s)| < w$. Furthermore, $t((-\infty, +\infty)) = (-\infty, +\infty)$.*

It is impossible to extend solutions beyond singularities of collision of more than two bodies, so Sundman's approach can not be generalized to solve the question for initial conditions of zero angular momentum for the three-body problem and for problems of more than three bodies.

(2) To solve the question of global power series solutions for the problems of more than three bodies, we introduced the following coordinate change in [15] to deal with the difficulty associated with singularities. First we write

$$\begin{aligned} u &= (2U(q) + h)^{-1}, \qquad && h > 0 \\ u &= (2U(q))^{-1} && h \le 0 \end{aligned}$$

where h is the total energy of the n-body system. Let F, G and τ be defined as

$$F = u^{-1} q, \qquad G = u^{\frac{1}{2}} p, \qquad \frac{d\tau}{dt} = u^{-\frac{3}{2}}. \tag{3}$$

Then F, G, u and t satisfies the following equations:

$$\begin{aligned} \frac{du}{dt} &= -2(M^{-1}G, DU(F))u \\ \frac{dF}{dt} &= M^{-1}G + 2(M^{-1}G, DU(F))F \\ \frac{dG}{dt} &= DU(F) - (M^{-1}G, DU(q))G \\ \frac{dt}{d\tau} &= u^{\frac{3}{2}} \end{aligned}$$

where

$$G^T M^{-1} G = 1, \qquad \frac{1}{2} - U(F) = uh, \quad h > 0;$$
$$G^T M^{-1} G = 1 + 2uh, \quad \frac{1}{2} - U(F) = 0, \quad h < 0.$$

First we have

Proposition 1 *([15]) The new time variable τ introduce as above has the property that, for any given initial condition (t_0, q_0, p_0) of the n-body problem,*

$$\tau((t_-(t_0, q_0, p_0), t_+(t_0, q_0, p_0))) = (-\infty, +\infty).$$

We further proved in [15] the following theorem that resolves the question of the construction of the global power series solutions for the n-body problem for all n.

Theorem 2 *([15]) For any given initial condition (t_0, q_0, p_0) of the n-body problem, there are constants $A, B > 0$* **explicitly given** *in (q_0, p_0) and the masses of the n bodies, such that the variables F, G, u and t introduced as above are analytic functions of τ on*

$$H: \qquad |Im\tau| < Ae^{-B|Re\tau|}.$$

We remark that (i) our solution covers the case of zero angular momentum for the three-body problem; (ii) the constants A and B are much easier to estimate than the constant w in Sundman's theorem; and (iii) the coordinate transform we introduced is a variation of McGehee's transform. Note that McGehee's transform has served as a starting point for many progresses made in the study of the n-body problem since early 1970's [8], [6], [17].

3. The Integral Manifold of the Three-body Problem

(1) Notations: To study the integral manifold of the three-body problem, we first use Jacobi's coordinate (R_1, R_2) to remove the integrals of linear momentum where R_1 is pointing from m_1 to m_2 and R_2 is pointing from the center of masses of m_1 and m_2 to m_3. The integrals of angular momentum and energy are written as

$$\mu_1 R_1 \times V_1 + \mu_2 R_2 \times V_2 = c \tag{4}$$
$$\frac{1}{2}(\mu_1 |V_1|^2 + \mu_2 |V_2|^2) - U = h \tag{5}$$

where $V_i = \frac{d}{dt} R_i$; c, h are the integral constants of angular momentum and energy and μ_i are certain functions of masses of the three bodies. To remove μ_i we further introduce $r_i = \sqrt{\mu_i} R_i$, $v_i = \sqrt{\mu_i} V_i$. (4) and (5) become

$$r_1 \times v_1 + r_2 \times v_2 = c \tag{6}$$
$$\frac{1}{2}(|v_1|^2 + |v_2|^2) - U = h. \tag{7}$$

Let $c_0(:= |c|)$, k $(:= c/|c|)$ be the magnitude and the direction of the total angular momentum. Without loss of generality we assume $k = (0,0,1)$. We will only consider the non-trivial case of negative energy.

Let $r = (r_1, r_2) \in \mathbb{R}^6$ be a *position vector* and $v = (v_1, v_2)$ be a *velocity vector*. For a rotation O in $\mathbb{R}^3$ we let $Or = (Or_1, Or_2)$, $Ov = (Ov_1, Ov_2)$. Let $S = \{r \in R^6 : |r| = 1\}$. For $r, r' \in S$ we say $r \sim r'$ if there is a rotation O such that $r = Or'$. The *space of three-body configurations* D is defined as the quotient $S/\sim$ and the map of projection is denoted as $\pi : S \to D$. We also denote $r_1 = (x_1, y_1, z_1)$, $r_2 = (x_2, y_2, z_2)$.

