1 Evaluating an integral with a branch cut

This is an elementary illustration of an integration involving a branch cut. It may be done also by other means, so the purpose of the example is only to show the method.

The integral is

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\, dx = \pi.$$

The essential point is to consider an appropriate analytic function. The choice of how to do this is explained below. For the moment, we take

$$f(z) = \frac{1}{z\sqrt{1-\frac{1}{z}}}.$$

The square root is taken with the cut along the negative axis. Thus the function is analytic except where $1 - 1/z$ is on the negative axis or where we are dividing by zero. This just says that $z = x$ where $0 \leq x \leq 1$.

If we take $z = x + iy$ with $0 < x < 1$, then $1 - 1/z = 1 - (x - iy)/(x^2 + y^2)$ with real part $1 - x/(x^2 + y^2)$ and imaginary part $iy/(x^2 + y^2)$. As y approaches zero the real part converges to $1 - 1/x < 0$ and the imaginary part approaches zero from the half-plane in which y is taken. So the boundary value is $i\sqrt{1/x - 1}$ when we approach from the upper half plane and is $-i\sqrt{1/x - 1}$ when we approach from the lower half plane. Thus the boundary values of $f(z)$ are

$$f(x + i0) = \frac{1}{\pm ix\sqrt{1-\frac{1}{x}}} = \frac{1}{\pm i\sqrt{x(1-x)}}.$$

Take a curve C going around the interval $0 \leq x \leq 1$ counterclockwise. We can replace C by such a curve that goes around the interval and stays a distance ϵ from it. Then by taking the limit $\epsilon \to 0$ we see that

$$\int_C f(z)\, dz = 2i \int_0^1 \frac{1}{\sqrt{x(1-x)}}\, dx.$$

Now we would like to evaluate the contour integral. The trick is to realize that there is a pole of $f(z)\, dz$ at $z = \infty$. The easy way to see this is to make the change of variable $z = 1/w$, $dz = -1/w^2\, dw$. Then we get

$$\int_C f(z)\, dz = -\int_{C'} f(1/w) \frac{dw}{w^2} = -\int_{C'} \frac{1}{\sqrt{1-w}} \frac{dw}{w}.$$

The contour C' goes around the pole at $w = 0$ in the clockwise direction. By the residue theorem the answer is $2\pi i$ times the residue, which is 1. So $2i$ times the original integral is $2\pi i$.

2 Defining the analytic function

The confusing aspect of this problem is the definition of the analytic function in the denominator. Throughout we take the square root to have a cut along the negative real axis. The function is

$$g(z) = z\sqrt{1 - 1/z}.$$

This has a cut on the real interval from 0 to 1. Notice that it is positive above one and negative below zero.

There are other ways to write this function. The most obvious is

$$g(z) = \sqrt{z}\sqrt{z-1}.$$

This is certainly the same for $z = x$ real with $x > 1$. So it should be the same for all z that are not real. The only question is the meaning of this for $z = x$ for $x < 0$. The problem is that the branch cut for the square root is along the negative axis. However in this case one can take either square root, and the result for the product is the same, since $i^2 = (-i)^2 = -1$. So we can think of this as the same function. Again it is positive above one and negative below zero. The boundary values on the unit interval are just the same as before.

Another possible expression that looks almost the same is

$$h(z) = \sqrt{z(z-1)}.$$

Notice, however, that it is positive both above one and below zero. Furthermore, the branch structure is quite different. This may be seen by writing it as

$$h(z) = \sqrt{(z - \frac{1}{2})^2 - \frac{1}{4}}.$$

This has a cut when when $z = x$ is real and in the unit interval, but also when $z - 1/2 = iy$ is pure imaginary. Thus there is a second branch cut when $z = 1/2 + iy$. Furthermore, the boundary values on the unit interval are quite different.

3 The treacherous square root

Again define the square root with the cut on the negative axis. The basic problem is that in general

$$\sqrt{zw} \neq \sqrt{z}\sqrt{w}.$$

In fact the right hand is undefined whenever z and w are both negative. The left hand side, on the other hand, is undefined when zw is negative, which is considerably more common. On the other hand, the value of the left hand side is always in the right half plane.