Definition 2 *$r \in S$ is a vector of standard position if*

$$z_1 = z_2 = x_1y_1 + x_2y_2 = 0, \quad x_1^2 + x_2^2 \geq y_1^2 + y_2^2.$$

We remark that, for any $r \in S$, there exists a rotation O in $\mathbb{R}^3$ such that Or is a vector of standard position.

We studied the integral manifolds in [7] by a projecting scheme. The following is a list of the intermediate objects created by projections.

(i) **The integral manifold**

$$M(c_0, h) = \{(r, v) : \textit{ such that } (6)(7) \textit{ hold}\};$$

(ii) **The Hill's region**

$$H(c_0, h) = \{r : \textit{ there exist } v, \textit{ such that } (r, v) \in M(c_0, h)\};$$

(iii) **The un-scaled Hill's region**

$$K(c_0, h) = \{r / \mid r \mid : \; r \in H(c_0, h)\};$$

(iv) **The set of admissible configuration**

$$AC(c_0, h) = \{p \in D : \; \pi^{-1}(p) \cap K(c_0, h) \neq \emptyset\}.$$

(2) Geometric descriptions: We now describe the geometry of the objects listed above. In the rest of this article, all the statements made are results of a rather lengthy geometric analysis that was carried out in [7].

Despite the fact that the integral manifolds are defined by the constant of angular momentum c_0k and the constant of energy h, there is only one bifurcation parameter $\mu = 2c_0^2|h|$. We will then denote the objects listed above as M_μ, H_μ, K_μ and AC_μ. We start from AC_μ, the set of admissible configurations.

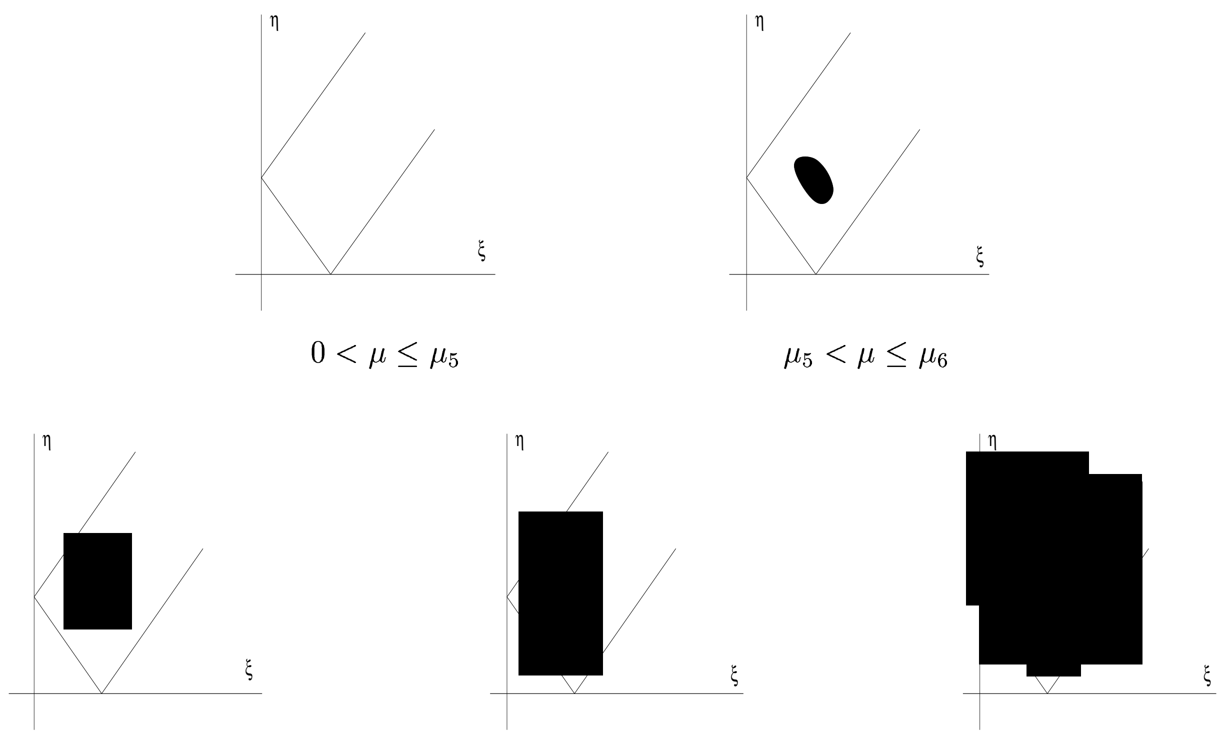
A description on AC_μ: For $p \in D$, let $r \in \pi^{-1}p$ be a vector of standard position that represents p. We have

Proposition 2 *$p \in AC_\mu$ if and only if*

$$U^2(r) \geq \mu. \tag{8}$$

D is actually the space of all abstract triangles. To represent a point $p \in D$, we fix one side of the triangle as being one and denote the other two sides as ξ and η. Then D is the region in the first quadrant of the (ξ, η)-plane determined by $\xi + \eta \geq 1, \xi + 1 \geq \eta, \eta + 1 \geq \xi$.

Let μ_5 be the value of μ corresponding to the solution of the equilateral triangle, and $\mu_6 \leq \mu_7 \leq \mu_8$ be the values corresponding to that of the collinear central configurations (μ_1-μ_4 are reserved for other potential values of bifurcation). It is shown in [7] that (8) imposes no restriction on AC_μ so $AC_\mu = D$ for $0 \leq \mu \leq \mu_5$. As μ passes μ_5, a small region emerges at the equilateral central configuration. Triangles in this region are excluded from AC_μ by (8). The boundary of this excluded region of triangles expands as μ increases, and touches the boundary of D at one of the collinear central configurations at $\mu = \mu_6$. This region keeps expanding, touching other two collinear central configurations at $\mu = \mu_7$ and μ_8. See Figure 1.



$0 < \mu \leq \mu_5$ $\quad$ $\mu_5 < \mu \leq \mu_6$

$\mu_6 < \mu \leq \mu_7$ $\quad$ $\mu_7 < \mu \leq \mu_8$ $\quad$ $\mu_8 < \mu$

Figure 1 The Geometry of the Admissible Configurations

A description on K_μ: We again let $r \in \pi^{-1}p$ be a vector of standard position.

Proposition 3 *K_μ is a singular fiber bundle over AC_μ. The fiber of K_μ over $p \in AC_\mu$ is determined as the following:*

(i) Let c_1, c_2 be such that

$$\frac{c_1^2}{y_1^2 + y_2^2} + \frac{c_2^2}{x_1^2 + x_2^2} \leq \frac{U^2(r)}{\mu} - 1 \tag{9}$$

and $c_1^2 + c_2^2 \leq 1$.

Define V_p as the set of unit vectors $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ with the indicated c_1, c_2 and $c_3 = \pm\sqrt{1 - c_1^2 - c_2^2}$.

(ii) For any $c \in V_p$, let

$$O(c) = \{O \in SO(3) : \text{ such that } Oc = k\}$$

where $SO(3)$ is the set of rotations in $\mathbb{R}^3$.

(iii) The fiber of K_μ over $p \in AC_\mu$ is

$$F_p = \{Or : \ O \in O(c) \text{ and } c \in V_p\}.$$

For a given triangle $p \in AC_\mu$, we have three degrees of freedom in placing p in $\mathbb{R}^3$. We can rotate p around k, move p closer or away from k and rotating the plane defined by p. The first rotation about k is always admissible but there are restrictions on the other two. These restrictions are represented by V_p. Depending on the way in which the ellipse (9) and the unit circle $c_1^2 + c_2^2 = 1$ intersect. V_p could be the entire unit sphere, a band or two caps. See Figure 2.

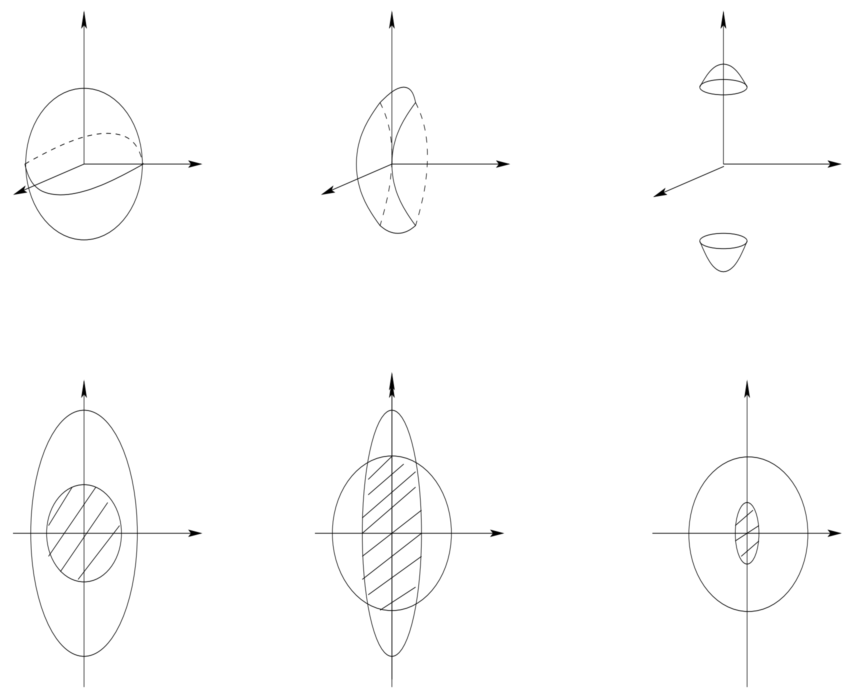
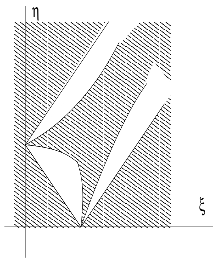


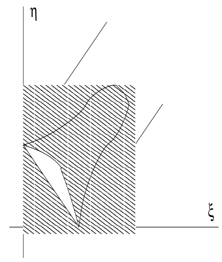
Figure 2 Three Possibilities for V_p

We need to further study the respective sets in AC_μ over which the fiberation is the entire unit sphere, a band or two caps. A detailed analysis was carried out in [7] and the results are presented in Figure 3. In these pictures, No shadowing indicates that V_p is a ring, single hashing indicates that V_p is the entire unit sphere and double hashing indicates that V_p is two caps. The solid black indicates the set of triangles that are not admissible, as shown in Figure 1.

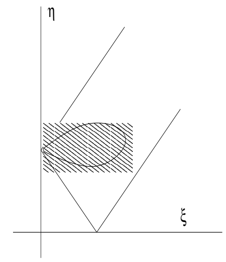
For small $\mu > 0$, the fiberation is shown in Figure 3(I). There are three-simple curves, very close to the boundary lines of D, connecting configurations of collisions of two bodies. Inside the region surrounded by these curves, V_p is the entire unit sphere and on the outside V_p is a band. These curves take the boundary lines of D as limit when $\mu \to 0$. As μ passes through $\mu_1 \leq \mu_2 \leq \mu_3$, these curves pull off from configurations of two-body collision, giving cases in Figure 3(II)-(IV). As μ increases to $\mu = \mu_4$, the single hashed region shrinks to disappear at the *critical configuration*, a triangle whose orthocenter is also the center of gravity. As μ passes μ_4, a double hashed region emerges immediately at the critical configuration. The next set of bifurcations are as described before in Figure 1.



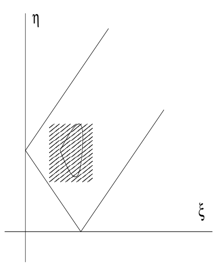
(I) $0 < \mu \leq \mu_1$



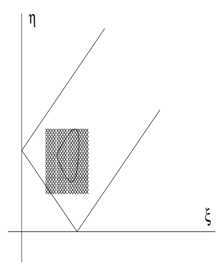
(II) $\mu_1 < \mu \leq \mu_2$



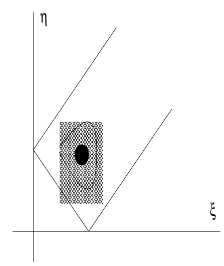
(III) $\mu_2 < \mu \leq \mu_3$



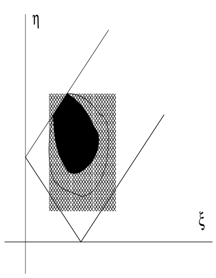
(IV) $\mu_3 < \mu \leq \mu_4$



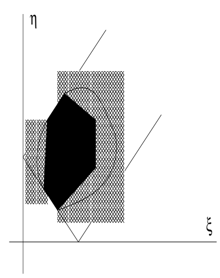
(V) $\mu_4 < \mu \leq \mu_5$



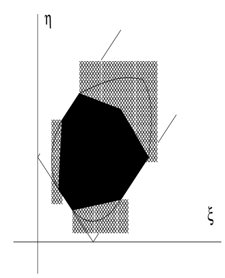
(VI) $\mu_5 < \mu \leq \mu_6$



(VII) $\mu_6 < \mu \leq \mu_7$



(VIII) $\mu_7 < \mu \leq \mu_8$



(IX) $\mu_8 < \mu$

Figure 3 The Fiberation in K_μ over AC_μ

A description on H_μ and M_μ: We further obtained the following propositions on the geometry of H_μ and M_μ in [7].

Proposition 4 *H_μ is a singular bundle over K_μ. For any $r \in K_\mu$, the fiber over r in H_μ is determined as follows:*

(i) According to the description on K_μ, there exists O, a rotation in $\mathbb{R}^3$; r', a vector of standard position; and c, a unit vector in $\mathbb{R}^3$; such that $r = Or'$ and $Oc = k$.

(ii) *Let I_r be the set of all positive real ρ satisfying*

$$2 \mid h \mid \rho^2 - 2\rho U(r) \leq c_0^2 (\frac{c_1^2}{\rho_y} + \frac{c_2^2}{\rho_x} + \frac{c_3^2}{\rho_x + \rho_y}), \tag{10}$$

then the fiber in $H(c_0, h)$ over $r \in K(c_0, h)$ is

$$H_r = \{\rho r : \quad \rho \in I_r\}.$$

Note that $\rho_x = x_1^2 + x_2^2$, $\rho_y = y_1^2 + y_2^2$ and x_i, y_i are from the vector of standard position r'.

(iii) *I_r is in general a closed interval, which shrinks to one point over the boundaries of K_μ.*

We also have the following proposition on M_μ.

Proposition 5 *M_μ is a singular fiber bundle over H_μ. For each $r \in H_\mu$, the fiber over r is a 2-sphere if r is not collinear. It is a 3-sphere if r is collinear. Each fiber shrinks to one point over the boundaries of H_μ.*

(3) The table of cohomology groups for M_μ: We further computed the cohomology groups of M_μ for different values of μ in [7]. See Table 1 for the result.

$H^p(M_\mu)$	0	1	2	3	4	5	6	7	8
$0 < \mu \leq \mu_1$	$\mathbb{Z}$	0	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^3$	0	$\mathbb{Z}^2$	0
$\mu_1 < \mu \leq \mu_2$	$\mathbb{Z}$	0	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	0
$\mu_2 < \mu \leq \mu_3$	$\mathbb{Z}$	0	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
$\mu_3 < \mu \leq \mu_4$	$\mathbb{Z}$	0	$\mathbb{Z}^3$	$\mathbb{Z}^4$	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0
$\mu_4 < \mu \leq \mu_5$	$\mathbb{Z}$	0	$\mathbb{Z}^3$	$\mathbb{Z}^4$	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0
$\mu_5 < \mu \leq \mu_6$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	0	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0
$\mu_6 < \mu \leq \mu_7$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
$\mu_7 < \mu \leq \mu_8$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0
$\mu_8 < \mu$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0	0	0	0

Table 1 The table of cohomology groups for M_μ

From this table we conclude that the topological types of M_μ change when μ passes through μ_1-μ_3 and μ_5-μ_8. No bifurcation is detected at the critical configuration ($\mu = \mu_4$). It was actually proved earlier by Albouy [1] that μ_4 is not a bifurcation value. This table clearly refutes Birkhoff's conjecture since μ_1, μ_2 and μ_3 are values of bifurcation related to collision, not central, configurations.

We conclude this discussion by listing the values of μ_1-μ_8.

$$\mu_{1,2,3} = \frac{1}{2}\frac{(m_1 m_2)^3}{m_1 + m_2}$$

$$\mu_4 = \frac{1}{2}\left(m_1 m_2\sqrt{\frac{m_1 m_2}{m_1+m_2}} + m_2 m_3\sqrt{\frac{m_2 m_3}{m_2+m_3}} + m_3 m_1\sqrt{\frac{m_3 m_1}{m_3+m_1}}\right)^2$$

$$\mu_5 = \frac{1}{2}\frac{(m_1 m_2 + m_2 m_3 + m_1 m_3)^3}{m_1 + m_2 + m_3}$$

$$\mu_{6,7,8} = \frac{1}{4}\frac{(m_1 m_2 r_{12}^2 + m_2 m_3 r_{23}^2 + m_1 m_3 r_{13}^2)}{m_1 + m_2 + m_3}\left(\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_1 m_3}{r_{13}}\right)^2$$

where $\mu_{1,2,3}$ means that the three values are obtained by permuting the masses in the formula given. $\mu_{6,7,8}$ are similar, with r_{ij} being the mutual distances of the three bodies at the collinear central configurations.

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